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Asymptotic expansion in 1d log-gas with many-body interactions

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1. Introduction : model and motivations

2. Main results and applications

3. Some elements of the proof

4. Conclusion

1d log-gas with mean-field interaction

- To a particle configuration $\lambda_1, \ldots, \lambda_N \in A \subseteq \mathbb{R}$

  we assign a Boltzmann weight $Z_N^{-1}\, e^{-\beta[E_C(\lambda)+E_I(\lambda)]}$ $\beta > 0$

Coulomb energy in 2d $E_C(\lambda) = -\sum_{i<j} \ln|\lambda_i - \lambda_j|$

Interaction potential $E_I(\lambda)$

★ regular

★ of same order than $E_C(\lambda)$ for typical finite λ's

1-body $E_I(\lambda) = N \sum_{i=1}^{N} T_1(\lambda_i)$

2-body $E_I(\lambda) = \frac{1}{2} \sum_{i,j} T_2(\lambda_i, \lambda_j)$

r-body $E_I(\lambda) = (N^{2-r}/r!) \sum_{\mathbf{i}} T_r(\lambda_{i_1}, \ldots, \lambda_{i_r})$

- Partition function

$$Z_N = \int_{A^N} \Big[\prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \Big]\, e^{-\beta E_I(\lambda)}$$

Sources of examples of $\left[\prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta\right] e^{-\beta E_I(\lambda)}$

- Invariant random matrix theory

$$\mathrm{d}M\, e^{N\,\mathrm{Tr}\,T(M)}$$

Wigner, Dyson, Mehta (50s-60s)

M = matrix of size N

$$\begin{cases} \beta = 1 & \text{real symmetric} \\ \beta = 2 & \text{hermitian} \\ \beta = 4 & \text{quaternionic self-dual} \end{cases}$$

- Statistical physics on 2d random lattices (orientable $\Leftrightarrow$ $\beta = 2$)
  - O(n) loop model $E_I(\lambda) = (n/4) \sum_{i,j} \ln(\lambda_i + \lambda_j)$ **Gaudin, Kostov '89**

    (realizes Ising for $n = 1$, Potts for $n = \sqrt{q}$, ...)

  - 6v-model $E_I(\lambda) = \frac{1}{4} \sum_{i,j} \left\{ \ln\left[\cosh(\lambda_i - \lambda_j) - \cos c\right] + \ln\left(\frac{\sinh(\lambda_i - \lambda_j)}{\lambda_i - \lambda_j}\right)\right\}$

    Ginsparg '91, Kostov '99

Sources of examples of $\left[\prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta\right] e^{-\beta E_I(\lambda)}$

- Coulomb energy in other 2d domains/deformations

$$\widetilde{E}_C(\lambda) = \sum_{i<j} G(\lambda_i, \lambda_j) = E_C(\lambda) + \sum_{i<j} \left[G(\lambda_i, \lambda_j) + \ln|\lambda_i - \lambda_j|\right]$$

  - Cylinder $\quad G_1(x,y) = -\ln\left|\sinh(x-y)\right|$
  - Torus $\quad G_2(x,y) = -\ln\left|\vartheta_1(x-y|\tau)\right|$
  - (q,t) deformation $\quad G_{q,t}(x,y) = -\frac{1}{2}\ln\left|\frac{(e^{x-y};q)_\infty (e^{y-x};q)_\infty}{(te^{x-y},q)_\infty (te^{y-x},q)_\infty}\right|$

- Chern-Simons theory on $\mathbb{S}_3/\Gamma$ $\qquad (a_1, \ldots, a_R)$ = Schäfli symbol of Γ

$$\widetilde{E}_C(\lambda) = \sum_{i<j} \left[(2-R)G_1(\lambda_i, \lambda_j) + \sum_{m=1}^{R} G_1(\lambda_i/a_m, \lambda_j/a_m)\right] \quad \text{for SU(N + 1)}$$

$$+\left[(2-r)G_1(\lambda_i, -\lambda_j) + \sum_{m=1}^{r} G_1(\lambda_i/a_m, -\lambda_j/a_m)\right] \quad \text{for SO(N)/Sp(2N)}$$

Sources of examples of $\left[\prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta\right] e^{-\beta E_I(\lambda)}$

- Building block for correlation functions of XXZ spin chain, N sites

$$\widetilde{E}_C(\lambda) = \tfrac{1}{2} \sum_{i<j} \ln \left|\sinh[N^\alpha \varpi_1(\lambda_i - \lambda_j)] \sinh[N^\alpha \varpi_2(\lambda_i - \lambda_j)]\right|$$

★ nature of singularity of the interaction changes at $N \to \infty$

The questions are ...

Find all-order asymptotic expansion when $N \to \infty$ up to $o(e^{-cN})$ of

- partition function $Z_N = \int_{A^N} \left[\prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta\right] e^{-\beta E_I(\lambda)}$

- k-point correlations $\mathbb{E}\left[\sum_{\mathbf{i}} f(\lambda_{i_1}, \ldots, \lambda_{i_k})\right]$

How does the (random) empirical measure $L_N^{(\lambda)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ behaves ?

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The leading order ... is given by a continuous approximation

- Define the energy functional on a proba. measure μ

$$\mathcal{T}(\mu) = \iint -\ln|x_1 - x_2| \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2) + \tfrac{1}{r!}\int T(x_1,\ldots,x_r)\prod_{i=1}^{r}\mathrm{d}\mu(x_i)$$

- Assumption 1 : uniqueness of minimizer := μ_{eq}
- Assumption 2 : positive Hessian at μ_{eq} (strict minimum)

for any ν = finite signed measure of mass 0

$$\mathcal{T}''(\mu_{\mathrm{eq}})[\nu,\nu] = \mathfrak{D}^2[\nu] \in [0,+\infty]$$

and = 0 iff $\nu = 0$

Lemma

$L_N^{(\lambda)} \longrightarrow \mu_{\mathrm{eq}}$ almost surely and in expectation

