Modified algebraic Bethe ansatz

XXX and XXZ chains on the segment

Samuel Belliard
RAQIS'14

Belliard S. and Crampé N., SIGMA 9 (2013), 072, arXiv:1309.6165
Belliard S., arXiv:1408.4840

Plan of the talk

- Motivations
- QISM for XXX chain on the segment
- Construction of a reference state
- Generalised ABA : XXX diagonal/diagonal boundaries
- MABA : XXX general/diagonal boundaries
- MABA : XXZ lower/upper boundaries
- Discussion

Motivations

State of the art ABA

- Usual ABA [Faddeev et al. 79]: Quantum groups, models on the circle
- Generalised ABA [Sklyanin 88]: Coideal sub-algebras of the quantum groups, models on the segment
- Correlation functions : Circle Sl(2) [Izergin, Korepin 80's; Slavnov 89; Lyon group 00], Segment Sl(2) [Lyon group 07] diagonal case !!

Problematic for models without U(1) symmetry:
- No referent state for the transfer matrix !!!
- No decomposition of the spectrum problem !!!

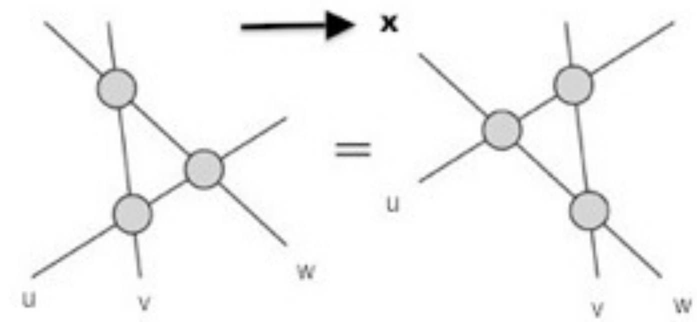
- Constraints on parameters : AnBA [Nepomechie 04], ABA [Cao et al 03], CBA [Crampé et al 10],...
- Alternatives : q-Onsager [Baseilhac, Koizumi 07], Functional approach [Galleas 08]
  SoV [Frahm et al. 08; Niccoli et al. 12],
  ODBA [Cao et al. 13] $\longrightarrow$ New term for the eigenvalues and BE!!!

New ABA presented here : the modified ABA for models without U(1) symmetry
[Belliard, Crampé 13], [Belliard 14],...

QISM for XXX chain on the segment [Sklyanin 88]

$$H_{XXX} = \frac{1}{q}\left(\sigma_1^z + \xi^+\sigma_1^+ + \xi^-\sigma_1^-\right) + \sum_{n=1}^{N-1}\left(\sigma_n^x \otimes \sigma_{n+1}^x + \sigma_n^y \otimes \sigma_{n+1}^y + \sigma_n^z \otimes \sigma_{n+1}^z\right) + \frac{1}{p}\left(\sigma_N^z + \eta^+\sigma_N^+ + \eta^-\sigma_N^-\right)$$

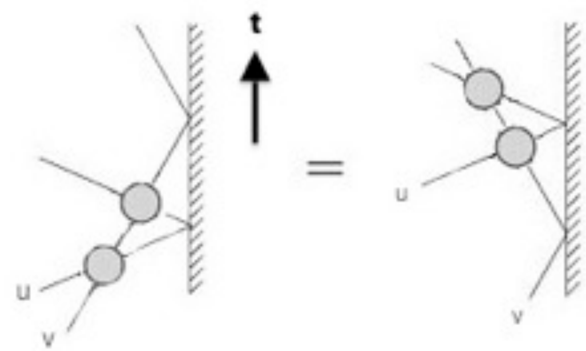
Yang-Baxter equation and rational R matrix:



$$R_{ab}(u-v)R_{ac}(u-w)R_{bc}(v-w) = R_{bc}(v-w)R_{ac}(u-w)R_{ab}(u-v)$$

$R(u) = u + P$ SU(2) invariant $[R(u), Q\otimes Q] = 0$

Reflexion equation, dual reflexion equation and general solutions [de Vega, Gonzalez-Ruiz 93]



$$R_{ab}(u-v)K_a^-(u)R_{ab}(u+v)K_b^-(v) = K_b^-(v)R_{ab}(u+v)K_a^-(u)R_{ab}(u-v)$$

$$K^-(u) = \begin{pmatrix} p+u & \eta^+ u \\ \eta^- u & p-u \end{pmatrix}$$

$$R_{ab}(-u+v)(K_a^+(u))^{t_a}R_{ab}(-u-v-2)(K_b^+(v))^{t_b} = (K_b^+(v))^{t_b}R_{ab}(-u-v-2)(K_a^+(u))^{t_a}R_{ab}(-u+v)$$

$$K^+(u) = \begin{pmatrix} q+u+1 & \xi^+(u+1) \\ \xi^-(u+1) & q-u-1 \end{pmatrix}$$

