Spin Chains by Separation of Variables Method

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Abstract

The poster at hand depicts the main results of a research project realized during a three-year PhD program. The study concerns mainly the set-up and fine tuning of a technique known as *Separation of Variables* (SoV) in its quantum formulation. This method permitted us to recover the full spectrum and solve completely two eigenproblems strictly related to the open XXZ and XYZ spin-1/2 chain with the most generic boundary conditions. The eigenstates, in the *inhomogeneous* case, are constructed in terms of solutions of a system of quadratic equations. The SoV representation permits to compute scalar products as well and can be useful for the eventual calculation of correlation functions.

Notation

The sigma operators σ_i^a, with $i \in \{1,\dots,N\}$ and $a \in \{x,y,z\}$, are the usual Pauli matrices acting non-trivially in the ith space of the tensor product

$$\sigma_i^a = \underset{1}{\mathbf{Id}} \otimes \cdots \otimes \underset{i}{\sigma^a} \otimes \cdots \otimes \underset{N}{\mathbf{Id}}.$$

with

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The open XXZ spin-1/2 chain: definition

The quantum system that we want to describe and analyze is defined by the following quantum Hamiltonian

$$\begin{aligned}\mathbf{H}_{\mathrm{xxz}} = & \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(\eta)\left(\sigma_i^z \sigma_{i+1}^z - 1\right)\right] \\ & + \frac{\sinh(\eta)}{\sinh(\xi_-)}\left(\sigma_1^z \cosh\xi_- + 2\kappa_-(\sigma_1^x \cosh\tau_- + i\sigma_1^y \sinh\tau_-)\right) \\ & + \frac{\sinh(\eta)}{\sinh(\xi_+)}\left(\sigma_N^z \cosh\xi_+ + 2\kappa_+(\sigma_N^x \cosh\tau_+ + i\sigma_N^y \sinh\tau_+)\right).\end{aligned}$$

The quantum system here defined lives in the Hilbert space $\mathcal{H} = \mathbb{C}^{2^{\otimes N}}$, which consists of a tensor product of N spin-1/2 representation spaces $\mathcal{H}_{1/2} = \mathbb{C}^2$. The parameters $\{\zeta_-, \zeta_+, \kappa_-, \kappa_+, \tau_-, \tau_+\} \in \mathbb{C}^6$ encode the interaction with the boundaries.

The Sklyanin's SoV method: general scheme

- Pinpoint a valid $\mathcal{B}(\lambda)$ operator s.t.

$$[\mathcal{B}(\lambda), \mathcal{B}(\mu)] = 0$$

- Find the *operator zeroes* $\{\hat{x}_n\}$ of $\mathcal{B}(\lambda)$: the **separated variables**;

$$\mathcal{B}(\lambda) = B_N \prod_{n=1}^{N} (\lambda - \hat{x}_n)$$

NB The other generators evaluated in these zeroes will be useful ladder operators;

- *Conjugated momenta* to the *coordinates* $\{x_n\}$

$$X_n^- = \sum_{p=1}^{N} x_n^p A_n = [A(\mu)]_{\mu = x_n},$$

$$X_n^+ = \sum_{p=1}^{N} x_n^p D_n = [D(\mu)]_{\mu = x_n};$$

- SoV-representation:

$$X_n^{\pm} |x_1, \dots, x_n, \dots, x_n\rangle \propto |x_1, \dots, x_n \pm \eta, \dots, x_n\rangle,$$

- Solve the eigenproblem associated to the transfer matrix $\mathcal{T}(\{X_n^{\pm}\}, \mathcal{B})$ $\forall n \in (1,\dots,N)$

$$\tau(x_n)\varphi(\mathbf{x}) = \Delta_n^+(\mathbf{x})\varphi(E_n^- \mathbf{x}) + \Delta_n^-(\mathbf{x})\varphi(E_n^+ \mathbf{x}),$$

- Separated *Baxter-like* equations, $\varphi(x_1,\dots,x_n) = \prod_{n=1}^{N} Q_n(x_n)$