$$Z_N = \exp\big\{ -N^2\beta(\mathcal{T}(\mu_{\mathrm{eq}}) + o(1))\big\}$$

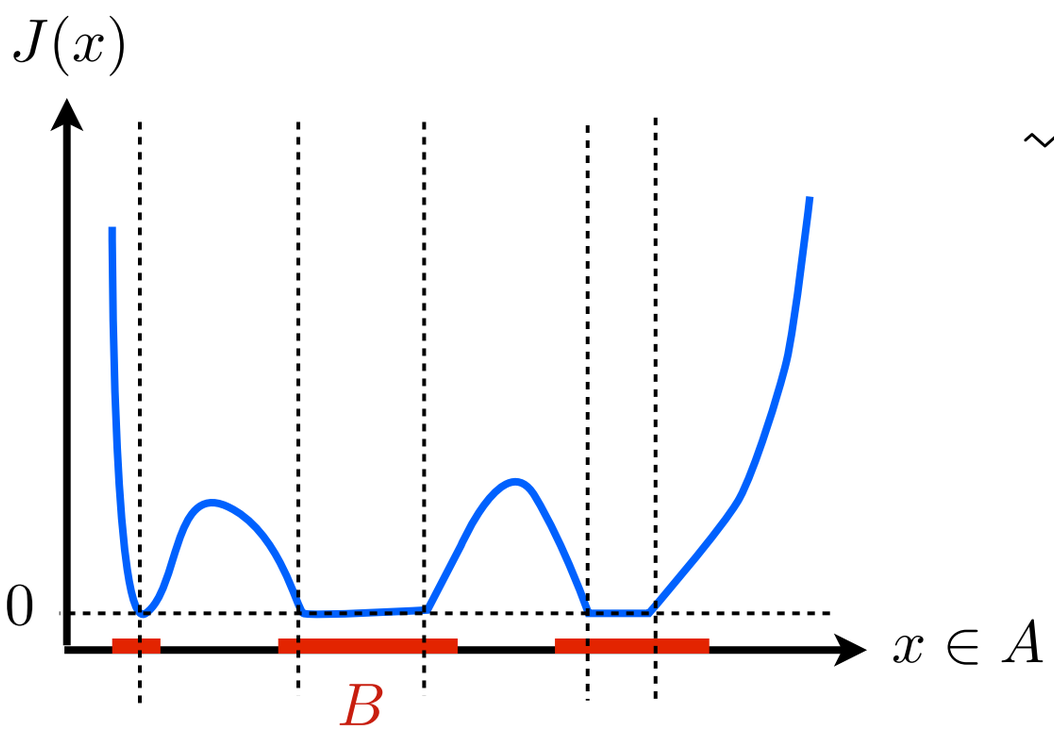
Large deviations for a single particle

- A particle at position x feels the effective potential

$$J(x) = \mathcal{T}'(\mu_{\mathrm{eq}})[\delta_x] - \inf_{x \in A} \mathcal{T}'(\mu_{\mathrm{eq}})[\delta_x]$$

Lemma

For any closed $F \subseteq A$ $\qquad \mathbb{P}[\exists i,\ \lambda_i \in F] \leq \exp\Big\{ -\beta N(\inf_{x \in F} J(x) + o(1))\Big\}$



$\rightsquigarrow$ One can restrict to a compact $B \subseteq A$ neighborhood of $\{J(x) = 0\}$

$$Z_N^B = Z_N^A(1 + o(e^{-cN}))$$

Large deviations of empirical measure

- Natural "distance" $\mathcal{T}''(\mu_{\rm eq})[\nu,\nu] = \mathfrak{D}^2[\nu] \in [0,+\infty]$

  but $\mathfrak{D}[L_N^{(\lambda)} - \mu_{\rm eq}] = +\infty$ because of atoms and log singularity

- Let us pick a nice regularization **idea from Maïda, Maurel-Segala**

$$L_N^{(\lambda)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} \rightsquigarrow \widetilde{L}_N^{(\lambda)}$$

Lemma

If $T(x_1,\ldots,x_r)$ is smooth, we have for N large enough

$$\mathbb{P}_N\left[\mathfrak{D}[\widetilde{L}_N^{(\lambda)} - \mu_{\rm eq}] > t\right] \leq \exp\left(N\ln N - N^2 t^2/2\right)$$

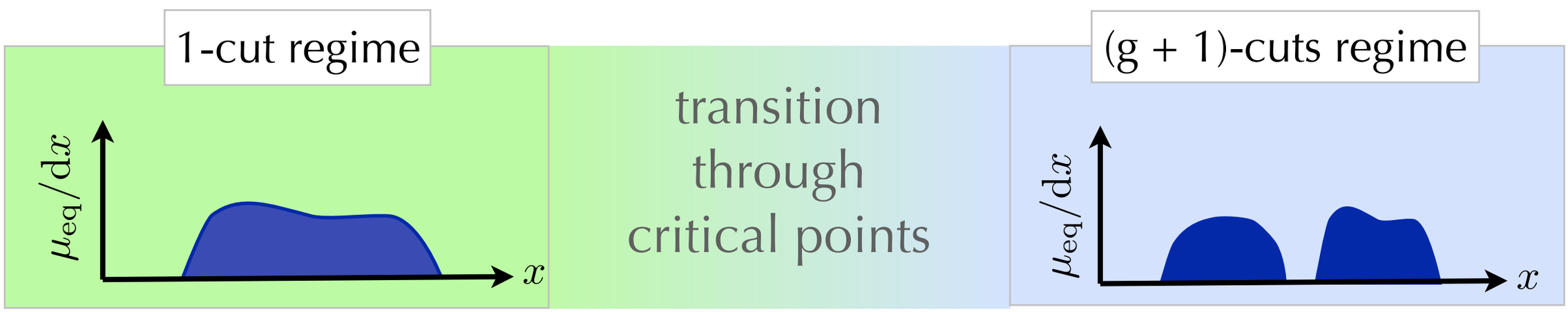
$\longrightarrow$ Concentration of measure at rate N^2

The equilibrium measure

- T real-analytic $\Longrightarrow \begin{cases} \mu_{\mathrm{eq}} \text{ is supported on a finite number of segments} \\ S = \bigcup_{h=0}^{g} [a_h, b_h] \end{cases}$

- $\alpha \in \partial S$ is a hard edge if $\alpha \in \partial A$, is a soft edge otherwise

$$\mathrm{d}\mu_{\mathrm{eq}}(x) = \frac{\mathbf{1}_S(x)\mathrm{d}x}{2\pi} M(x) \prod_{\alpha \text{ soft}} |x-\alpha|^{1/2} \prod_{\alpha \text{ hard}} |x-\alpha|^{-1/2}$$



- We say that μ_{eq} is off-critical when $M(x) > 0$ on A

Finite size corrections : we assume ...

- Uniqueness of minimizer μ_{eq}
- Energy functional has positive hessian at μ_{eq}
- $T = T^{(0)} + N^{-1}T^{(1)} + \cdots$ $\begin{cases} T^{(0)} & \text{real analytic on } A \\ T^{(1)} & \text{complex analytic on } A \end{cases}$
- Control of large deviations $J(x) > 0$ for $x \in A \setminus S$
- μ_{eq} is off-critical
- f = test function, analytic on A

Some examples in which assumptions hold ...

Uniqueness of minimizer $\mu_{\rm eq}$

Energy functional has positive hessian at $\mu_{\rm eq}$

- 1-matrix model

$$\mathrm{d}M\, e^{N\,\mathrm{Tr}\,T(M)}$$

OK

- O(n) loop model

$$\int \prod_{i<j} \frac{\lambda_i-\lambda_j}{(\lambda_i+\lambda_j)^{n/2}} \prod_i e^{NT_1(\lambda_i)}\mathrm{d}\lambda_i$$

OK for $|n| \leq 2$

- Chern-Simons theory on $\mathbb{S}_3/\Gamma$

$$\int \left\{ \prod_{i<j} \left[\sinh\!\left(\lambda_i-\lambda_j\right)\right]^{2-R} \prod_{m=1}^{R} \sinh\!\left(\frac{\lambda_i-\lambda_j}{a_m}\right)\right\} \prod_i e^{NT_1(\lambda_i)}\mathrm{d}\lambda_i$$

OK for

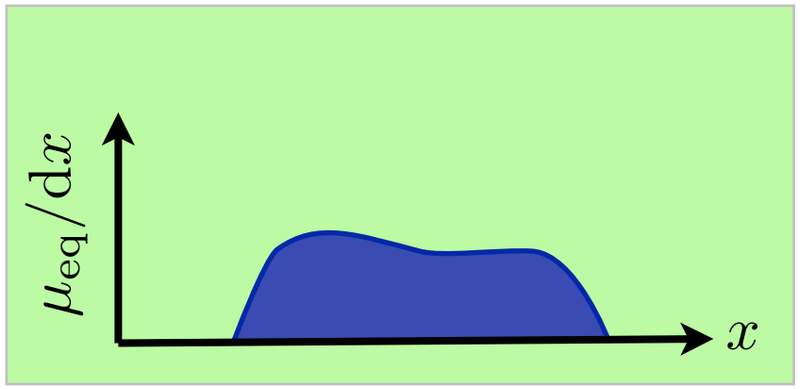
$$2-R+\sum_{m=1}^{R}\frac{1}{a_m} \geq 0$$