Monodromy matrices:

$$T_0(u) = R_{01}(u-\theta_1)\cdots R_{0N}(u-\theta_N)$$
$$\hat{T}_0(u) = R_{0N}(u+\theta_N)\cdots R_{01}(u+\theta_1)$$

$$K_a(u) = T_a(u)\,K_a^-(u)\,\hat{T}_a(u) = \begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) + \frac{1}{2u+1}\mathscr{A}(u) \end{pmatrix}$$

Transfer matrix: $t(u) = \mathrm{Tr}_a\left(K_a^+(u)K_a(u)\right) = \dfrac{2(u+q)(u+1)}{2u+1}\,\mathscr{A}(u) + (u+1)\big(\xi^-\mathscr{B}(u) + \xi^+\mathscr{C}(u)\big) + (q-u-1)\,\mathscr{D}(u)$

$[t(u), t(v)] = 0$ $H_{XXX} \sim \dfrac{d}{du}t(u)|_{u=0,\theta_i=0}$

Construction of a reference state

From **SU(2) invariance**, by a similarity transformation , we can put right boundary in a diagonal form
[Galleas, Martins 04] [Crampé et al. 04]

$$(Q^-)^{-1}K^-(u)Q^- = D^-(u) = \begin{pmatrix} p+u(1-\tau) & 0 \\ 0 & p-u(1-\tau) \end{pmatrix}, \quad Q^- = \begin{pmatrix} \eta^+ & \tau \\ -\tau & \eta^- \end{pmatrix}$$

with $\tau = 1 \pm \sqrt{1+\eta^+\eta^-}$

Highest weight vector (reference state) : $\mathscr{A}(u)|\Omega\rangle = \Lambda_1(u)|\Omega\rangle, \quad \mathscr{D}(u)|\Omega\rangle = \Lambda_2(u)|\Omega\rangle, \quad \mathscr{C}(u)|\Omega\rangle = 0$

where $\Lambda_1(u) = (p+u(1-\tau))\Lambda(u)$ and $\Lambda_2(u) = \frac{2u}{2u+1}(p-(u+1)(1-\tau))\Lambda(-u-1)$,

with $\Lambda(u) = \prod_{j=1}^{N}(u-\theta_j+1)(u+\theta_j+1)$

Remarks:
- The action of the creation operator is nilpotent $\mathscr{B}(u)\mathscr{B}(\lambda_1)\dots\mathscr{B}(\lambda_N)|\Omega\rangle = 0$
- We can always put the right K matrix into a triangular form.
  In this case the creation operator has an **off-shell action**

$$2\xi^-(u+1)\mathscr{B}(u)\mathscr{B}(\bar{\lambda})|\Omega\rangle = \Lambda^N_{gt}(u,\bar{\lambda})\mathscr{B}(\bar{\lambda})|\Omega\rangle + \sum_{i=1}^{N} F(u,\lambda_i)\mathrm{BE}^N_{gt}(\lambda_i,\bar{\lambda}_i)\mathscr{B}(u)\mathscr{B}(\bar{\lambda}_i)|\Omega\rangle$$

$$\Lambda^N_{gt}(u,\bar{\lambda}) = 4\xi^-\eta^+u\,(u+1)\;\Lambda(u)\Lambda(-u-1)m(u,\bar{\lambda}), \quad \mathrm{BE}^N_{gt}(\lambda_i,\bar{\lambda}_i) = \lim_{u\to\lambda_i}\left((u-\lambda_i)\Lambda^N_{gt}(u,\bar{\lambda})\right)$$

- For the XXZ chain, no SU(2) invariance, one use a **Face/Vertex transformation** to find an highest weight vector [Cao et al. 2003], [Filali, Kitanine 2011].

