Asymptotic expansion in 1d log-gas with many-body interactions

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based on joint works with
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Asymptotic expansion in 1d log-gas with many-body interactions

1. Introduction: model and motivations

2. Main results and applications

3. Some elements of the proof

4. Conclusion
To a particle configuration \( \lambda_1, \ldots, \lambda_N \in A \subseteq \mathbb{R} \),
we assign a Boltzmann weight
\[
Z_N^{-1} e^{-\beta [E_C(\lambda) + E_I(\lambda)]} \quad \beta > 0
\]

Coulomb energy in 2d
\[
E_C(\lambda) = -\sum_{i<j} \ln |\lambda_i - \lambda_j|
\]

Interaction potential
\[
E_I(\lambda)
\]

- regular
- of same order than \( E_C(\lambda) \) for typical finite \( \lambda \)'s

1-body
\[
E_I(\lambda) = N \sum_{i=1}^N T_1(\lambda_i)
\]

2-body
\[
E_I(\lambda) = \frac{1}{2} \sum_{i,j} T_2(\lambda_i, \lambda_j)
\]

r-body
\[
E_I(\lambda) = (N^{2-r} / r!) \sum_i T_r(\lambda_{i_1}, \ldots, \lambda_{i_r})
\]

Partition function
\[
Z_N = \int_{A^N} \left[ \prod_{i=1}^N d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta \right] e^{-\beta E_I(\lambda)}
\]
Sources of examples of

\[ \left[ \prod_{i=1}^{N} d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta \right] e^{-\beta E_I(\lambda)} \]

- Invariant random matrix theory

\[ dM \ e^{N \text{Tr} T(M)} \]

\[ M = \text{matrix of size } N \]

\[ \{ \begin{array}{ll}
\beta = 1 & \text{real symmetric} \\
\beta = 2 & \text{hermitian} \\
\beta = 4 & \text{quaternionic self-dual} \\
\end{array} \]

Wigner, Dyson, Mehta
(50s-60s)

- Statistical physics on 2d random lattices (orientable \iff \beta = 2)

- O(n) loop model

\[ E_I(\lambda) = \left( \frac{n}{4} \right) \sum_{i,j} \ln(\lambda_i + \lambda_j) \quad \text{Gaudin, Kostov '89} \]

(realizes Ising for \( n = 1 \), Potts for \( n = \sqrt{q} \), ...)

- 6v-model

\[ E_I(\lambda) = \frac{1}{4} \sum_{i,j} \left\{ \ln \left[ \cosh(\lambda_i - \lambda_j) - \cos c \right] + \ln \left( \frac{\sinh(\lambda_i - \lambda_j)}{\lambda_i - \lambda_j} \right) \right\} \quad \text{Ginsparg '91, Kostov '99} \]
Sources of examples of

\[ \left[ \prod_{i=1}^{N} d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^\beta \right] e^{-\beta E_I(\lambda)} \]

- **Coulomb energy in other 2d domains/deformations**

\[ \tilde{E}_C(\lambda) = \sum_{i<j} G(\lambda_i, \lambda_j) = E_C(\lambda) + \sum_{i<j} \left[ G(\lambda_i, \lambda_j) + \ln |\lambda_i - \lambda_j| \right] \]

  - Cylinder
    \[ G_1(x, y) = -\ln |\sinh(x - y)| \]
  - Torus
    \[ G_2(x, y) = -\ln |\vartheta_1(x - y|\tau)| \]
  - (q,t) deformation
    \[ G_{q,t}(x, y) = -\frac{1}{2} \ln \left| \frac{(e^{x-y};q)_\infty (e^{y-x};q)_\infty}{(te^{x-y},q)_\infty (te^{y-x},q)_\infty} \right| \]

- **Chern-Simons theory on** \( S_3/\Gamma \) \( (a_1, \ldots, a_R) = \) Schläfli symbol of \( \Gamma \)

\[ \tilde{E}_C(\lambda) = \sum_{i<j} \left[ (2-R)G_1(\lambda_i, \lambda_j) + \sum_{m=1}^{R} G_1(\lambda_i/a_m, \lambda_j/a_m) \right] \quad \text{for SU}(N + 1) \]

\[ + \left[ (2-r)G_1(\lambda_i, -\lambda_j) + \sum_{m=1}^{r} G_1(\lambda_i/a_m, -\lambda_j/a_m) \right] \quad \text{for SO}(N)/\text{Sp}(2N) \]
Sources of examples of

\[ \left[ \prod_{i=1}^{N} d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \right] e^{-\beta E_I(\lambda)} \]

- Building block for correlation functions of XXZ spin chain, N sites

\[ \widetilde{E}_C(\lambda) = \frac{1}{2} \sum_{i<j} \ln |\sinh[N^\alpha \varpi_1(\lambda_i - \lambda_j)] \sinh[N^\alpha \varpi_2(\lambda_i - \lambda_j)]| \]

★ nature of singularity of the interaction changes at \( N \to \infty \)

The questions are ...