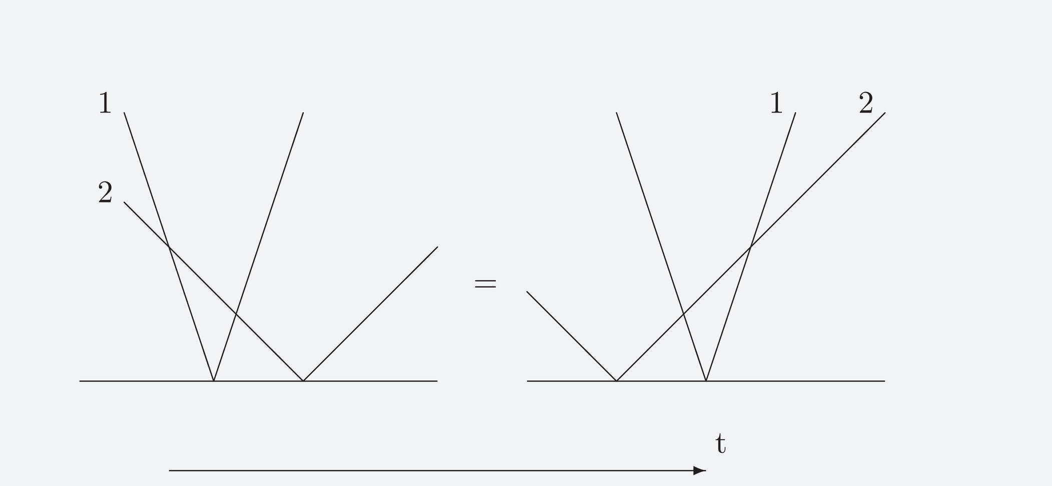
$$\tau(x_n)Q_n(x_n) = \Delta_+(x_n)Q_n(x_n - \eta) + \Delta_-(x_n)Q_n(x_n + \eta)$$

Historical Background

Open chains

In the 80's **Cherednik** ('84) studied $(1+1)$-dim scattering theories on the half-line in the **S-matrix** theory. Alongside the usual two particle scattering matrix he introduced a reflection matrix describing the scattering against a boundary

- Reflection equation -



Cherednik's work motivated **Sklyanin** to *translate* the theory in the algebraic framework of *QISM*. This applies successfully to the study of open spin-chains with integrable boundary conditions characterized by a *boundary monodromy matrix* satisfying the reflection equation

$$R_{12}(\lambda-\mu)\overset{1}{\mathcal{U}}(\lambda)R_{12}(\lambda+\mu-\eta)\overset{2}{\mathcal{U}}(\mu) = \overset{2}{\mathcal{U}}(\mu)R_{12}(\lambda+\mu-\eta)\overset{1}{\mathcal{U}}(\lambda)R_{12}(\lambda-\mu)$$

where as usual

$$R_{ij}(\lambda) \in \mathrm{End}(V_i \otimes V_j), \quad \overset{i}{\mathcal{U}}(\lambda) \in \mathrm{End}(\mathcal{H} \otimes V_i)$$

QISM

- The **Quantum Inverse Scattering Method** (QISM) is a direction of the theory of quantum integrable systems who has its roots in summer 1978:
  - Leningrad, U.S.S.R. → Faddeev, Sklyanin, Takhtajan et al.;
  - Fermilab, U.S.A. → Tacker, Creamer, Wilkinson;
- QISM as synthesis of three traditions:
  - *Bethe Ansatz* technique. Hans Bethe (1931) and devoloped by : Hulten, Lieb, Yang, Yang, Baxter ...;
  - *CISM*. Works of Gardner, Greene, Kruskal and Miura (1967-74) on KdV and developed by: Lax, Ablowitz, Kaupp, Newell, Zakharov, Shabat, Faddeev ...;
  - *S-matrix* theory. Works of A.B. Zamolodchikov and Al.B. Zamolodchikov in the 60's in the context of $(1+1)$-dim massive QFT as deformation of massless CFT

The reflection algebra

This model belongs to a class of quantum integrable systems characterized, in the framework of QISM, by monodromy matrices $\mathcal{U}(\lambda)$ solutions of a **reflection equation**, with the **6-vertex trigonometric** solution to the **Yang-Baxter** equation

$$R_{12}(\lambda)R_{13}(\lambda+\mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda+\mu)R_{12}(\lambda)$$

$$R_{12}^{6\mathrm{v}}(\lambda) = \begin{pmatrix} \sinh(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \sinh\lambda & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh\lambda & 0 \\ 0 & 0 & & \sinh(\lambda+\eta) \end{pmatrix} \in \mathrm{End}(V_1 \otimes V_2)$$