- XXZ building block

$$\int \left\{ \prod_{i<j} \sinh[N^\alpha \varpi_1(\lambda_i-\lambda_j)]\sinh[N^\alpha \varpi_2(\lambda_i-\lambda_j)]\right\} \prod_i e^{T_1(\lambda;N)}\mathrm{d}\lambda_i$$

no because of N dependence

Result in the 1-cut regime



- 1/N asymptotic expansion

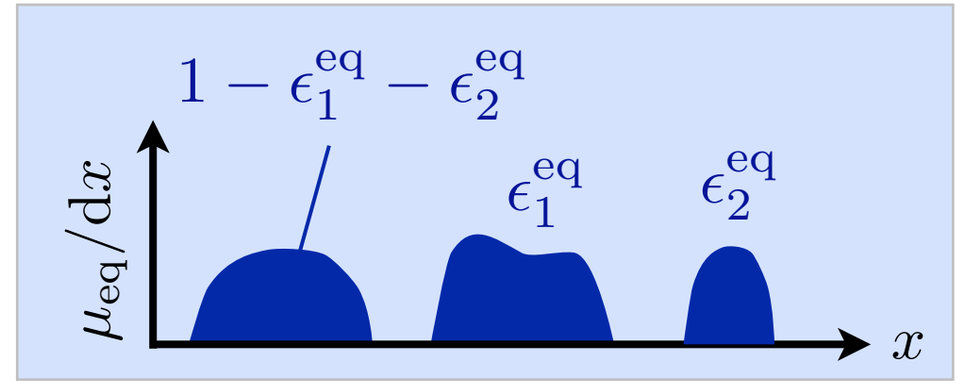
$$Z_N = N^{(\beta/2)N+\gamma'} \exp\Big[\sum_{m\geq -2} N^{-m}\, F^{[m]} + O(N^{-\infty})\Big]$$

and for soft edges $\gamma' = \frac{3+\beta/2+2/\beta}{12}$
etc.

- Central limit theorem

$$\Big(\sum_{i=1}^{N} f(\lambda_i) - N\int_A f(\xi)\mathrm{d}\mu_{\rm eq}(\xi)\Big) \longrightarrow \text{ (non-centered) gaussian}$$

Result in the (g + 1)-cuts regime



- Oscillatory asymptotic expansion

$$Z_N = N^{(\beta/2)N+\gamma'}(\mathcal{D}_N\Theta_{-N\epsilon^{\rm eq}})(F^{[-1]'}|F^{[-2]''})\,\exp\Big[\sum_{m\geq -2} N^{-m}\,F^{[m]} + O(N^{-\infty})\Big]$$

where $\mathcal{D}_N = \sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\ell_1,\dots,\ell_p\geq 1\\ m_1,\dots,m_p\geq -2\\ \sum_i(m_i+\ell_i)>0}} N^{-\sum_i(m_i+\ell_i)}\prod_{i=1}^{p}\frac{F_{\rm eq}^{[m_i],(\ell_i)}\cdot\nabla_{\mathbf{w}}^{\otimes\ell_i}}{\ell_i!}$

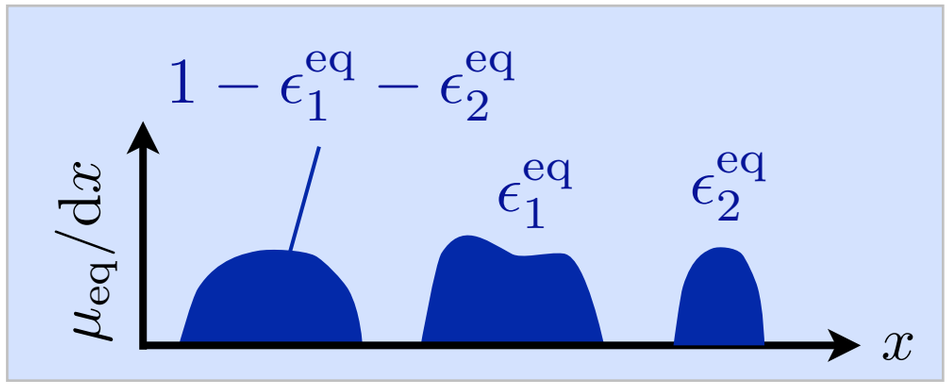
acts as a differential operator on the Siegel theta function

$$\Theta_\mu(\mathbf{w}|\mathbf{Q}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} e^{\mathbf{w}\cdot(\mathbf{m}+\mu)+\frac{1}{2}(\mathbf{m}+\mu)\cdot\mathbf{Q}\cdot(\mathbf{m}+\mu)}$$

→ (pseudo)-periodicity due to $\mu = -N\epsilon^{\rm eq} \bmod \mathbb{Z}^g$

- E.g. for soft edges $\gamma' = (g+1)\frac{\beta/2+2/\beta+3}{12}$

Result in the (g + 1)-cuts regime



- No central limit theorem in general ...

$$\mathbb{E}\left[e^{\mathrm{i}s\left(\sum_{i=1}^{N} f(\lambda_i)-N\int f(x)\mathrm{d}\mu_{\mathrm{eq}}(x)\right)}\right] \underset{N\infty}{\sim} e^{\mathrm{i}s\, m_1[f]-m_2[f]s^2/2} \; \frac{\Theta_{-N\epsilon_{\mathrm{eq}}}\left(F^{[-1]'}+\mathrm{i}sv[f]\,\middle|\,F^{[-2]''}\right)}{\Theta_{-N\epsilon_{\mathrm{eq}}}\left(F^{[-1]'}\,\middle|\,F^{[-2]''}\right)}$$

(non-centered) gaussian

\+ discrete Gaussian, centered at $\mu = -N\epsilon_{\mathrm{eq}} \bmod \mathbb{Z}^g$

step $v[f] \propto \left(\int_S \frac{f(x)\,x^i\,\mathrm{d}x}{\prod_\alpha |x-\alpha|^{1/2}}\right)_{0\le i\le g-1}$

Topological recursion (β = 2)

x

- The coefficients of expansion can be computed recursively by a universal recursion

$$\text{initial data} = \begin{cases} \mu_{\text{eq}} \text{ :=large N limit of } L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} \\ \text{2-pt function = large N covariance of } N L_N \end{cases}$$

- Diagrammatic techniques to describe the expansion

History of 1-body interactions : 1-cut regime

$\beta = 2$

- If 1/N expansion exists, then $Z_N = N^{\gamma N + \gamma'} \exp\Big[\sum_{m \geq -1} N^{-2m} F^{\{m\}} \Big]$

  and $F^{\{m\}}$ can be computed by the moment method

  Ambjørn, Chekhov, Kristjansen, Makeenko 90s

- Rewriting of $F^{\{m\}}$ in terms of a universal topological recursion

  Eynard '04

- Diagrammatics of topo. rec. related to CFT of a free boson

  Kostov 90's, Kostov, Orantin, '09

- Existence of 1/N expansion by

  - analysis of SD equations **Albeverio, Pastur, Shcherbina '01**
  - RH techniques **Ercolani, McLaughlin '02**
  - analysis of int. system **Bleher, Its, '05**

