Integrability vs exact solvability in the quantum Rabi model and beyond

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in collaboration with Huan-Qiang Zhou, arXiv:1408.3816

Outline of this talk

0) interaction between light and matter

1) Rabi model

2) eigenspectrum of the quantum Rabi model

3) phenomenological criterion for quantum integrability

4) Yang-Baxter integrability

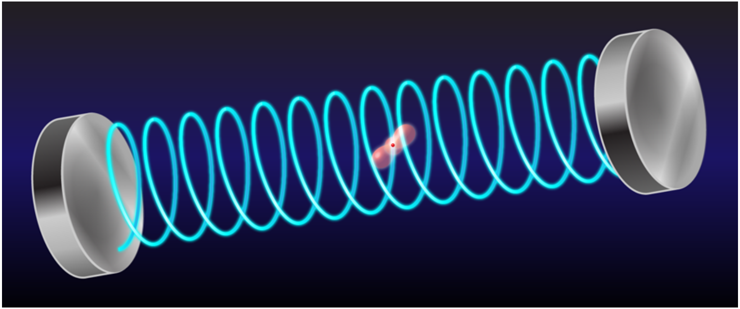
5) Dicke model

Interaction between quantum light and matter

The Rabi model describes a two-level atom coupled to a quantised, single mode harmonic oscillator (bosonic field).

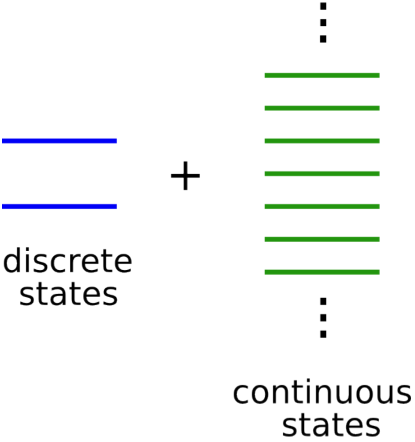
Applicable to a wide range of physical systems:

- interaction between light and trapped ions or quantum dots
- interaction between microwaves and superconducting qubits
- cavity QED
- circuit QED



APS/Alan Stonebraker

different class of models



The quantum Rabi model

I Rabi, Phys. Rev. **49**, 324 (1936); **51**, 652 (1937)

The Hamiltonian ($\hbar = 1$) reads

$$H_R = \Delta\,\sigma_z + \omega\, a^\dagger a + g\,\sigma_x (a + a^\dagger)$$

where

• σ_x and σ_z are Pauli matrices for the two-level system with level splitting 2Δ, and

• $a^\dagger$ (a) denote creation (destruction) operators for a single bosonic mode with $[a, a^\dagger] = 1$ and frequency ω.

• g is the coupling between the two systems.

The Rabi model has Z_2 symmetry (parity).

$\Longrightarrow$ simplest system of quantum light interacting with matter.

The Jaynes-Cummings model

E T Jaynes & F W Cummings, Proc. IEEE **51**, 89 (1963)

The JC model is the rotating-wave approximation to the Rabi model, with Hamiltonian

$$H_{JC} = \Delta\,\sigma_z + \omega\,a^\dagger a + g\,(\sigma^+ a + \sigma^- a^\dagger)$$

with $\sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

$$\left[H_R = \Delta\,\sigma_z + \omega\,a^\dagger a + g\,(\sigma^+ a + \sigma^- a^\dagger) + g\,(\sigma^+ a^\dagger + \sigma^- a)\right]$$

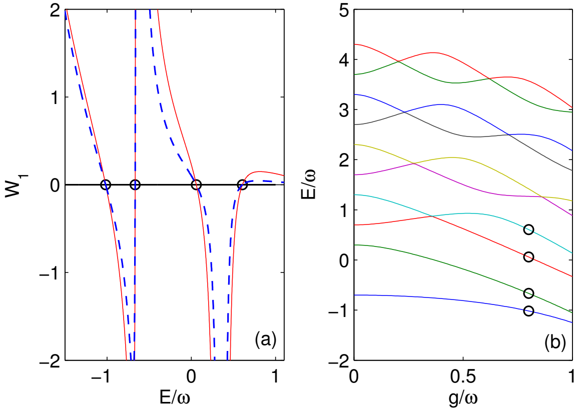
• applicable because the conditions of near resonance, $2\Delta \approx \omega$ and weak coupling $g \ll \omega$, for the rotating-wave approximation apply to many experiments.