- The boundary matrices solution of the **RE**

$$K\pm(\lambda) = \frac{1}{\sinh\zeta_\pm}\begin{pmatrix} \sinh(\lambda+\zeta_\pm \pm \eta/2) & \kappa_\pm e^{\tau_\pm}\sinh(2\lambda \pm \eta) \\ \kappa_\pm e^{-\tau_\pm}\sinh(2\lambda \pm \eta & \sinh(\zeta_\pm - \lambda \mp \eta/2) \end{pmatrix}$$

- The *bulk* monodromy matrix solution of the Yang-Baxter relation

$$R_{12}(\lambda-\mu)M_1(\lambda)M_2(\mu) = M_2(\mu)M_1(\lambda)R_{12}(\lambda-\mu)$$

$$M_0(\lambda) = R_{0N}(\lambda-\xi_N-\eta/2)\dots R_{01}(\lambda-\xi_1-\eta/2) \in \mathrm{End}(\mathcal{H}\otimes V), \ V = \mathbb{C}^2$$

  where we introduced a set of N inhomogeneities $\{\xi_i\}_{i=1,\dots,N}$.

- The boundary monodromy matrix solution of the **RE**

$$\mathcal{U}_- = M_0(\lambda)K_-(\lambda)M_0^{-1}(-\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix} \in \mathrm{End}(\mathcal{H}\otimes V)$$

- The transfer matrix: $\boxed{\mathcal{T}(\lambda) = \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}}$

Trace identity

$$H_{\mathrm{xxz}} = \frac{2(\sinh\eta)^{(1-2N)}}{\mathrm{tr}\{K_+(\eta/2)\}\,\mathrm{tr}\{K_-(\eta/2)\}}\frac{d}{d\lambda}\ln(\mathcal{T}(\lambda))\Big|_{\substack{\lambda=\eta/2 \\ \xi_1,\dots,\xi_N=0}} + \text{const.}$$

The gauge transformations

In order to keep the boundary terms unconstrained we need to introduce some gauge transformations. The gauge transformations that we employed are acting purely at a representation level, the auxiliary space $\mathcal{V}_0 \simeq \mathbb{C}^2$, while the Hilbert space will be left unchanged. The main idea is that the transfer matrix, and then its spectrum, should be invariant under the action of such transformations. The *naive* representation

$$\mathcal{T}_{\mathrm{gauge}}(\lambda) = \mathrm{tr}_0\{S^{-1}M(\lambda)SS^{-1}K_-(\lambda)SS^{-1}\hat{M}(\lambda)SS^{-1}K_+(\lambda)S\} = \mathcal{T}(\lambda)$$

Definitions

The form of the gauge transformations looks like:

$$\bar{G}(\lambda|\beta) = (X(\lambda|\beta), Y(\lambda|\beta)), \ \tilde{G}(\lambda|\beta) = (X(\lambda|\beta+1), Y(\lambda|\beta-1))$$

where the column vectors X and Y are defined as

$$X(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha+\beta)\eta]} \\ 1 \end{pmatrix}, \quad Y(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha-\beta)\eta]} \\ 1 \end{pmatrix}$$

The gauged algebra

The gauge transformations define a new set of operators in the following way:

$$M(\lambda|\beta) = \tilde{G}^{-1}(\lambda-\eta/2|\beta)M(\lambda)\tilde{G}(\lambda-\eta/2|\beta+N) = \begin{pmatrix} A(\lambda|\beta) & B(\lambda|\beta) \\ C(\lambda|\beta) & D(\lambda|\beta) \end{pmatrix},$$

$$\hat{M}(\lambda|\beta) = \bar{G}^{-1}(\eta/2-\lambda|\beta+N)\hat{M}(\lambda)\bar{G}(\eta/2-\lambda|\beta) = \begin{pmatrix} \hat{A}(\lambda|\beta) & \hat{B}(\lambda|\beta) \\ \hat{C}(\lambda|\beta) & \hat{D}(\lambda|\beta) \end{pmatrix},$$

$$\mathcal{U}_-(\lambda|\beta) = \tilde{G}^{-1}(\lambda-\eta/2|\beta+N)\mathcal{U}_-(\lambda)\tilde{G}(\eta/2-\lambda|\beta+N) = \begin{pmatrix} \hat{\mathcal{A}}(\lambda|\beta+2) & \hat{\mathcal{B}}(\lambda|\beta) \\ \hat{\mathcal{C}}(\lambda|\beta+2) & \hat{\mathcal{D}}(\lambda|\beta) \end{pmatrix}.$$