Find all-order asymptotic expansion when \( N \to \infty \) up to \( o(e^{-cN}) \) of

- partition function

\[ Z_N = \int_{A^N} \left[ \prod_{i=1}^{N} d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \right] e^{-\beta E_I(\lambda)} \]

- k-point correlations

\[ \mathbb{E} \left[ \sum_i f(\lambda_{i_1}, \ldots, \lambda_{i_k}) \right] \]

How does the (random) empirical measure \( L_{N}^{(\lambda)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \) behaves?
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Define the energy functional on a probability measure $\mu$

$$\mathcal{T}(\mu) = \int \int -\ln |x_1 - x_2| d\mu(x_1) d\mu(x_2) + \frac{1}{r!} \int T(x_1, \ldots, x_r) \prod_{i=1}^{r} d\mu(x_i)$$

Assumption 1: uniqueness of minimizer := $\mu_{eq}$

Assumption 2: positive Hessian at $\mu_{eq}$ (strict minimum)

for any $\nu$ = finite signed measure of mass 0

$$\mathcal{T}''(\mu_{eq})[\nu, \nu] = \mathcal{D}^2[\nu] \in [0, +\infty]$$

and = 0 iff $\nu = 0$

Lemma

$$L_N^{(\lambda)} \rightarrow \mu_{eq} \quad \text{almost surely and in expectation}$$

$$Z_N = \exp \left\{ -N^2 \beta (\mathcal{T}(\mu_{eq}) + o(1)) \right\}$$
Large deviations for a single particle

- A particle at position \( x \) feels the effective potential
  \[ J(x) = T'(\mu_{eq})[\delta_x] - \inf_{x \in A} T'(\mu_{eq})[\delta_x] \]

**Lemma**

For any closed \( F \subseteq A \)
\[ \mathbb{P}[\exists i, \lambda_i \in F] \leq \exp \left\{ -\beta N \left( \inf_{x \in F} J(x) + o(1) \right) \right\} \]

\( J(x) \)

\[ Z_N^B = Z_N^A (1 + o(e^{-cN})) \]

\( \leadsto \) One can restrict to a compact \( B \subseteq A \) neighborhood of \( \{J(x) = 0\} \)
Let us pick a nice regularization

Let us pick a nice regularization

**Lemma**

If $T(x_1, \ldots, x_r)$ is smooth, we have for $N$ large enough

$$
P_N \left[ \mathcal{D} [\widetilde{L}_N^{(\lambda)} - \mu_{eq}] > t \right] \leq \exp \left( N \ln N - N^2 t^2 / 2 \right)
$$

→ Concentration of measure at rate $N^2$
The equilibrium measure

- $T$ real-analytic $\implies \left\{ \begin{array}{l} \mu_{eq} \text{ is supported on a finite number of segments} \\ S = \bigcup_{h=0}^{g} [a_h, b_h] \end{array} \right.$

- $\alpha \in \partial S$ is a hard edge if $\alpha \in \partial A$, is a soft edge otherwise

$$d\mu_{eq}(x) = \frac{1_{S(x)}dx}{2\pi} M(x) \prod_{\alpha \text{ soft}} |x - \alpha|^{1/2} \prod_{\alpha \text{ hard}} |x - \alpha|^{-1/2}$$

- We say that $\mu_{eq}$ is off-critical when $M(x) > 0$ on $A$
Finite size corrections: we assume ...

- Uniqueness of minimizer $\mu_{eq}$
- Energy functional has positive hessian at $\mu_{eq}$

- $T = T^{(0)} + N^{-1}T^{(1)} + \ldots$
  \begin{align*}
  T^{(0)} & \text{ real analytic on } A \\
  T^{(1)} & \text{ complex analytic on } A
  \end{align*}

- Control of large deviations $J(x) > 0$ for $x \in A \setminus S$

- $\mu_{eq}$ is off-critical

- $f$ = test function, analytic on $A$
Some examples in which assumptions hold ...