• the JC model is integrable in the Yang-Baxter sense.

nice review in N M Bogoliubov & P P Kulish, J. Math. Sciences **192**, 14 (2013)

Eigenspectrum of the quantum Rabi model

- For a long time the Rabi model has been known to exhibit isolated exact solutions. B Judd, J Phys C **12**, 1685 (1979)
- Proven to be of the form $E = n\omega - g^2/\omega$ with n level crossings for each n. M Kus, J Math Phys **26**, 2792 (1985)



Aside: quasi-exactly solved models

- In quantum mechanics quasi-exactly solved (QES) systems are systems with potentials for which it is possible to find a finite number of exact eigenvalues and associated eigenfunctions in closed form.
- There is a correspondence between QES models in quantum mechanics and the set of orthogonal polynomials $P_m(E)$, which are polynomials in energy E. CM Bender & GV Dunne, JMP **37**, 6 (1996)
- The Rabi eigenvalue problem can be written as a 2nd order ODE.
- Under certain conditions this ODE can be solved in terms of orthogonal polynomials $\Rightarrow$ gives the isolated exact solutions. E.g., for $n = 1$, $E = 1 - g^2$ with $\Delta^2 + 4g^2 = 1$ (in units of ω).

  R Koc, M Koca & H Tütüncüler, J Phys A **35**, 9425 (2002)
- The Rabi model has been called a *quasi-exactly solved model*.

  A Moroz, Ann Phys (NY) **338**, 319 (2013), Y-Z Zhang, JMP **54**, 102104 (2013)

  only "a finite no. of their eigenvalues and corresponding eigenfunctions can be determined algebraically"

Braak's solution of the quantum Rabi model

D Braak, PRL **107**, 100401 (2011) & Ann. Phys. (Berlin) **525**, L23 (2013)

Using the representation of the bosonic operators in the Bargmann space of analytic functions

$$a^\dagger \to z, \qquad a \to \frac{d}{dz}$$

the regular eigenvalues are given in terms of the zeros $x_n^\pm$ of

$$G_\pm(x) = \sum_{n=0}^{\infty} K_n(x) \left[1 \mp \frac{\Delta}{x - n\,\omega}\right] \left(\frac{g}{\omega}\right)^n$$

where $K_n(x)$ are defined recursively by $nK_n = f_{n-1}(x)\,K_{n-1} - K_{n-2}$ with initial conditions $K_0 = 1$, $K_1(x) = f_0(x)$, and

$$f_n(x) = \frac{2g}{\omega} + \frac{1}{2g}\left(n\omega - x + \frac{\Delta^2}{x - n\,\omega}\right) .$$

The eigenvalues follow from $E_n^\pm = x_n^\pm - g^2/\omega$.

The function $G_\pm(x)$ is obtained as a consistency condition for two different series expansions for the eigenstates.

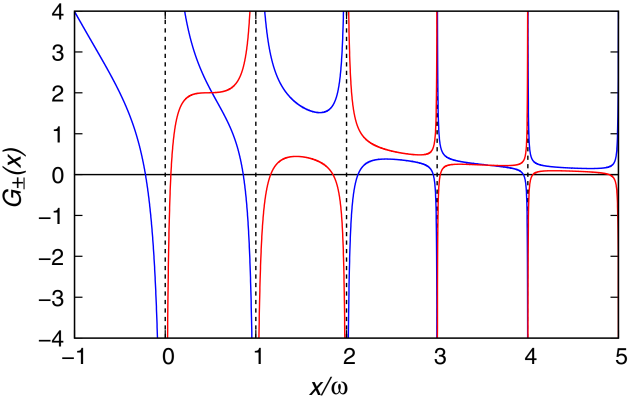


Figure from Braak for $\omega = 1, g = 0.7, \Delta = 0.4$

Red lines: $+$ parity, blue lines: $-$ parity.