Generalised ABA : XXX diagonal/diagonal boundaries [Sklyanin 88]

$$t_d(u) = \alpha(u)\mathscr{A}(u) + \delta(u)\mathscr{D}(u) \qquad [t_d(u), S_z] = 0 \qquad \longrightarrow \qquad \mathcal{H} = \oplus_{M=0}^{N} \mathcal{W}_M$$

Highest weight vector (reference state) :

$$\mathscr{A}(u)|\Omega\rangle = \Lambda_1(u)|\Omega\rangle, \quad \mathscr{D}(u)|\Omega\rangle = \Lambda_2(u)|\Omega\rangle, \quad \mathscr{C}(u)|\Omega\rangle = 0$$

Commutation relations :

$$\mathscr{A}(u)\mathscr{B}(v) = f(u,v)\mathscr{B}(v)\mathscr{A}(u) + g(u,v)\mathscr{B}(u)\mathscr{A}(v) + w(u,v)\mathscr{B}(u)\mathscr{D}(v)$$

$$\mathscr{D}(u)\mathscr{B}(v) = h(u,v)\mathscr{B}(v)\mathscr{D}(u) + k(u,v)\mathscr{B}(u)\mathscr{D}(v) + n(u,v)\mathscr{B}(u)\mathscr{A}(v)$$

Bethe Vector of subspace M : $|\Phi^M_{diag}(\bar\lambda)\rangle = \mathscr{B}(\lambda_1)\dots\mathscr{B}(\lambda_M)|\Omega\rangle = \mathscr{B}(\bar\lambda)|\Omega\rangle \quad \in \mathcal{W}_M$

Off-shell action of the transfer matrix for subspace M :

$$t_d(u)|\Phi^M_{diag}(\bar\lambda)\rangle = \Lambda^M_{diag}(u)|\Phi^M_{diag}(\bar\lambda)\rangle + \sum_{k=1}^{M} F(u,\lambda_k)\mathrm{BE}^M_{diag}(\lambda_k,\bar\lambda_k)|\Phi^M_{diag}(u,\bar\lambda_k)\rangle$$

$$\Lambda^M_{diag}(u) = \alpha(u)\Lambda_2(u)f(u,\bar\lambda) + \delta(u)\Lambda_2(u)h(u,\bar\lambda) \qquad \mathrm{BE}^M_{diag}(\lambda_k,\bar\lambda_k) = Res_{u=\lambda_k}\left(\Lambda^M_{diag}(u)\right)$$

Remarks:

- For generic boundaries and imposing constraints on the parameters one can reduce to this case [Galleas, Martins 04] [Crampé 04]
- The generalised ABA can also be apply to diagonal/triangular case.

MABA : XXX general/diagonal boundaries I

From SU(2) invariance we can put right boundary in a diagonal form and have an **highest weight vector !**

$$\mathscr{A}(u)|\Omega\rangle = \Lambda_1(u)|\Omega\rangle, \quad \mathscr{D}(u)|\Omega\rangle = \Lambda_2(u)|\Omega\rangle, \quad \mathscr{C}(u)|\Omega\rangle = 0$$

We can make a rotation for the left boundary to have a **diagonal form** :

$$\left(Q^+\right)^{-1} K^+(u) Q^+ = D^+(u) = \begin{pmatrix} q+(u+1)(1-\rho) & 0 \\ 0 & q-(u+1)(1-\rho) \end{pmatrix}, \quad Q^+ = \begin{pmatrix} \xi^+ & \rho \\ -\rho & \xi^- \end{pmatrix}$$

with $\rho^2 - 2\rho + \xi^+\xi^- = 0$

It allows to construct **new double rows monodromy matrix** and **new operators** :

$$\overline{K}_a(u) = (Q_a^+)^{-1} K_a(u) Q_a^+ = \begin{pmatrix} \overline{\mathscr{A}}(u) & \overline{\mathscr{B}}(u) \\ \overline{\mathscr{C}}(u) & \overline{\mathscr{D}}(u) + \frac{1}{2u+1}\overline{\mathscr{A}}(u) \end{pmatrix}$$

$$\overline{\mathscr{A}}(u) = \left(\frac{\xi^+}{\xi^-} + \frac{1}{2u+1}\left(\frac{\rho}{\xi^-}\right)^2\right)\mathscr{A}(u) - \frac{\rho}{\xi^-}\left(\mathscr{B}(u) + \frac{\xi^+}{\xi^-}\mathscr{C}(u)\right) + \left(\frac{\rho}{\xi^-}\right)^2 \mathscr{D}(u)$$

$$\overline{\mathscr{D}}(u) = \left(\frac{\xi^+}{\xi^-} - \frac{1}{2u+1}\left(\frac{\rho}{\xi^-}\right)^2\right)\mathscr{D}(u) + \frac{2(u+1)}{2u+1}\frac{\rho}{\xi^-}\left(\mathscr{B}(u) + \frac{\xi^+}{\xi^-}\mathscr{C}(u)\right) + \frac{4u(u+1)}{(2u+1)^2}\left(\frac{\rho}{\xi^-}\right)^2 \mathscr{A}(u)$$

$$\overline{\mathscr{B}}(u) = \mathscr{B}(u) + \frac{\rho}{\xi^-}\left(\frac{2u}{2u+1}\mathscr{A}(u) - \mathscr{D}(u)\right) - \left(\frac{\rho}{\xi^-}\right)^2 \mathscr{C}(u)$$