Uniqueness of minimizer $\mu_{eq}$
Energy functional has positive hessian at $\mu_{eq}$

- 1-matrix model
  \[ dM \ e^{N \text{Tr} T(M)} \]
  \[ \text{OK} \]

- $O(n)$ loop model
  \[ \int \prod_{i<j} \frac{\lambda_i - \lambda_j}{(\lambda_i + \lambda_j)^{n/2}} \prod_i e^{NT_1(\lambda_i)} d\lambda_i \]
  \[ \text{OK for } |n| \leq 2 \]

- Chern-Simons theory on $S_3/\Gamma$
  \[ \int \left\{ \prod_{i<j} \left[ \sinh(\lambda_i - \lambda_j) \right]^{2-R} \prod_{m=1}^R \sinh\left(\frac{\lambda_i - \lambda_j}{a_m}\right) \right\} \prod_i e^{NT_1(\lambda_i)} d\lambda_i \]
  \[ \text{OK for } 2 - R + \sum_{m=1}^R \frac{1}{a_m} \geq 0 \]

- XXZ building block
  \[ \int \left\{ \prod_{i<j} \sinh[N^\alpha \varphi_1(\lambda_i - \lambda_j)] \sinh[N^\alpha \varphi_2(\lambda_i - \lambda_j)] \right\} \prod_i e^{T_1(\lambda_i;N)} d\lambda_i \]
  \[ \text{no because of } N \text{ dependence} \]
Result in the 1-cut regime

- 1/N asymptotic expansion

\[
Z_N = N^{(\beta/2)N+\gamma'} \exp \left[ \sum_{m \geq -2} N^{-m} F[m] + O(N^{-\infty}) \right]
\]

and for soft edges

\[
\gamma' = \frac{3+\beta/2+2/\beta}{12}
\]

eq etc.

- Central limit theorem

\[
\left( \sum_{i=1}^{N} f(\lambda_i) - N \int_{A} f(\xi) d\mu_{eq}(\xi) \right) \longrightarrow \text{ (non-centered) gaussian}
\]
Result in the \((g + 1)\)-cuts regime

- Oscillatory asymptotic expansion

\[
Z_N = N^{(\beta/2)N + \gamma'} (\mathcal{D}_N \Theta_{-N\epsilon_{eq}})(F[-1]'|F[-2]''\right) \exp \left[ \sum_{m \geq -2} N^{-m} F[m] + O(N^{-\infty}) \right]
\]

where \( \mathcal{D}_N = \sum_{p \geq 0} \frac{1}{p!} \sum_{\ell_1, \ldots, \ell_p \geq 1} \sum_{m_1, \ldots, m_p \geq -2} \sum_{\sum_i (m_i + \ell_i) > 0} N^{-\sum_i (m_i + \ell_i)} \prod_{i=1}^p F_{\epsilon_{eq}}[m_i], (\ell_i) \cdot \nabla_{\mathbf{w}} \otimes \ell_i \ell_i!
\]

acts as a differential operator on the Siegel theta function

\[
\Theta_{\mu}(\mathbf{w}|\mathbf{Q}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} e^{\mathbf{w} \cdot (\mathbf{m} + \mu) + \frac{1}{2} \mathbf{m} \cdot \mathbf{Q} \cdot \mathbf{m} + \mu}
\]

→ (pseudo)-periodicity due to \( \mu = -N\epsilon_{eq} \mod \mathbb{Z}^g \)

- E.g. for soft edges \( \gamma' = (g + 1) \frac{\beta/2 + 2/\beta + 3}{12} \)
Result in the \((g + 1)\)-cuts regime

- No central limit theorem in general ...

\[
\mathbb{E}\left[e^{is \left( \sum_{i=1}^{N} f(\lambda_i) - N \int f(x) d\mu_{eq}(x) \right)} \right] \sim_{N \to \infty} e^{is \left( m_1[f] - m_2[f]s^2/2 \right)}
\]

(non-centered) gaussian

+ discrete Gaussian, centered at \( \mu = -N\epsilon_{eq} \mod \mathbb{Z}^g \)

step \( \nu[f] \propto \left( \int_{S} \frac{f(x) x^i \, dx}{\prod_{\alpha} |x - \alpha|^{1/2}} \right)_{0 \leq i \leq g-1} \)

\[
\Theta - N\epsilon_{eq} \left( \frac{F[-1]' + i s \nu[f] \mid F[-2]''}{\Theta - N\epsilon_{eq} (F[-1]' \mid F[-2]'' )} \right)
\]
Topological recursion ($\beta = 2$)