Simple poles at $x = 0, \omega, 2\omega, \ldots$ correspond to the eigenvalues of the uncoupled bosonic modes.

Rabi spectrum

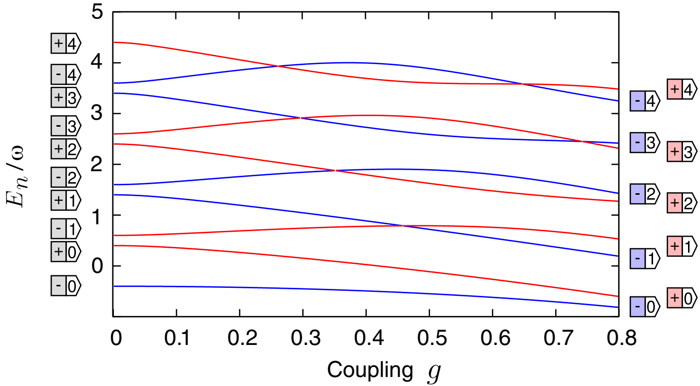


Figure from Braak for $\omega = 1$, $\Delta = 0.4$

Eigenstates $|\psi\rangle$ of the quantum Rabi model

H Zhong, Q Xie, MTB & C Lee, J. Phys. A **46**, 415302 (2013)

$$|\psi\rangle = \psi_1(a^\dagger)|0\rangle\,|\uparrow\rangle + \psi_2(a^\dagger)|0\rangle\,|\downarrow\rangle$$

where $\psi_{1,2}$ are analytical functions of the creation operator $a^\dagger$, $|0\rangle$ is the vacuum state for the bosonic mode, and $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z with eigenvalues 1 and -1.

Using the relations $a|0\rangle = 0$, $[a^\dagger, \psi_{1,2}] = 0$ and $[a, \psi_{1,2}] = \frac{d\psi_{1,2}}{dz}$ with $z = a^\dagger$, the operator functions satisfy

$$z\frac{d\psi_1}{dz} + g\left(\frac{d\psi_2}{dz} + z\psi_2\right) + \Delta\psi_1 = E\psi_1$$
$$z\frac{d\psi_2}{dz} + g\left(\frac{d\psi_1}{dz} + z\psi_1\right) - \Delta\psi_2 = E\psi_2$$

These equations are equivalent to the those in the Bargmann space of analytical functions.

Writing $f_1 = \psi_1 + \psi_2$ and $f_2 = \psi_1 - \psi_2$ and eliminating f_2, the coupled first-order differential equations are equivalent to a second-order differential equation for $f_1(z)$:

$$\frac{d^2 f_1}{dz^2} + p(z)\frac{df_1}{dz} + q(z) f_1 = 0$$

$$\begin{aligned} p(z) &= \frac{(1 - 2E - 2g^2)z - g}{z^2 - g^2}, \\ q(z) &= \frac{-g^2 z^2 + gz + E^2 - g^2 - \Delta^2}{z^2 - g^2}. \end{aligned}$$

• Symmetric, anti-symmetric and asymmetric solutions for the eigenstates are given in terms of confluent Heun functions.

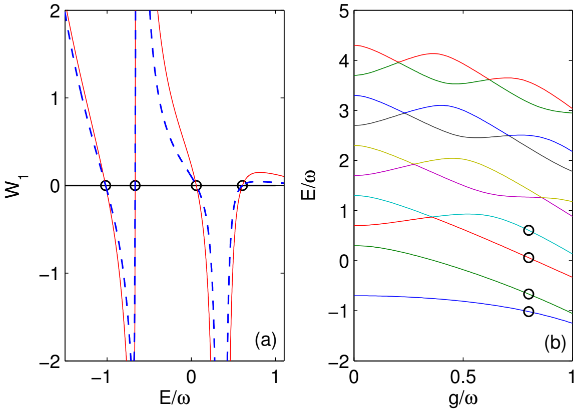
See also A J Maciejewski, M Przybylska & T Stachowiak, Phys. Lett. A **378**, 16 (2014)

The Heun functions satisfy a 2nd order linear ODE (Heun 1889). See "The Heun Project" theheunproject.org

• The exceptional parts of the eigenspectrum – the Judd isolated exact solutions – appear naturally as truncations of the confluent Heun functions.

