Quantization and dynamization of Trace-Poisson brackets

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Introduction: The problem

- Starting point (Van den Bergh construction) define relevant Poisson structures on representation spaces of intersecting algebras.
- Question: deformations? dynamical ? Gervais-Neveu-Felder?
- Strategy:
  Define classical r-matrix formulation
  Quantize it; Identify as Reflection Algebra type
  then Dynamize it (known procedure; three GNF deformations already exist, see RAQIS '12)

Introduction: Preliminaries

$\rightarrow$ Notion of Double Lie Bracket; Double Poisson Bracket; Trace Poisson Bracket.

On vector space: Double Lie Bracket:

Definition

Let V be any $\mathbb{C}-$vector space. A **double Lie bracket** *is a $\mathbb{C}-$linear map $\{\!\{,\}\!\} : V \otimes V \mapsto V \otimes V$ satisfying the following two conditions:* $\{\!\{u, v\}\!\} = -\{\!\{v, u\}\!\}^o$

$$\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l + \sigma\left(\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l\right) + \sigma^2\left(\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l\right) = 0$$

On Algebra: Double Poisson Bracket (add Leibniz Rule)

$$\{\!\{a, bc\}\!\} = b\,\{\!\{a, c\}\!\} + \{\!\{a, b\}\!\}\,c$$

Go to Trace Poisson Brackets on Trace Space $A_\natural := A/[A, A]$: quotient vector space (the "trace space of A").
Representation variety:
A : free associative algebra over $\mathbb{C}$,$A = \mathbb{C} < x_1, ..., x_n >$
Then space of $N-$dimensional representations is

$$\mathrm{Rep}_N(A) = \{M = (M_1, \ldots, M_n) \in \mathbb{M}_N(\mathbb{C}) \oplus ... \oplus \mathbb{M}_N(\mathbb{C})\},$$

where $\mathbb{M}_N(\mathbb{C}) = \mathrm{End}(\mathbb{C}^N)$.
Van Den Bergh construction in this case:

Theorem

Given a double Poisson bracket on A one defines a bracket $[-,-] : A \times A \to A$

$$[a,b] := \{\!\{a,b\}\!\}^{(1)}\{\!\{a,b\}\!\}^{(2)} = \sum_i \{\!\{a,b\}\!\}_i^{(1)} \{\!\{a,b\}\!\}_i^{(2)} \qquad (1.1)$$

such that

- *(1.1) satisfies the following derivation property:*

$$[a,[b,c]] = [a,b],c]] + [b,[a,c]];$$

- *The restriction $[-,-] : A_\natural \times A_\natural \to A_\natural$ defines on $A_\natural$ a Lie algebra structure:*

$$[p(a),p(b)] := p([a,b]);$$

Theorem

- *Given a double Poisson bracket on A one defines a Poisson structure on the representation variety* $\mathrm{Rep}_N(A)$ *i.e. a Poisson bracket* $\{-,-\} : A_N \times A_N \to A_N$ *such that on generators of* A_N *(defined by elements a and b of A) we have*

$$\{a_{ij}, b_{kl}\} := \{\!\{a, b\}\!\}^{(1)}_{kj} \{\!\{a, b\}\!\}^{(2)}_{il}$$

- *The map* $\mathrm{Tr} : A_\natural \to A_N$ *is a morphism of Lie algebras* $A_\natural$ *and* $A_N = \mathbb{C}[\mathrm{Rep}_N(A)]$. *Namely:*

$$\{\mathrm{Tr}p(a), \mathrm{Tr}p(b)\} = \mathrm{Tr}([p(a), p(b)]).$$

From mappings to endomorphisms

Proposition

For V a vector space, let $r \in \mathrm{End}(V \otimes V)$ defines the operation

$$\{\!\{u, v\}\!\} := r(u \otimes v).$$

This operation induces a double Lie bracket on V iff r is skew-symmetric *and satisfies the* **Associative Yang–Baxter Equation (AYBE)** *in $V \otimes V \otimes V$:*

$$AYBE(r) := \quad r^{12} r^{13} - r^{23} r^{12} + r^{13} r^{23} = 0,$$

where, as usual, r^{ij} acts in $V^{\otimes 3}$, non trivially on (i, j) spaces and as identity elsewhere.

r is analogous of classical r matrix.

Remark: $AYBE - P_{13} AYBE P_{13} \equiv$ Classical YBE for r skew symmetric.