History of 1-body interactions : 1-cut regime

$\beta > 0$

- if 1/N expansion exists, then $Z_N = N^{\gamma N + \gamma'} \exp\Big[\sum_{m \geq -2} N^{-m} F^{[m]} \Big]$

  and $F^{[m]}$ computed by a β-topological recursion

  Chekhov, Eynard '06

- Central limit theorem

  Johansson '98

- Existence of 1/N expansion (analysis of SD eqn)

  GB, Guionnet '11

History of 1-body interactions : multi-cut regime

$\beta = 2$

- numerous observations of oscillatory behavior
  physicists, '90s
- Riemann-Hilbert techniques up to o(1)
  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou, ...
- heuristic derivation up to o(1)
  Bonnet, David, Eynard '00
- generalization to all orders
  Eynard '07
- observation of "no CLT"
  Pastur '06

$\beta > 0$

- Proof of "no CLT" and asymptotics of Z_N^A up to o(1)
  Shcherbina '12
- General proof
  GB, Guionnet '13

History of r-body interactions

- all results extend to r-body models

  GB, Guionnet, Kozlowski '13

- computation of expansion by topological recursion

  2-body **GB, Eynard, Orantin '13**

  r-body **GB '13**

- application to Chern-Simons theory // knot theory

  lens spaces // torus knots

  Halmagyi, Yasnov '03

  Tierz '06 - ....

  Brini, Eynard, Mariño '11

  $\mathbb{S}_3/\Gamma$

  GB, Eynard '14

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① A digression : Chern-Simons/int. system ?

- Application to Chern-Simons theory/knot theory on $\mathbb{S}_3/\Gamma$

$$\int \left\{ \prod_{i<j} \left[\sinh(\lambda_i - \lambda_j)\right]^{2-R} \prod_{m=1}^{R} \sinh[(\lambda_i - \lambda_j)/a_m] \right\} \prod_i e^{-N\lambda_i^2/2u}$$

$\rightsquigarrow$ conjectural relation between $\mu_{\rm eq}$ and
char. poly. of Lax matrix of an ADE relativistic Toda chain
Brini (+ GB, Klemm, in progress)

Γ	$(a_1, \ldots, a_R)$	Weyl group	after symm. breaking
cyclic	$(1, p)$	A_{p-1}	A_{p-1}
dihedral	$(2, 2, p)$	D_{p+2}	A_p (even) D_{p+1} (odd)
tetrahedral	$(2, 3, 3)$	E_6	D_4
octahedral	$(2, 3, 4)$	E_7	E_6
icosahedral	$(2, 3, 5)$	E_8	E_8

② Orthogonal polynomials and random matrices

For $\beta = 2$

$$V(x) = V_0(x) + \sum_{j\geq 1} \frac{t_j}{j}\, x^j$$

measure over the space of $N \times N$ hermitian matrices

$$\frac{1}{Z_{N,n}}\, \mathrm{d}M\, e^{-n\,\mathrm{Tr}\,V(M)}$$

- $P_{N,n}(x) = \mathbb{E}_{N\times N}\big[\det(x - M)\big]$
  is the Nth orthogonal polynomial for the weight $\mathrm{d}x\, e^{-nV(x)}$ on $\mathbb{R}$

- Let $h_{N,n}$ = norm of $P_{N,n}$
  $\widehat{P}_{N,n} = P_{N,n}/\sqrt{h_{N,n}}$ satisfies a 3-term recurrence relation

$$(x - \beta_{N,n})\widehat{P}_{N,n}(x) = \sqrt{h_{N,n}}\,\widehat{P}_{N+1,n}(x) + \sqrt{h_{N-1,n}}\,\widehat{P}_{N-1,n}(x)$$

Orthogonal polynomials and random matrices

For $\beta = 2$

measure over the space of $N \times N$ hermitian matrices

$$V(x) = V_0(x) + \sum_{j\geq 1} \frac{t_j}{j}\, x^j$$

$$\frac{1}{Z_{N,n}}\, \mathrm{d}M\, e^{-n\,\mathrm{Tr}\, V(M)}$$

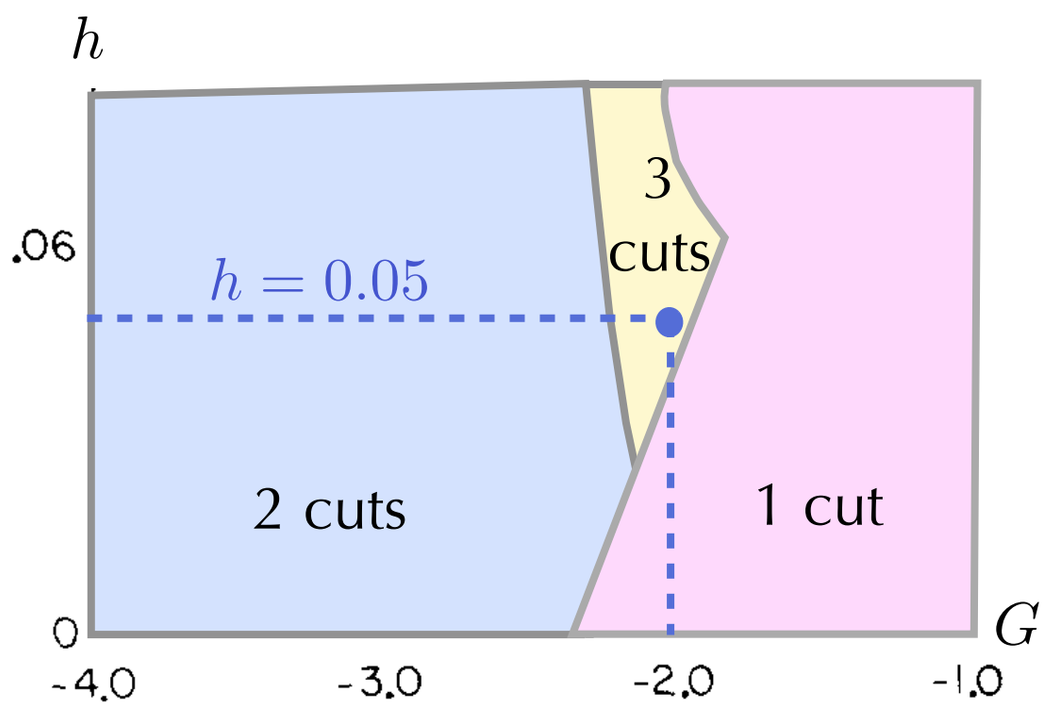
- The coefficients are solutions of a Toda chain :

$$\begin{cases} u_{N,n} = \ln h_{N,n} \\ v_{N,n} = -\beta_{N,n} \end{cases} \qquad \begin{cases} \partial_{t_1} u_{N,n} = v_{N,n} - v_{N-1,n} \\ \partial_{t_1} v_{N,n} = e^{u_{N+1,n}} - e^{u_{N,n}} \end{cases}$$