• The conditions proposed by Braak are a type of sufficiency condition for determining the regular solutions.

• The figure shows the Wronskian $W_1(E, z)$ as a fn of E/ω for $z = 0$ (red lines) and $z = 0.5$ (blue lines) with $\omega = 1$, $\Delta = 0.7$ and $g = 0.8$.



Integrability of the Rabi model??

Selected for a Viewpoint in *Physics*

PRL **107,** 100401 (2011) PHYSICAL REVIEW LETTERS week ending 2 SEPTEMBER 2011

Integrability of the Rabi Model

D. Braak

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(Received 22 April 2011; published 29 August 2011)

The Rabi model is a paradigm for interacting quantum systems. It couples a bosonic mode to the smallest possible quantum model, a two-level system. I present the analytical solution which allows us to consider the question of integrability for quantum systems that do not possess a classical limit. A criterion for quantum integrability is proposed which shows that the Rabi model is integrable due to the presence of a discrete symmetry. Moreover, I introduce a generalization with no symmetries; the generalized Rabi model is the first example of a nonintegrable but exactly solvable system.

Phenomenological criterion for quantum integrability

D Braak, PRL **107**, 100401 (2011)

Inspired by the classical integrability of the hydrogen atom, QI is stated to be equivalent to the existence of f "quantum numbers" to classify eigenstates uniquely.

"If each eigenstate of a quantum system with f_1 discrete and f_2 continuous degrees of freedom can be uniquely labelled by $f_1 + f_2 = f$ quantum numbers $\{d_1, \ldots, d_{f_1}, c_1, \ldots, c_{f_2}\}$, such that the d_j can take on $\dim(\mathcal{H}_j)$ different values, where $\mathcal{H}_j$ is the state space of the jth discrete degree of freedom and the c_k range from 0 to infinity, then this system is quantum integrable."

"The Rabi model has $f_1 = f_2 = 1$ and degeneracies take place between levels of states with different parity, whereas within the parity subspaces no level crossings occur. ... The global label (valid for all values of g) is two dimensional as $f = f_1 + f_2 = 2$; the Rabi model belongs therefore to the class of integrable systems."

Centre for Modern Physics Director and Teamaster Prof. Huan-Qiang Zhou

29 April 2014

But what about Yang-Baxter integrability?

If the Rabi model is integrable, is it Yang-Baxter integrable?

The concept of Yang-Baxter integrability is ideally suited to (1+1)-dimensional quantum systems.

Variables can either be discrete or continuous.

The master key to integrability!

Mural at Simons Centre, Stony Brook

Yang-Baxter integrability in the Rabi model

$$H_R = 2\Delta\, s_z + \omega\, a^\dagger a + g(s^+ + s^-)(a + a^\dagger)$$

SUMMARY

We find the model is Yang-Baxter integrable for two cases:

1) $\Delta = 0 \Rightarrow$ known as the degenerate atomic limit.
2) $\omega = 0 \Rightarrow$ not included in Braak's solution.

In both cases, there is an extra conserved quantity C,
i.e., $[H_R, C] = 0$, where:

1) for $\Delta = 0$, $C = s^+ + s^-$.
2) for $\omega = 0$, $C = a^\dagger + a$.

The key idea is to introduce an operator-valued twist, which yields a solution to the Yang-Baxter relation.

1) $\Delta = 0$

We construct $\tau(u) = \mathrm{tr}\, T(u)$, where the monodromy matrix

$$T(u) = \begin{bmatrix} 1 & s^+ + s^- \\ s^+ + s^- & -1 \end{bmatrix} \begin{bmatrix} 1 + \eta u + \eta^2 N & \eta a \\ \eta a^+ & 1 \end{bmatrix}$$

satisfies the intertwining relation

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v)$$

with the R-matrix

$$R_{12}(u) = \begin{bmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{bmatrix}$$

satisfying the Yang-Baxter relation

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v).$$

Thus by construction $[\tau(u), \tau(v)] = 0$, with

$$\tau(u) = \eta[u + \eta N + (s^+ + s^-)(a^\dagger + a)] = \eta[u + g^{-1} H_R],$$

where we have identified $\eta = \omega/g$ and $N = a^\dagger a$.