The *quantum determinant* relation is still valid:

$$\mathcal{U}_-(\lambda+\eta/2|\beta)\mathcal{U}_-(-\lambda+\eta/2|\beta) = \frac{\text{q-det}\ (\mathcal{U}_-(\lambda)}{\sinh(2\lambda-2\eta)}$$

The Left-SoV representation

The reference state

First of all one has to introduce a proper *reference state*

$$\langle\beta| = \prod_{n=1}^{N}\left({}_n\langle\uparrow| + g_\beta(\xi_n)\,{}_n\langle\downarrow|\right)$$

The bulk operators act on it as

$$\langle\beta|B(\lambda|\beta) = \langle\beta|\bar{B}(\lambda|\beta) = 0;$$

$$\langle\beta|A(\lambda|\beta) = \frac{\sinh(N+\beta)\eta}{\sinh\beta\eta}\prod_{n=1}^{N}\sinh(\lambda-\xi_n+\eta/2)\langle\beta-1|;$$

$$\langle\beta|D(\lambda|\beta) = \prod_{n=1}^{N}\sinh(\lambda-\xi_n-\eta/2)\langle\beta+1|;$$

$$\langle\beta|\bar{A}(\lambda|\beta) = \frac{\sinh\beta\eta}{\sinh(N+\beta)\eta}\prod_{n=1}^{N}\sinh(\lambda+\xi_n-\eta/2)\langle\beta+1|;$$

$$\langle\beta|\bar{D}(\lambda|\beta) = \prod_{n=1}^{N}\sinh(\lambda+\xi_n-\eta/2)\langle\beta-1|;$$

N.B. the reference state is a *pseudo-eigenvector*: $\boxed{\langle\beta|\mathcal{B}_-(\lambda|\beta) = \mathsf{B}_0(\lambda)\langle\beta-2|}$.

The Left-SoV eigenbasis

The following set of states

$$\begin{cases} \langle\beta,\mathbf{h}| = \langle\beta|\prod_{n=1}^{N}\left(\mathcal{A}_-(\eta/2-\xi_n|\beta+2)/N_n^1\right)^{h_n} \\ \textbf{Cond.} \quad \xi_a \neq \xi_b + r\eta, \ \forall a \neq b \in \{1\dots N\}, \ r \in \{-1,0,1\} \end{cases}$$

is a set of pseudo-eigenvectors : $\boxed{\langle\beta,\mathbf{h}|\mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\mathbf{h}}(\lambda)\langle\beta-2,\mathbf{h}|}$

Main results

SoV identity decomposition

The set of states defined above is indeed complete since the following holds

$$\mathbf{Id} = \frac{1}{\mathcal{N}}\sum_{h_1,\dots,h_n=0}^{1}\prod_{1\leq b<a\leq N}(\eta_a^{h_a}-\eta_b^{h_b})|\beta,h_1,\dots,h_n\rangle\langle\beta-2,h_1,\dots,h_n|$$

where $\eta_a^{h_a} = \cosh 2\,[\xi_a + (h_a - 1/2)\eta]$ and

$$\langle\beta-2,h_1,\dots,h_n|\beta,h_1,\dots,h_n\rangle = \mathcal{N}\prod_{1\leq b<a\leq N}\frac{1}{(\eta_a^{h_a}-\eta_b^{h_b})}$$

The eigenvectors of the transfer matrix can be defined in this basis

$$|\tau\rangle = \sum_{h_1,\dots,h_n=0}^{1}\prod_{a=1}^{N}Q_\tau(\zeta_a^{h_a})\prod_{1\leq b<a\leq N}(\eta_a^{h_a}-\eta_b^{h_b})|\beta,h_1,\dots,h_n\rangle$$

given that $\zeta_n^{h_n} = \phi_n(\xi_n + (h_n - 1/2)\eta)$.