MABA : XXX general/diagonal boundaries II

Off diagonal action on the highest weight vector:

$$\overline{\mathscr{A}}(u)|\Omega\rangle = \Lambda_1(u)|\Omega\rangle - \frac{\rho}{\xi^-}\overline{\mathscr{B}}(u)|\Omega\rangle \qquad \overline{\mathscr{D}}(u)|\Omega\rangle = \Lambda_2(u)|\Omega\rangle + \frac{2(u+1)}{2u+1}\frac{\rho}{\xi^-}\overline{\mathscr{B}}(u)|\Omega\rangle$$

Commutation relations:

$$\overline{\mathscr{A}}(u)\overline{\mathscr{B}}(v) = f(u,v)\overline{\mathscr{B}}(v)\overline{\mathscr{A}}(u) + g(u,v)\overline{\mathscr{B}}(u)\overline{\mathscr{A}}(v) + w(u,v)\overline{\mathscr{B}}(u)\overline{\mathscr{D}}(v)$$

$$\overline{\mathscr{D}}(u)\overline{\mathscr{B}}(v) = h(u,v)\overline{\mathscr{B}}(v)\overline{\mathscr{D}}(u) + k(u,v)\overline{\mathscr{B}}(u)\overline{\mathscr{D}}(v) + n(u,v)\overline{\mathscr{B}}(u)\overline{\mathscr{A}}(v)$$

Modified diagonal transfer matrix and Bethe vectors:

$$t(u) = \mathrm{Tr}_a\left(D_a^+(u)\,\overline{K}_a(u)\right) = \alpha(u)\overline{\mathscr{A}}(u) + \delta(u)\overline{\mathscr{D}}(u) \qquad |\Phi^N(\bar{\lambda})\rangle = \overline{\mathscr{B}}(\bar{\lambda})|\Omega\rangle$$

Modified off-shell action:

$$t(u)|\Phi^N(\bar{\lambda})\rangle = \frac{\rho(1-\rho)}{\xi^-}2(u+1)\overline{\mathscr{B}}(u)|\Phi^N(\bar{\lambda})\rangle + \Lambda_d^N(u,\bar{\lambda})|\Phi^N(\bar{\lambda})\rangle + \sum_{k=1}^{M} F(u,\lambda_k)\mathrm{BE}_d^N(\lambda_k,\bar{\lambda}_k)|\Phi^N(\lambda_k,\bar{\lambda}_k)\rangle$$

MABA : XXX general/diagonal boundaries III

Conjecture of the action of the new creation operator on the BV !!

$$\frac{\rho(1-\rho)}{\xi^-}2(u+1)\overline{\mathscr{B}}(u)|\Phi^N(\bar\lambda)\rangle = \Lambda_g^N(u,\bar\lambda)|\Phi^N(\bar\lambda)\rangle + \sum_{k=1}^{M} F(u,\lambda_k)\mathrm{BE}_g^N(\lambda_k,\bar\lambda_k)|\Phi^N(u,\bar\lambda_k)\rangle$$

$$\Lambda_g^N(u,\bar\lambda) = -\rho\, 2u(u+1)\ \Lambda(u)\Lambda(-u-1)m(u,\bar\lambda)\ , \quad \mathrm{BE}_g^N(\lambda_i,\bar\lambda_i) = \lim_{u\to\lambda_i}\Big((u-\lambda_i)\Lambda_g^N(u,\bar\lambda)\Big)$$