2) $\omega = 0$

We construct $\tau(u) = \mathrm{tr}\, T(u)$, with now the monodromy matrix

$$T(u) = \begin{bmatrix} 1+\lambda & a+a^{+} \\ a+a^{+} & 1-\lambda \end{bmatrix} \begin{bmatrix} u+\eta s^{z} & \eta s^{-} \\ \eta s^{+} & u-\eta s^{z} \end{bmatrix}$$

where $\lambda = \frac{\Delta}{g}$, which satisfies the intertwining relation

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

with the same R-matrix

$$R_{12}(u) = \begin{bmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{bmatrix}$$

satisfying the Yang-Baxter relation. Here

$$\tau(u) = 2u + \eta[2\lambda s^{z} + (a^{\dagger}+a)(s^{+}+s^{-})] = 2u + \eta g^{-1} H_R\,.$$

Yang-Baxter integrability in general?

- Although integrable at the two parameter values $\Delta = 0$ and $\omega = 0$, the fully quantised Rabi model does *not* appear to be Yang-Baxter integrable in general.
- Others have tried in the past to look for integrable extensions of the Rabi model beyond the random-wave approximation, i.e., beyond the Jaynes-Cummings model.

  L Amico, H Frahm, A Osterloh & GAP Ribeiro, NPB **787**, 283 (2007)

  L Amico, H Frahm, A Osterloh & T Wirth, NPB **839**, 604 (2010)

  They could not construct the fully quantised Rabi model in this way.
- We contend that the Rabi model is not Yang-Baxter integrable in general.

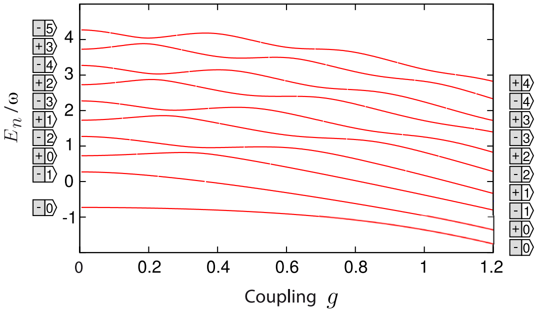
“First example of a nonintegrable but exactly solvable system”??

Braak also considers the generalised Rabi model

$$H_\epsilon = \Delta\,\sigma_z + \omega\,a^\dagger a + g\,\sigma_x(a + a^\dagger) + \epsilon\,\sigma_x$$

- the term $\epsilon\,\sigma_x$ breaks the parity symmetry.
- Braak solves this model in the same way.
- Eigenstates also obtained in terms of confluent Heun functions.

H Zhong, Q Xie, X W Guan, MTB, K Gao & C Lee, J Phys A **47**, 045301 (2014)

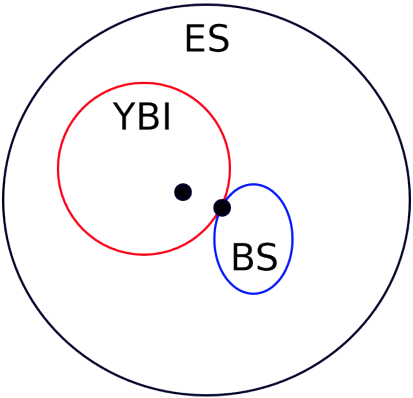


Braak figure for $\Delta = 0.7$, $\epsilon = 0.2$

Contradiction?

- The generalised Rabi model is non-integrable according to Braak's criterion.
- Can also show, by modifying the operator-valued twists, that the generalised Rabi model is YBI at each of the two points $\Delta = 0$ and $\omega = 0$.