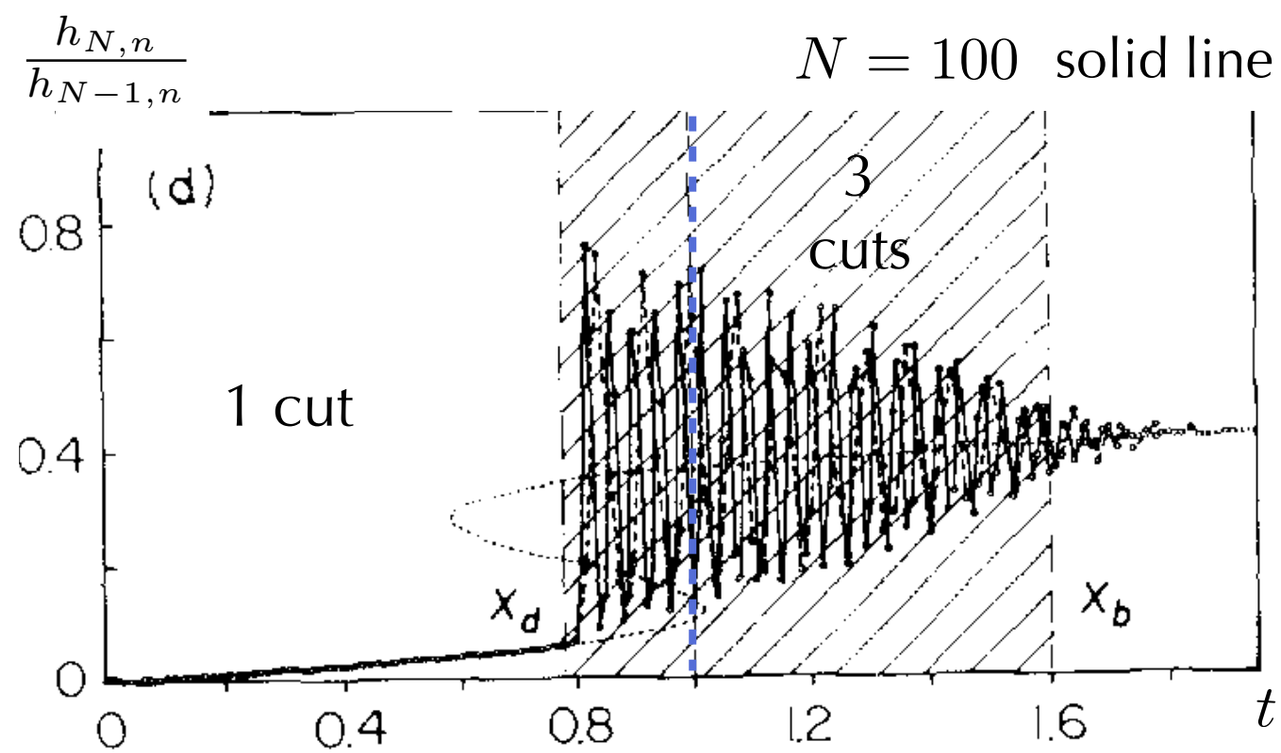
- ∂_{t_j} are the higher Toda flows
- initial condition prescribed by the string equations
- $Z_{N,n} = N! \prod_{j=0}^{N-1} h_{j,n}$ is the Tau function

The continuum limit of Toda

$N, n \to \infty \qquad N/n = t$ fixed



$V(x) = \frac{x^2}{2} + hG\,\frac{x^4}{4} + h^2\,\frac{x^6}{6}$



from Jurkiewicz '91 Phys. Lett. B, 261, 3

- if the model with V/t has (g + 1)-cuts and is off-critical

main result & $h_{N,n} = \dfrac{1}{N+1}\,\dfrac{Z_{N+1,nN/(N+1)}}{Z_{N,n}}$

$\Downarrow$

all-order oscillatory asymptotics for $u_{N,n} = \ln h_{N,n}$

Asymptotics of orthogonal polynomials

$N, n \to \infty$
$N/n = t$ fixed

- main result + $P_{N,n}(x) = \dfrac{Z_{N,n}^{V-(1/n)\ln(x-\bullet)}}{Z_{N,n}^{V}}$

  $\Longrightarrow$ all-order asymptotics of $P_{N,n}(x)$ for x away from its zero locus

- $\beta = 1, 4$ are related to skew orthogonal polynomials

  $\langle P_{j,n} | P_{k,n} \rangle = (\delta_{j,k-1} - \delta_{j-1,k}) h_{j,n}$

$$\beta = 1 \quad \begin{cases} M = \textit{real symmetric} \\ \langle f | g \rangle_{\beta=1} = \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \, e^{-n(V(x)+V(y))} \, \mathrm{sgn}(x-y) \, f(x) g(y) \\ N_{\beta=1} = 2N \end{cases}$$

$$\beta = 4 \quad \begin{cases} M = \textit{quaternionic self-dual} \\ \langle f | g \rangle_{\beta=4} = \int_{\mathbb{R}} \mathrm{d}x \, e^{-nV(x)} \big(f(x) g'(x) - f'(x) g(x) \big) \\ N_{\beta=4} = N \end{cases}$$

$$\begin{aligned} P_{2N,n}(x) &= \mathbb{E}_{N_\beta \times N_\beta} \big[\det(x - M) \big] \\ P_{2N+1,n}(x) &= \mathbb{E}_{N_\beta \times N_\beta} \big[(x + \mathrm{Tr}\, M) \det(x - M) \big] \end{aligned} \quad \Longrightarrow \text{ similar asymptotic results}$$

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What are Schwinger-Dyson equations ?

= relations between expectation values from integration by parts

- In the model $$Z_N^{-1}\, e^{-N^2\beta\mathcal{T}_I(L_N^{(\lambda)})} \prod_{1\le i<j\le N} |\lambda_i-\lambda_j|^\beta \prod_{i=1}^N \mathbf{1}_A(\lambda_i)\mathrm{d}\lambda_i$$

  we find for any smooth test function h
  and smooth functional $\mathcal{O}$

$$\mathbb{E}\Bigg[\Big(-\beta\sum_i h(\lambda_i)\,\mathcal{T}_I'(L_N^{(\lambda)})[\delta_{\lambda_i}] \;+\beta\sum_{i<j}\frac{h(\lambda_i)-h(\lambda_j)}{\lambda_i-\lambda_j}+\sum_i h'(\lambda_i)\Big)\mathcal{O}(L_N^{(\lambda)})$$

$$+\sum_i N^{-1}\,h(\lambda_i)\,\mathcal{O}'(L_N^{(\lambda)})[\delta_{\lambda_i}]\Bigg] + \text{boundary} = 0$$

Reminder : $L_N^{(\lambda)} = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$

What are Schwinger-Dyson equations ?