Gauge fixing

Before completing the solution of the eigenproblem we have to consider the decomposition of the transfer matrix

$$\begin{aligned}\mathcal{T}(\lambda) = \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\} = & [K_+]^{11}(\lambda|\beta-1)\mathcal{A}(\lambda|\beta) + [K_+]^{12}(\lambda|\beta-1)\mathcal{D}(\lambda|\beta) \\ & + [K_+]^{21}(\lambda|\beta-1)\mathcal{B}(\lambda|\beta-2) + [K_+]^{12}(\lambda|\beta-1)\mathcal{C}(\lambda|\beta+2).\end{aligned}$$

We exploit the gauge freedom to put the fourth term to zero

$$[K_+(\lambda|\beta-1)]^{12} = 0$$

SoV characterization of the spectrum

Everything has been set-up now and we can compute the action of the transfer matrix evaluated in the zeroes of $\mathcal{B}_-(\lambda)$, *i.e.* the separated variables

$$\langle\beta-2,h_1,\dots,h_N|t(\zeta_n^{h_n})|\tau\rangle \quad \forall n \in \{1,\dots,N\}$$

$$\boxed{\tau(\zeta_n^{h_n})\Psi_\tau(\mathbf{h}) = \boldsymbol{\Delta}(\zeta_{\mathbf{n}}^{\mathbf{h_n}})\boldsymbol{\Psi}_\tau(\mathbf{E}_{\mathbf{n}}^-(\mathbf{h})) + \boldsymbol{\Delta}(-\zeta_{\mathbf{n}}^{\mathbf{h_n}})\boldsymbol{\Psi}_\tau(\mathbf{E}_{\mathbf{n}}^+(\mathbf{h}))}$$

where

$$\Psi_\tau(\mathbf{h}) = \langle\beta-\mathbf{2},\mathbf{h_1},\dots,\mathbf{h_N}|\tau\rangle = \prod_{\mathbf{a}=\mathbf{1}}^{\mathbf{N}}\mathbf{Q}_\tau(\zeta_{\mathbf{a}}^{\mathbf{h_a}})$$

and the characterization of the spectrum reads

$$\Sigma_t \equiv \left\{\tau(\lambda): \ \tau(\lambda) = f(\lambda) + \sum_{a=1}^{N}g_a(\lambda)x_a, \ \forall\{x_n\} \in \Omega\right\}$$

$$x_n\sum_{a=1}^{N}g_a(\zeta_n^{(1)})x_a + x_n f(\zeta_n^{(1)}) = q_n, \ \forall n \in \{1,\dots,N\}$$

Scalar Products

Scalar products. Given the generic SoV state (and the right counterpart)

$$\langle\alpha| = \sum_{h_1,\dots,h_n=0}^{1}\prod_{a=1}^{N}\alpha_\tau(\zeta_a^{h_a})\prod_{1\leq b<a\leq N}(\eta_a^{h_a}-\eta_b^{h_b})\langle\beta,h_1,\dots,h_n|$$

$$\langle\alpha|\beta\rangle = \det_N||\mathcal{M}_{a,b}^{(\alpha,\beta)}|| \quad \text{with} \quad \mathcal{M}_{a,b}^{(\alpha,\beta)} \equiv \sum_{h=0}^{1}\alpha(\zeta_a^h)\beta(\zeta_a^h)(\eta_a^{(h)})^{(b-1)}$$

Remarks and Future works

- The whole set of results displayed here were obtained for the generic open boundary XYZ spin-1/2 chain as well;
- Recently *Kitanine, Maillet and Niccoli* managed to build a Q-operator satisfying a inhomogeneous Baxter equation

$$\mathcal{T}(\lambda)Q(\lambda) = \Delta(\lambda)Q(\lambda-\eta/2) + \Delta(-\lambda)Q(\lambda+\eta/2) + F(\lambda),$$

The construction of the Q-op. for the XYZ chain constitutes my work in progress;
- Form factors and correlation functions ;
- Completeness of the spectrum in the homogeneous limit.

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