$\longrightarrow$ Done from explicit calculation for small N =1,2,3

Off-shell action of the transfer matrix on the BV

$$t(u)|\Phi^N(\bar\lambda)\rangle = \Lambda^N(u,\bar\lambda)|\Phi^N(\bar\lambda)\rangle + \sum_{k=1}^{M} F(u,\lambda_k)\mathrm{BE}^N(\lambda_k,\bar\lambda_k)|\Phi^N(u,\bar\lambda_k)\rangle$$

$$\Lambda^N(u,\bar\lambda) = \Lambda_d^N(u,\bar\lambda) + \Lambda_g^N(u,\bar\lambda), \quad \mathrm{BE}^N(\lambda_i,\bar\lambda_i) = \mathrm{BE}_d^N(\lambda_i,\bar\lambda_i) + \mathrm{BE}_g^N(\lambda_i,\bar\lambda_i)$$

MABA : XXZ lower/upper boundaries

$$H_{XXZ}=\epsilon\,\sigma_1^z+\kappa^-\,\sigma_1^-+\kappa^+\,\sigma_1^++\sum_{k=1}^{N-1}\left(\sigma_k^x\otimes\sigma_{k+1}^x+\sigma_k^y\otimes\sigma_{k+1}^y+\Delta\sigma_k^z\otimes\sigma_{k+1}^z\right)+\nu\,\sigma_N^z+\tau^-\,\sigma_N^-+\tau^+\,\sigma_N^+,$$

R matrix and K matrices for XXZ chain

$$R_{ab}(u)=\begin{pmatrix}\frac{qu-q^{-1}u^{-1}}{q-q^{-1}} & 0 & 0 & 0\\ 0 & \frac{u-u^{-1}}{q-q^{-1}} & 1 & 0\\ 0 & 1 & \frac{u-u^{-1}}{q-q^{-1}} & 0\\ 0 & 0 & 0 & \frac{qu-q^{-1}u^{-1}}{q-q^{-1}}\end{pmatrix}$$

$$K^-(u)=\begin{pmatrix}\nu_-u+\nu_+u^{-1} & \tau\,(u^2-u^{-2})\\ \tilde{\tau}\,(u^2-u^{-2}) & \nu_-u^{-1}+\nu_+u\end{pmatrix}$$

$$K^+(u)=\begin{pmatrix}\epsilon_+qu+\epsilon_-q^{-1}u^{-1} & \tilde{\kappa}(q^2u^2-q^{-2}u^2)\\ \kappa\,(q^2u^2-q^{-2}u^2) & \epsilon_+q^{-1}u^{-1}+\epsilon_-qu\end{pmatrix}$$

Off shell action of the transfer matrix for left lower and right upper triangular boundaries

$$t_{lo/up}(u)\Phi^N_{lo/up}(\bar{u})=\kappa\,c(qu)\mathscr{B}(u)\Phi^N_{lo/up}(\bar{u})+\Lambda^N_d(u,\bar{u})\Phi^N_{lo/up}(\bar{u})+\sum_{i=1}^{N}F(u,u_i)E^N_d(u_i,\bar{u}_i)\Phi^N_{lo/up}(\{u,\bar{u}_i\})$$

Off shell action of the creation operator for right upper triangular boundary

$$\kappa\,c(qu)\mathscr{B}(u)\Phi^N_{lo/up}(\bar{u})=\Lambda^N_g(u,\bar{u})\Phi^N_{lo/up}(\bar{u})+\sum_{i=1}^{N}F(u,u_i)E^N_g(u_i,\bar{u}_i)\Phi^N_{lo/up}(\{u,\bar{u}_i\})$$

$$\Lambda^N_g(u,\bar{u})=-\tau\,\kappa\,c(u)c(q^{-1}u^{-1})\Lambda(u)\Lambda(q^{-1}u^{-1})m(u,\bar{u})\qquad E^N_g(u_i,\bar{u}_i)=\tau\,\kappa\,\frac{c(u_i)c(q^{-1}u_i^{-1})}{b(qu_i^2)}\Lambda(u_i)\Lambda(q^{-1}u_i^{-1})m(u_i,\bar{u}_i)$$

Discussion

Results:
- XXX : MABA for general/general boundaries [Belliard Crampé 2013]
- XXZ : MABA for lower/upper triangular boundaries [Belliard 2014]

Projects:
- XXX : MABA Dual Bethe vector and scalar product [Belliard Pimenta],
  MABA XXX on the circle [Belliard Nepomechie].
- XXZ : MABA for general/general and general/diagonal boundaries [Avan Belliard Pimenta]
  MABA XXZ on the circle [Avan Belliard].

Some open problems:
- Proof of the conjecture: SoV, ODBA
- Slavnov formula, norm $\longrightarrow$ correlation functions, form factor,...
- Thermodynamic limit and applications to physics, mathematics... link with Onsager approach