- The coefficients of expansion can be computed recursively by a universal recursion

\[
\mu_{eq} := \text{large N limit of } L_N = \frac{1}{N} \sum_{i=1}^{N} \delta \lambda_i
\]

initial data = \[
\begin{cases}
\mu_{eq} := \text{large N limit of } L_N = \frac{1}{N} \sum_{i=1}^{N} \delta \lambda_i \\
2\text{-pt function} = \text{large N covariance of } N L_N
\end{cases}
\]

- Diagrammatic techniques to describe the expansion
History of 1-body interactions: 1-cut regime

\[ \beta = 2 \]

- If 1/N expansion exists, then
  \[ Z_N = N^{\gamma N + \gamma'} \exp \left[ \sum_{m \geq -1} N^{-2m} F^{\{m\}} \right] \]

  and \( F^{\{m\}} \) can be computed by the moment method

  Ambjørn, Chekhov, Kristjansen, Makeenko 90s

- Rewriting of \( F^{\{m\}} \) in terms of a universal topological recursion
  Eynard ’04

- Diagrammatics of topo. rec. related to CFT of a free boson
  Kostov 90’s, Kostov, Orantin, ’09

- Existence of 1/N expansion by
  - analysis of SD equations \( \text{Albeverio, Pastur, Shcherbina ’01} \)
  - RH techniques \( \text{Ercolani, McLaughlin ’02} \)
  - analysis of int. system \( \text{Bleher, Its, ’05} \)
if $1/N$ expansion exists, then \[ Z_N = N^{\gamma N + \gamma'} \exp \left[ \sum_{m \geq -2} N^{-m} F[m] \right] \]
and $F[m]$ computed by a $\beta$-topological recursion

Chekhov, Eynard ’06

- Central limit theorem
  Johansson ’98

- Existence of $1/N$ expansion (analysis of SD eqn)
  GB, Guionnet ’11
**History of 1-body interactions: multi-cut regime**

\( \beta = 2 \)
- Numerous observations of oscillatory behavior
  - Physicists, ‘90s
- Riemann-Hilbert techniques up to \( o(1) \)
  - Deift, Kriecherbauer, McLaughlin, Venakides, Zhou, ...
- Heuristic derivation up to \( o(1) \)
  - Bonnet, David, Eynard ’00
- Generalization to all orders
  - Eynard ’07
- Observation of “no CLT”
  - Pastur ’06

\( \beta > 0 \)
- Proof of “no CLT” and asymptotics of \( Z_N^A \) up to \( o(1) \)
  - Shcherbina ’12
- General proof
  - GB, Guionnet ’13
History of r-body interactions

■ all results extend to r-body models
  GB, Guionnet, Kozlowski ’13

■ computation of expansion by topological recursion
  2-body GB, Eynard, Orantin ’13
  r-body GB ’13

■ application to Chern-Simons theory // knot theory

  lens spaces // torus knots
  Halmagyi, Yasnov ’03
  Tierz ’06 - ....
  Brini, Eynard, Mariño ’11

$S_3/\Gamma$
GB, Eynard ’14
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A digression: Chern-Simons/int. system?

- Application to Chern-Simons theory/knot theory on $\mathbb{S}_3/\Gamma$
  $$\int \{ \prod_{i<j} [\sinh(\lambda_i - \lambda_j)]^{2-R} \prod_{m=1}^R \sinh[(\lambda_i - \lambda_j)/a_m] \} \prod_i e^{-N\lambda_i^2/2u}$$

$\rightsquigarrow$ conjectural relation between $\mu_{eq}$ and
char. poly. of Lax matrix of an ADE relativistic Toda chain

Brini (+ GB, Klemm, in progress)

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$(a_1, \ldots, a_R)$</th>
<th>Weyl group</th>
<th>after symm. breaking</th>
</tr>
</thead>
<tbody>
<tr>
<td>cyclic</td>
<td>$(1, p)$</td>
<td>$A_{p-1}$</td>
<td>$A_{p-1}$</td>
</tr>
<tr>
<td>dihedral</td>
<td>$(2, 2, p)$</td>
<td>$D_{p+2}$</td>
<td>$A_p$ (even) $D_{p+1}$ (odd)</td>
</tr>
<tr>
<td>tetrahedral</td>
<td>$(2, 3, 3)$</td>
<td>$E_6$</td>
<td>$D_4$</td>
</tr>
<tr>
<td>octahedral</td>
<td>$(2, 3, 4)$</td>
<td>$E_7$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>icosahedral</td>
<td>$(2, 3, 5)$</td>
<td>$E_8$</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>
Orthogonal polynomials and random matrices