- We define the k-points correlators

$$W_k(x_1,\dots,x_k) = \text{Cumulant}\Big(\int \frac{N\,\mathrm{d}L_N^{(\lambda)}(\xi_1)}{x_1-\xi_1},\dots,\int \frac{N\,\mathrm{d}L_N^{(\lambda)}(\xi_k)}{x_k-\xi_k}\Big)$$

- Choose $h_z(x) = \dfrac{1}{z-x}$ and $\mathcal{O}_{z_2,\dots,z_k}(L_N^{(\lambda)}) = \displaystyle\prod_{i=2}^{k}\int \frac{\mathrm{d}L_N^{(\lambda)}(\xi_i)}{z_i-\xi_i}$

  for $z, z_i \in \mathbb{C}\setminus A$

$\longrightarrow$ family of functional relations between $W_1,\dots,W_{r+k-1}$
indexed by $k \geq 1$

Reminder : r-body interactions

The master operator

- Decompose $W_1(z) = N\big(W_{\rm eq}(z) + \delta_{-1} W_1(z)\big)$

  with $W_{\rm eq}(z) = \int \frac{{\rm d}\mu_{\rm eq}(\xi)}{z-\xi}$

- The Schwinger-Dyson equations can be recast

$$(\mathcal{K} + \delta\mathcal{K})[\delta_{-1} W_1](z) = A_1 + \text{boundary}$$

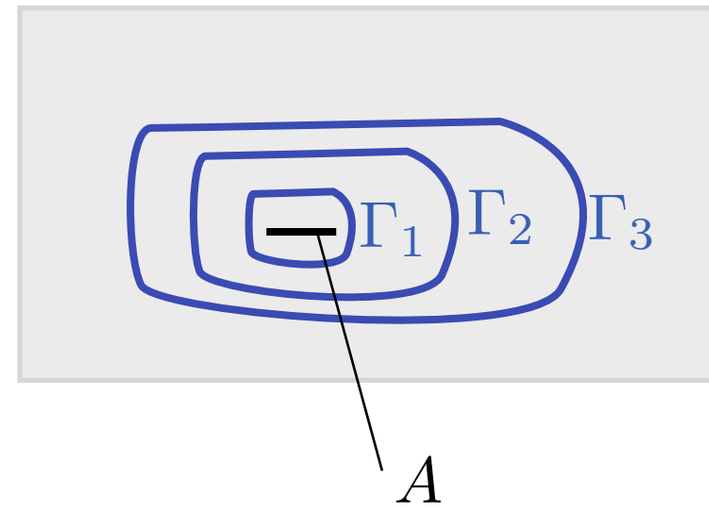
$$(\mathcal{K} + \delta\mathcal{K})[W_n(\cdot, z_2, \ldots, z_n)](z) = A_n + \text{boundary}$$

with : $\mathcal{K}[f](z) = 2W_{\rm eq}(z) f(z) - \mathcal{T}_I'(\mu_{\rm eq})\Big[\frac{f(\lambda){\rm d}\lambda}{z-\lambda}\Big]$

$$\delta\mathcal{K}[f](z) = 2\delta_{-1} W_1(z) f(z) + N^{-1}(1 - 2/\beta)\partial_z f(z) + \cdots$$

Cornerstone of the asymptotic analysis

- Introduce norms $\|f\|_\Gamma = \sup_{z\in \mathrm{Ext}(\Gamma)} \left|f(z)\right|$



- Large deviations of empirical measure

$$\|N\delta_{-1}W_1\|_{\Gamma_1} \leq C_1\,(N\ln N)^{1/2}$$
$$\|W_k\|_{\Gamma_k} \leq C_k\,(N\ln N)^{k/2}$$

- Large deviation of single eigenvalue : boundary effects $\in o(e^{-cN})$

- Rigidity of SD equations : if $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c\,\|f\|_{\Gamma_i}$

$$\left\{\begin{array}{rcl} \|N\delta_{-1}W_1\|_{\Gamma_{i_1}} & \leq & c_1\big(\eta_N\kappa_N+1\big) \\ \|W_k\|_{\Gamma_{i_k}} & \leq & c_k\big(\eta_N^k\kappa_N+N^{2-k}\big) \end{array}\right\}$$

$$\Downarrow$$

$$\left\{\begin{array}{rcl} \|N\delta_{-1}W_1\|_{\Gamma_{i_1+2}} & \leq & c_1'\big(\eta_N(\eta_N/N)\kappa_N+1\big) \\ \|W_k\|_{\Gamma_{i_k+2}} & \leq & c_k'\big(\eta_N^k(\eta_N/N)\kappa_N+N^{2-k}\big) \end{array}\right\}$$

Cornerstone of the asymptotic analysis

Large deviations of empirical measure

+ Rigidity of SD equations

Corollary

If $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c\,\|f\|_{\Gamma_i}$

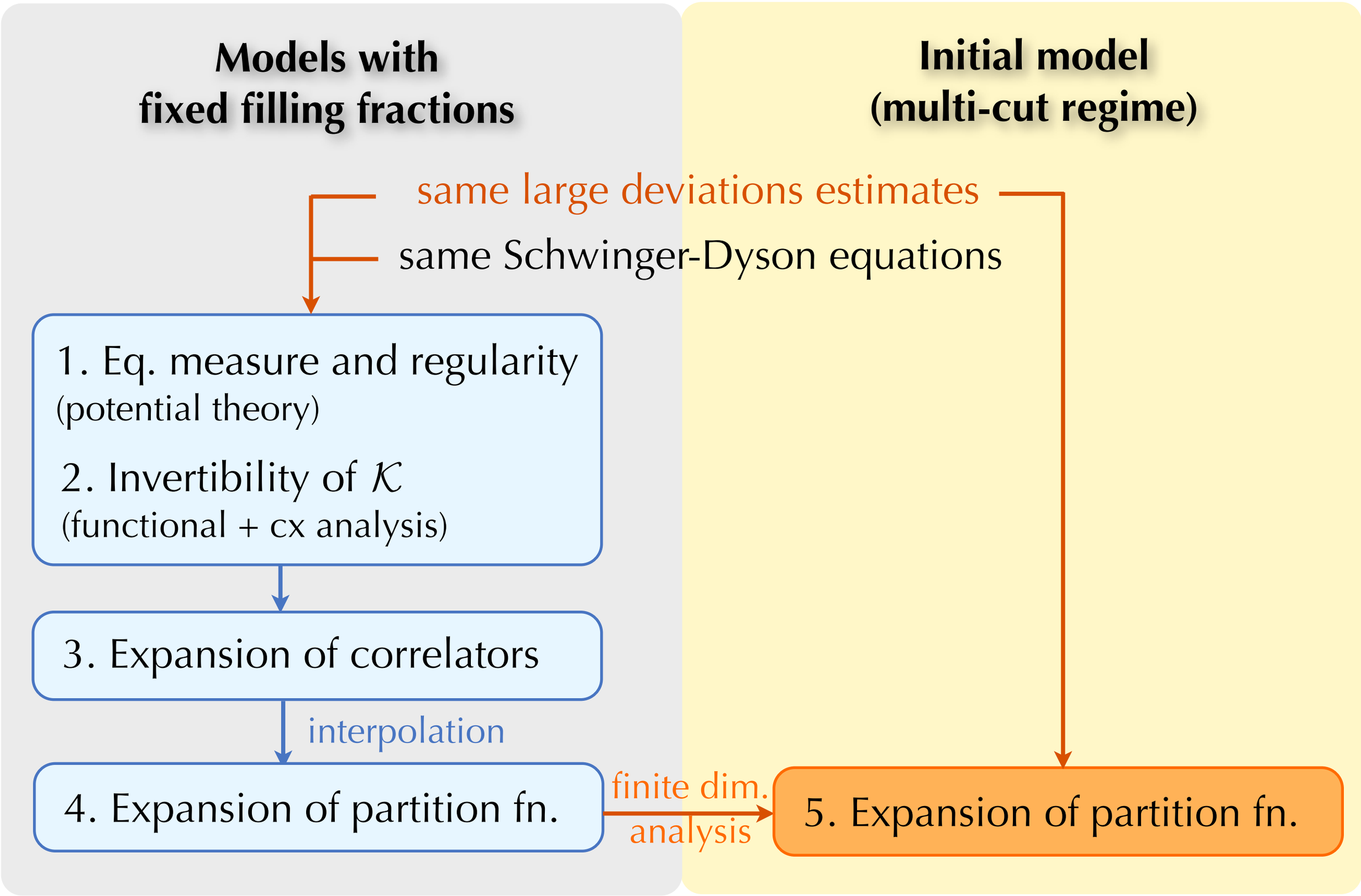
we have, for any $M \geq 0$ an asymptotic expansion

$$W_k = \sum_{m=k-2}^{M-1} N^{-m}\, W_k^{[m]} + O(N^{-M}; \Gamma_{M,k})$$

- Remark :