Particularize again to specific endomorphisms for V a free associative algebra A:

Let $A = \mathbb{C} < x_1, \ldots, x_m >$ be the free associative algebras

Constant, Linear, and Quadratic double brackets are defined by

$$\{\!\{x_\alpha, x_\beta\}\!\} = c_{\alpha\beta} 1 \otimes 1, \quad \text{with} \quad c_{\alpha,\beta} = -c_{\beta,\alpha},$$

$$\{\!\{x_\alpha, x_\beta\}\!\} = b^\gamma_{\alpha\beta} x_\gamma \otimes 1 - b^\gamma_{\beta\alpha} 1 \otimes x_\gamma,$$

$$\{\!\{x_\alpha, x_\beta\}\!\} = r^{uv}_{\alpha\beta}\, x_u \otimes x_v + a^{vu}_{\alpha\beta}\, x_u x_v \otimes 1 - a^{uv}_{\beta\alpha}\, 1 \otimes x_v x_u,$$

where

$$r^{\sigma\epsilon}_{\alpha\beta} = -r^{\epsilon\sigma}_{\beta\alpha},$$

Then trace brackets on representation variety are:
Constant:

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = c_{\alpha\beta}\,\delta^{j'}_i\,\delta^j_{i'} \qquad \Leftarrow \qquad \{\!\{x_\alpha, x_\beta\}\!\} = c_{\alpha\beta} 1 \otimes 1$$

Linear:

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} \;=\; b^\gamma_{\alpha\beta} x^{j'}_{i,\gamma} \delta^j_{i'} - b^\gamma_{\beta\alpha} x^j_{i',\gamma} \delta^{j'}_i$$

Quadratic:

$$\{x^j_{i,\alpha}, x^{j'}_{i',\beta}\} = r^{\gamma\epsilon}_{\alpha\beta} x^{j'}_{i,\gamma} x^j_{i',\epsilon} + a^{\gamma\epsilon}_{\alpha\beta} x^k_{i,\gamma} x^{j'}_{k,\epsilon} \delta^j_{i'} - a^{\gamma\epsilon}_{\beta\alpha} x^k_{i',\gamma} x^j_{k,\epsilon} \delta^{j'}_i \quad (2.\text{-}1)$$

Proposition

The quadratic Poisson brackets (2.-1) can be rewritten as

$$\{X_1 \overset{\otimes}{,} X_2\} = \mathfrak{r}_{12} X_1 X_2 + (X_2^t \mathfrak{a}_{12} X_1)^{t_2} - (X_1^t \mathfrak{a}_{21} X_2)^{t_1}$$

$$\begin{aligned}
X &= \sum_{\alpha=1}^{m} \sum_{ij=1}^{N} x_{i\alpha}^{j}\, e_\alpha \otimes e_{ji}\,, \\
\mathfrak{r}_{12} &= \sum_{\alpha,\beta,\gamma,\epsilon}^{m} \sum_{ij=1}^{N} r_{\alpha\beta}^{\gamma\epsilon}\, e_{\alpha\gamma} \otimes e_{ij} \otimes e_{\beta\epsilon} \otimes e_{ji}\,, \\
\mathfrak{a}_{12} &= \sum_{\alpha,\beta,\gamma,\epsilon}^{m} \sum_{ij=1}^{N} a_{\alpha\beta}^{\gamma\epsilon} e_{\alpha\gamma} \otimes e_{ij} \otimes e_{\beta\epsilon} \otimes e_{ji}\,.
\end{aligned}$$

Here two sets of indices: usual color indices(Roman letters) and supplementary "flavor indices" where non trivial behavior occurs.
Initial examples have trivial action on color indices.
General Trace Poisson bracket now assumed to have NO SUCH RESTRICTION!
Transposition here on flavor space (Greek indices).
Yang Baxter equations?