Integrability vs exact solvability



• • Yang-Baxter integrable points of the quantum Rabi model
ES:= Exactly Solvable
BS:= Braak Solvable
YBI:= Yang-Baxter Integrable

The Dicke model

R H Dicke, Phys. Rev. **93**, 99 (1954)

Extension of the Rabi model to N two-level qubits:

$$H_D = 2\Delta\, S_z + \omega\, a^\dagger a + g(S^+ + S^-)(a + a^\dagger)$$

where now

$$S^z = \sum_{j=1}^{N} s_j^z, \quad S^x = \sum_{j=1}^{N} s_j^x, \quad S^\pm = \sum_{j=1}^{N} s_j^\pm.$$

- Of great interest for both small and large N:
  a) For $N = 2$ it constitutes the simplest model of the universal quantum gate for ion trap quantum computing.
  b) Transition to super-radiant state for large N.
- Shown to be Braak solvable for $N = 2$ and $N = 3$.

J Peng, Z-Z Ren, D Braak, G-J Guo, G-X Ju, X Zhang & X-Y Guo, J. Phys. A **47**, 265303 (2014)

H Wang, S He, L-W Duan, Y Zhao & Q-H Chen, EPL **106**, 54001 (2014)

D Braak, J. Phys. B **46**, 224007 (2013)

Yang-Baxter integrability in the Dicke model

- Applying the RWA leads to the Tavis-Cummings model:

$$H_{TC} = 2\Delta\, S^z + \omega\, a^\dagger a + g\,(S^+ a + S^- a^\dagger).$$

M Tavis & F W Cummings, Phys. Rev. **170**, 379 (1968)

- The TC model reduces to the JC model for $N = 1$.
- The TC model is Yang-Baxter integrable for general N and can be solved by the algebraic Bethe Ansatz.

see N M Bogoliubov & P P Kulish, J. Math. Sciences **192**, 14 (2013)

$\Rightarrow$ so what about the Dicke model?

Not integrable for $N > 1$ according to Braak's criterion.

1) $\Delta = 0$

We construct $\tau(u) = \mathrm{tr}\, T(u)$, where the monodromy matrix

$$T(u) = \begin{bmatrix} 1 & S^+ + S^- \\ S^+ + S^- & -1 \end{bmatrix} \begin{bmatrix} 1 + \eta u + \eta^2 N & \eta a \\ \eta a^+ & 1 \end{bmatrix}$$

satisfies the intertwining relation

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

with the R-matrix

$$R_{12}(u) = \begin{bmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{bmatrix}$$

satisfying the Yang-Baxter relation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

Thus

$$\tau(u) = \eta[u + g^{-1}H_D],$$

where $\eta = \omega/g$ and $N = a^\dagger a$.

2) $\omega = 0$

We construct $\tau(u) = \mathrm{tr}\, T(u)$, with the monodromy matrix

$$T(u) = \begin{bmatrix} 1+\lambda & a+a^{+} \\ a+a^{+} & 1-\lambda \end{bmatrix} \begin{bmatrix} u+\eta S^{z} & \eta S^{-} \\ \eta S^{+} & u-\eta S^{z} \end{bmatrix}$$

where $\lambda = \frac{\Delta}{g}$, which satisfies the same intertwining and Yang-Baxter relations.

• In this case the spin part of the monodromy matrix can be factorised into N terms.

• Now the operator $\tau(u)$ is a polynomial of degree N, with

$$\tau(u) = 2u^{N} + \eta g^{-1} u^{N-1} H_D + \dots .$$

• The parameter value $\omega = 0$ of the Dicke model has been found to be YBI using another approach. In particular, a Bethe Ansatz solution has been obtained from the elliptic Gaudin model through a limiting procedure.

A. Kundu, Phys. Lett. A **350**, 210 (2006)

Conclusion

1) The fully quantised Rabi and Dicke models do not appear to be Yang-Baxter integrable in general.
2) They are however, YBI at two special parameter values. More work can be done at these points.
3) For systems of this kind, should the terms exact solved, integrable and YBI all be synonymously interchangeable?
4) Braak's phenomenological criterion for quantum integrability is questionable.