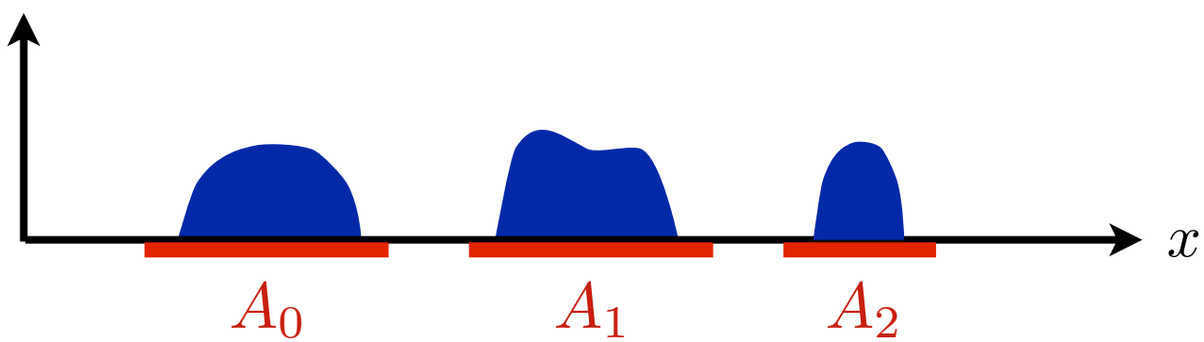
(g + 1) cuts
c = nb. critical conditions $\longrightarrow$ $\dim \operatorname{Ker} \mathcal{K} = g + c$

Scheme of the proof



Conditioning on the filling fractions

- From large deviations on single eigenvalue : up to $o(e^{-cN})$, we can choose



$$A = \bigcup_{h=0}^{g} A_h$$

- We will study $\mu^{(A_0,\dots,A_g)}_{(N_0,\dots,N_g)} = \mu^A_N$ conditioned to have $\begin{cases} N_0 \text{ first } \lambda\text{'s in } A_0 \\ N_1 \text{ next } \lambda\text{'s in } A_1 \\ \text{etc.} \end{cases}$

The partition function decomposes $$Z^A_N = \sum_{N_0+\cdots N_g=N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z^{(A_0,\dots,A_g)}_{(N_0,\dots,N_g)}$$

- $\epsilon_h = N_h/N$ are the filling fractions

The return of the master operator

- The correlators W_k in the initial model
  $W_{k;\epsilon}$ in the conditioned model

  satisfy the same Schwinger-Dyson equations

- We have $\oint_{A_{h_1}} \cdots \oint_{A_{h_k}} W_{k;\epsilon}(z_1, \ldots, z_k) \prod_{i=1}^{k} \frac{\mathrm{d}z_i}{2\mathrm{i}\pi} = \delta_{k,1}\, N \epsilon_{h_1}$

$\Longrightarrow$ we need the restriction $\mathcal{K}_{0;\epsilon}$ of $\mathcal{K}_\epsilon$ to the codim. = g subspace

$$\left\{ f, \quad \forall h, \quad \oint_{A_h} f(z)\mathrm{d}z = 0 \right\}$$

Lemma

For ϵ close enough to ϵ_{eq}

$\mathcal{K}_{0;\epsilon}$ is continuously invertible, and $\mathcal{K}_{0;\epsilon}^{-1}$ depends smoothly on ϵ

Asymptotic expansion of correlators in the conditioned model

Corollary

For ϵ close enough to ϵ_{eq}

we have, for any $M \geq 0$, an asymptotic expansion

$$W_{k;\epsilon} = \sum_{m=k-2}^{M-1} N^{-m} W_{k;\epsilon}^{[m]} + O(N^{-M}; \Gamma_{M,k})$$

depending smoothly on ϵ, with remainder uniform in ϵ

Partition function of the conditioned model

$$\frac{Z_{N;\epsilon}^{(T_1)}}{Z_{N;\epsilon}^{(T_0)}} = \exp\Big(-\frac{\beta N^{2-r}}{r!} \int \partial_t T_t(x_1,\dots,x_r) \prod_{i=1}^{r} \mathrm{d}L_N^{(\lambda),T_t}(x_i)\Big)$$

can be expressed in terms of $W_{j;\epsilon}^{T_t}$ for the model with interaction T_t

- We can find a interpolating family $(T_t)_{t\in[0,1]}$
  - respecting uniformly our assumptions
  - for which $Z_{N;\epsilon}^{(T_0)}$ is known (Selberg integrals)

so we deduce an expansion

$$Z_{N;\epsilon}^{(T_1)} = Z_{N;\epsilon}^{(T_0)} \times \exp\Big(\sum_{m=-2}^{M-1} N^{-m}\, F_\epsilon^{[m]} + O(N^{-M})\Big)$$

with remainder uniform in ϵ close enough to ϵ_{eq}

Sums and interferences - 1/3

We initially wanted to compute $Z_N = \sum_{N_0+\cdots N_g=N} \frac{N!}{\prod_{h=0}^g N_h!} Z_{(N_0,\ldots,N_g)}$

- From large deviations of empirical measures :

$$Z_N = \Big(\sum_{|\mathbf{N}-N\epsilon^\star|\leq \ln N} \frac{N!}{\prod_{h=0}^g N_h!} \; Z_{(N_0,\ldots,N_g)} \Big) \big(1+O(e^{-cN})\big)$$

- For $\mathbf{N} - N \quad \xi\, o(N)$, we just proved, with $\epsilon = (N_h/N)_{1\leq h\leq g}$

$$\frac{N!}{\prod_{h=0}^g N_h!} Z_{N\epsilon} = N^{\gamma N+\gamma'} \exp\Big[\sum_{m=-2}^{M-1} N^{-m} F_\epsilon^{[m]} + O(N^{-M}) \Big]$$

where $F_\epsilon^{[m]}$ depend smoothly on $\epsilon \approx \epsilon_{\rm eq}$

- Extra lemma : $(\nabla_\epsilon F^{[-2]})_{\epsilon_{\rm eq}} = 0$ and $(\nabla_\epsilon \nabla_\epsilon F^{[-2]})_{\epsilon_{\rm eq}} < 0$

Sums and interferences - 2/3

We plug the asymptotic formula and use a Taylor expansion at $\epsilon \approx \epsilon_{\mathrm{eq}}$

- E.g. up to o(1) :

$$Z_N = N^{\gamma N+\gamma'}\, e^{N^2 F_{\mathrm{eq}}^{[-2]}+N F_{\mathrm{eq}}^{[-1]}+F_{\mathrm{eq}}^{[0]}}$$

$$\times\Big(\sum_{|\mathbf{N}-N\epsilon_{\mathrm{eq}}|\leq \ln N} e^{\frac{1}{2}(\nabla^{\otimes 2}F^{[-2]})_{\mathrm{eq}}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})^{\otimes 2}+(\nabla F^{[-1]})_{\mathrm{eq}}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})}\Big)\big(1+O(e^{-c'(\ln N)^3/N})\big)$$