Proposition

If (r, a) is a solution of the AYBE relations

$$\begin{aligned}
&r^{\lambda\sigma}_{\alpha\beta} r^{\mu\nu}_{\sigma\tau} + r^{\mu\sigma}_{\beta\tau} r^{\nu\lambda}_{\sigma\alpha} + r^{\nu\sigma}_{\tau\alpha} r^{\lambda\mu}_{\sigma\beta} = 0,\\
&a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\mu\sigma}_{\tau\alpha} a^{\nu\lambda}_{\sigma\beta},\\
&a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\sigma\tau} = a^{\mu\sigma}_{\alpha\beta} r^{\lambda\nu}_{\tau\sigma} + a^{\mu\nu}_{\alpha\sigma} r^{\sigma\lambda}_{\beta\tau},\\
&a^{\lambda\sigma}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\sigma\nu}_{\alpha\beta} r^{\lambda\mu}_{\sigma\tau} + a^{\mu\nu}_{\sigma\beta} r^{\sigma\lambda}_{\tau\alpha}.
\end{aligned}$$

such that $a_{12} = a_{21}$ then $(\mathfrak{r}, \mathfrak{a})$, as defined above, is a solution of the set of direct and adjoint classical Yang–Baxter equation.

Classical to quantum

Proposition

Let R and A be two matrices acting on the tensor product of two copies of an auxiliary space $V = \mathbb{C}^N \otimes \mathbb{C}^m$. The space $\mathbb{C}^N$ (resp. $\mathbb{C}^m$) will be called the color (resp. flavor) space.
We define an associative algebra $\mathcal{A}$ through the following relation

$$(R_{12}(K_1{}^t A_{21} K_2)^{t_1})^{t_2} = ((K_2{}^t A_{12} K_1)^{t_1} R_{12}{}^{t_1 t_2})^{t_1},$$

where the transposition acts solely on the flavor indices, and $K \in \mathcal{A} \otimes End(\mathbb{C}^N) \otimes \mathbb{C}^m$.
Then $\mathcal{A}$ is a quantization of the trace Poisson algebra.

Generalized (following Freidel-Maillet):

Definition

The quantum trace Reflection Algebra relations read:

$$(A_{12}(K_1^t B_{12})^{t_1} K_2)^{t_2} = (K_2^t (C_{12} K_1)^{t_1} D_{12})^{t_1} \tag{4.1}$$

Auxiliary spaces 1 *and* 2 *are tensor spaces of a flavor and a color space. Transposition acts solely on the flavor indices. Dual relation exchange the auxiliary spaces 1 and 2:*

$$(A_{21}(K_2^t B_{21})^{t_2} K_1)^{t_1} = (K_1^t (C_{21} K_2)^{t_2} D_{21})^{t_2}$$

is a priori not *equivalent to* (4.1) *and must be considered simultaneously.*

Bivectorialize (Freidel Maillet) yields useful form of Reflection Algebra for YB and dynamization (link with RA as Twist of QA, Kulish-Mudrov)

Proposition

The quantum reflection trace algebra can be reformulated as:

$$\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}\,\mathcal{K}^{\mathrm{I}}_{11'}\mathcal{K}^{\mathrm{II}}_{22'} = \mathcal{K}^{\mathrm{II}}_{22'}\mathcal{K}^{\mathrm{I}}_{11'}$$

$$\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'} = (C^{T_{2'}}_{12'})^{-1}(D^{T_{1'}\,T_{2'}}_{1'2'})^{-1}A_{12}B^{T_{1'}}_{1'2}\,,$$

YB equation now:

Theorem

A sufficient condition for associativity of the R.A. is given by the Quantum Yang–Baxter equation for $\mathcal{R}$:

$$\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}\,\mathcal{R}^{\mathrm{I},\mathrm{III}}_{11',33'}\,\mathcal{R}^{\mathrm{II},\mathrm{III}}_{22',33'} = \mathcal{R}^{\mathrm{II},\mathrm{III}}_{22',33'}\,\mathcal{R}^{\mathrm{I},\mathrm{III}}_{11',33'}\,\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}$$

where $\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}$ is previous combined R-matrix.

Finding solutions of YBE required to get interesting example. Full decoupling flavor/color OK but too trivial. Other possibilities currently investigated.

Dynamization a la GNF

i.e. Deformation by coordinates on dual of Cartan algebra.
Starting point: canonical form of dynamical RA with pure color indices:
All three dynamical R.A.'s are represented by similar-looking quadratic exchange algebra relations:

$$A_{12}K_1(\lambda - \epsilon_R h_2)B_{12}K_2(\lambda + \epsilon_L h_1) = K_2(\lambda - \epsilon_R h_1)C_{12}K_1(\lambda + \epsilon_L h_2)D_{12}$$

Depending on value of ϵ_L, ϵ_R yields: Boundary Dynamical Algebra (-1,1); Semi-Dynamical Algebra (1,0) or (0,1); Twisted Yangian DRA (1,1).