It is the general term of a super-exponentially fast converging series :

$$Z_N = N^{\gamma N+\gamma'}\, e^{N^2 F_{\mathrm{eq}}^{[-2]}+N F_{\mathrm{eq}}^{[-1]}+F_{\mathrm{eq}}^{[0]}}$$

$$\times\Big(\sum_{\mathbf{N}\in\mathbb{Z}^g} e^{\frac{1}{2}(\nabla^{\otimes 2}F^{[-2]})_{\mathrm{eq}}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})^{\otimes 2}+(\nabla F^{[-1]})_{\mathrm{eq}}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})}\Big)\big(1+O(e^{-c''(\ln N)^3/N})\big)$$

- We recognize $\Theta_{-N\epsilon_{\mathrm{eq}}}\big((\nabla F^{[-1]})_{\mathrm{eq}}\,\big|\,(\nabla^{\otimes 2}F^{[-2]})_{\mathrm{eq}}\big)$

Sums and interferences - 3/3

- Including higher orders yields terms of the form

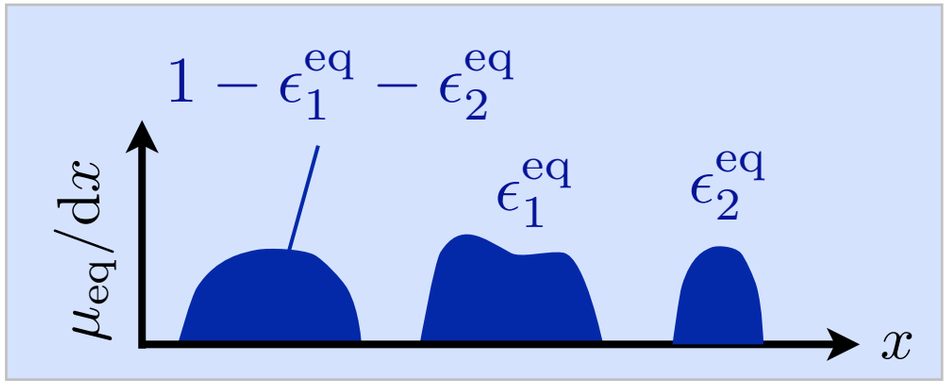
$$\sum_{\mathbf{N}\in\mathbb{Z}^g} \frac{1}{p!}\Big(\prod_{i=1}^{p} \frac{(\nabla^{\otimes \ell_i} F^{[m_i]})_{\mathrm{eq}}}{\ell_i!}\Big) \cdot (\mathbf{N} - N\epsilon_{\mathrm{eq}})^{\otimes(\sum_i \ell_i)}\, e^{\frac{1}{2}\mathbf{Q}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})^{\otimes 2}+\mathbf{w}\cdot(\mathbf{N}-N\epsilon_{\mathrm{eq}})}$$

We recognize $\displaystyle\sum_{\mathbf{N}\in\mathbb{Z}^g} \frac{1}{p!}\Big(\prod_{i=1}^{p} \frac{(\nabla^{\otimes \ell_i} F^{[m_i]})_{\mathrm{eq}}}{\ell_i!}\Big) \cdot (\nabla_{\mathbf{w}}^{\otimes(\sum_i \ell_i)}\Theta_{-N\epsilon_{\mathrm{eq}}})(\mathbf{w}|\mathbf{Q})$

Here $\mathbf{Q} = (\nabla^{\otimes 2} F^{[-2]})_{\mathrm{eq}}$ and $\mathbf{w} = (\nabla F^{[-1]})_{\mathrm{eq}}$

- We justified step by step the heuristics of **Bonnet, David, Eynard '00, Eynard '07**

Summary : the (g + 1)-cuts regime



- Oscillatory asymptotic expansion

$$Z_N = N^{\gamma N+\gamma'}(\mathcal{D}_N\Theta_{-N\epsilon_{\rm eq}})\big((\nabla F^{[-1]})_{\rm eq}\,\big|\,(\nabla^{\otimes 2}F^{[-2]})_{\rm eq}\big)\exp\Big[\sum_{m\geq -2}N^{-m}\,F^{[m]}+O(N^{-\infty})\Big]$$

where $$\mathcal{D}_N=\sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\ell_1,\dots,\ell_p\geq 1\\ m_1,\dots,m_p\geq -2\\ \sum_i(m_i+\ell_i)>0}}N^{-\sum_i(m_i+\ell_i)}\prod_{i=1}^{p}\frac{(\nabla^{\otimes\ell_i}F^{[m_i]})_{\rm eq}\cdot\nabla_{\mathbf{w}}^{\otimes\ell_i}}{\ell_i!}$$

acts as a differential operator on the Siegel theta function

$$\Theta_\mu(\mathbf{w}|\mathbf{Q})=\sum_{\mathbf{m}\in\mathbb{Z}^g}e^{\mathbf{w}\cdot(\mathbf{m}+\mu)+\frac{1}{2}(\mathbf{m}+\mu)\cdot\mathbf{Q}\cdot(\mathbf{m}+\mu)}$$

- Moving characteristics $\mu=-N\epsilon_{\rm eq}\bmod\mathbb{Z}^g$

  Quadratic form $\mathbf{Q}=-{\rm Hessian}_{\epsilon=\epsilon_{\rm eq}}[\mathcal{T}(\mu_{{\rm eq};\epsilon})]$

All order asymptotics for β-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices

2. Applications to orthogonal polynomials

3. Sketch of the proof of the main result

4. Conclusion

A general method ...

in progress with Guionnet, Kozlowski

- Large deviation estimates + rigidity of Schwinger-Dyson equations
- 1-cut = 1/N expansion multi-cut = oscillatory expansion
- concentration of measure at rate N^2 + particle tunneling
  $\longrightarrow$ Siegel theta function as "inteference function"

with currently some limitations ...

- require positive real weight at leading order
- require off-criticality
  $\rightarrow$ cannot address universality, ortho poly in the bulk, ...
  $\rightarrow$ cannot address (dis)appearance of singularities at $N = \infty$
- require analytic interactions (smooth should be enough)

XXZ building block

in progress with Guionnet, Kozlowski

$$Z_N = \int \left\{ \prod_{i<j} \sinh[N^\alpha \varpi_1(\lambda_i - \lambda_j)] \sinh[N^\alpha \varpi_2(\lambda_i - \lambda_j)] \right\} \prod_i e^{T_1(\lambda;N)} \mathrm{d}\lambda_i$$

- Scale where repulsion balanced by interactions : $T_1(x;N) = N^{1+\alpha} T(x)$
- Concentration of measure at rate $N^{2+\alpha}$ but multi-scale effects
- Strategy : define $\mu_{\mathrm{eq}}^{(P)}, \mathcal{K}^{(P)}$ depending on $P = N^\alpha$

  + fine asymptotic analysis of $(\mathcal{K}^{(P)})^{-1}$ when $P \to \infty$

  (auxiliary 2x2 Riemann-Hilbert problem)

Outcome : assuming $\begin{cases} T(x) \text{ smooth, strictly concave, confining, sub-linear} \\ \alpha \in [0, 1/6[\end{cases}$

$\longrightarrow$ formula for $\ln Z_N$ up to o(1)

$\longrightarrow$ asymptotic expansion in $1/N$ and $1/N^\alpha$