First step: Bivectorialize DRA to get nearest expression to Quantized TPB:

Theorem

The dynamical R.A. are represented in the bivector formalism by the following expression:

$$A_{12}(-\epsilon_L(h_{2'}+h_{1'}))(B^{t_{1'}}_{1'2}(-\epsilon_L h_{2'}))^{\mathrm{sr}_{1'}(\epsilon_L)}\bar{K}_{11'}(-\epsilon_R h_2-\epsilon_L h_{2'})\bar{K}_{22'}=$$

$$(D^{t_{1'}t_{2'}}_{1'2'})^{\mathrm{sr}_{1'}(\epsilon_L)\mathrm{sr}_{2'}(\epsilon_L)}(C^{t_{2'}}_{12'}(-\epsilon_L h_{1'}))^{\mathrm{sr}_{2'}(\epsilon_L)}\bar{K}_{22'}(-\epsilon_R h_1-\epsilon_L h_{1'})\bar{K}_{11'}$$

Notations sr and sl allude to shifts on dynamical variables along non-trivial directions of R matrix:

Definition

On matrix objects:

$$((e^{(\epsilon h_a \partial)} M)^{t_a} e^{(-\epsilon h_a \partial)})^{t_a} := M^{sr(a)}$$

$$((e^{(\epsilon h_a \partial)} (M e^{(-\epsilon h_a \partial)})^{t_a} := M^{sc(a)}$$

On single vector indices:

$$\bar{K}_{aa'} := K_{aa'}^{\mathrm{s}_{a'}(-\epsilon_L)}, \quad a \in \{1,2\} := ((e^{(\epsilon h_{a'} \partial)} K_{aa'})^{t'_a} e^{(-\epsilon h_{a'} \partial)})^{t_{a'}}$$

Second step: Conjecture form of consistent GNF type deformations based on pure color case. Consistency criteria viewed as:
Criterion 1 Shifts separate into shifts labeled by vector-type indices in K (line indices, unprimed) weighted by $-\epsilon_R$; and shifts labeled by the tensorial factors corresponding to covector-type indices in K (column indices, primed) weighted by $-\epsilon_L$.
Criterion 2 Structure matrices A, B, C, D are accordingly zero-weighted according to the nature of their tensorial labels : zero-weight conditions are written for the *partially transposed* matrices such as occur in the bivector formulation.

$$\epsilon_R\,[h^{(1)} + h^{(2)}\,,\,A_{12}] \;=\; \epsilon_L\,[h^{(1)} + h^{(2)}\,,\,D_{12}] \;=\; 0$$
$$[\epsilon_R\,h^{(1)} - \epsilon_L\,h^{(2)}\,,\,C_{12}] \;=\; [\epsilon_L\,h^{(1)} - \epsilon_R\,h^{(2)}\,,\,B_{12}] \;=\; 0\,,$$

Criterion 3 All four structure matrices are shifted ONLY along both primed-labeled directions, weighted by $-\epsilon_L$. Depending whether these labels occur or not in the matrix the shifts are inside shifts (resp. outside).

Criterion 4 The K matrices are shifted along three directions: the two respective outside shifts and the inside prime (transposed covector) shift occur with their respective consistent weight

Yields conjectured dynamical deformation of quantized double Poisson bracket:

$$A_{12}(-\epsilon_L(h_{2'}+h_{1'}))(B_{1'2}^{T_{1'}}(-\epsilon_L h_{2'}))^{\mathrm{sr}_{1'}(\epsilon_L)} \bar{K}_{11'}(-\epsilon_R h_2-\epsilon_L h_{2'}-\epsilon_f h_{\mathrm{II}})\bar{K}_{22'}$$

$$=(D_{1'2'}^{T_{1'}T_{2'}})^{\mathrm{sr}_{1'}(\epsilon_L)\mathrm{sr}_{2'}(\epsilon_L)}(C_{12'}^{T_{2'}}(-\epsilon_L h_{1'}))^{\mathrm{sr}_{2'}(\epsilon_L)} \bar{K}_{22'}(-\epsilon_R h_1-\epsilon_L h_{1'}-\epsilon_f h_{\mathrm{I}})\bar{K}_{11'}$$

Examples (from known examples of dynamical R matrices)? Use ?
To be continued............