For \( \beta = 2 \)

\[
V(x) = V_0(x) + \sum_{j \geq 1} \frac{t_j}{j} x^j
\]

measure over the space of \( N \times N \) hermitian matrices

\[
\frac{1}{Z_{N,n}} \ dM \ e^{-n \text{Tr} \ V(M)}
\]

- \( P_{N,n}(x) = \mathbb{E}_{N \times N} \left[ \det(x - M) \right] \)

is the \( N \)th orthogonal polynomial for the weight \( \text{d}x \ e^{-nV(x)} \) on \( \mathbb{R} \)

- Let \( h_{N,n} = \text{norm of } P_{N,n} \)

\[
\hat{P}_{N,n} = P_{N,n}/\sqrt{h_{N,n}} \quad \text{satisfies a 3-term recurrence relation}
\]

\[
(x - \beta_{N,n}) \hat{P}_{N,n}(x) = \sqrt{h_{N,n}} \hat{P}_{N+1,n}(x) + \sqrt{h_{N-1,n}} \hat{P}_{N-1,n}(x)
\]
The coefficients are solutions of a Toda chain:

\[
\begin{align*}
\left\{ \begin{array}{l}
  u_{N,n} = \ln h_{N,n} \\
  v_{N,n} = -\beta_{N,n}
\end{array} \right. \\
\left\{ \begin{array}{l}
  \partial_{t_1} u_{N,n} = v_{N,n} - v_{N-1,n} \\
  \partial_{t_1} v_{N,n} = e^{u_{N+1,n}} - e^{u_{N,n}}
\end{array} \right.
\]

- $\partial_{t_j}$ are the higher Toda flows
- initial condition prescribed by the string equations
- $Z_{N,n} = N! \prod_{j=0}^{N-1} h_{j,n}$ is the Tau function

For $\beta = 2$

\[
V(x) = V_0(x) + \sum_{j \geq 1} \frac{t_j}{j} x^j
\]

\[
\frac{1}{Z_{N,n}} \, dM \, e^{-n \, \text{Tr} \, V(M)}
\]
The continuum limit of Toda

\[ N, n \to \infty \quad N/n = t \quad \text{fixed} \]

\[ h = \frac{1}{N+1} \frac{Z_{N+1,n}N/(N+1)}{Z_{N,n}} \]

- if the model with \( V/t \) has \((g + 1)\)-cuts and is off-critical

main result & \( u_{N,n} = \ln h_{N,n} \)

\[ V(x) = \frac{x^2}{2} + hG \frac{x^4}{4} + h^2 \frac{x^6}{6} \]

\[ h = 0.05 \]

\[ h = 0.06 \]

\[ 1 \text{ cut} \]

\[ 2 \text{ cuts} \]

\[ 3 \text{ cuts} \]

from Jurkiewicz '91 Phys. Lett. B, 261, 3
Asymptotics of orthogonal polynomials

\[ N, n \to \infty \]

\[ N/n = t \text{ fixed} \]

**main result** + \( P_{N,n}(x) = \frac{\frac{Z_{V-(1/n)\ln(x-\bullet)}}{Z_{V}}}{Z_{N,n}} \)

\[ \implies \text{all-order asymptotics of } P_{N,n}(x) \text{ for } x \text{ away from its zero locus} \]

**\( \beta = 1, 4 \)** are related to skew orthogonal polynomials

\[ \langle P_{j,n} | P_{k,n} \rangle = (\delta_{j,k-1} - \delta_{j-1,k}) h_{j,n} \]

\[ \left\{ \begin{array}{l}
M = \text{real symmetric} \\
\beta = 1 \\
\langle f | g \rangle_{\beta=1} = \int_{\mathbb{R}^2} dx dy \, e^{-n(V(x)+V(y))} \, \text{sgn}(x-y) \, f(x)g(y) \\
N_{\beta=1} = 2N
\end{array} \right. \]

\[ \left\{ \begin{array}{l}
M = \text{quaternionic self-dual} \\
\beta = 4 \\
\langle f | g \rangle_{\beta=4} = \int_{\mathbb{R}} dx \, e^{-nV(x)} (f(x)g'(x) - f'(x)g(x)) \\
N_{\beta=4} = N
\end{array} \right. \]

\[ P_{2N,n}(x) = \mathbb{E}_{N_\beta \times N_\beta} \left[ \text{det}(x-M) \right] \]

\[ P_{2N+1,n}(x) = \mathbb{E}_{N_\beta \times N_\beta} \left[ (x + \text{Tr } M) \text{det}(x-M) \right] \]

\[ \implies \text{similar asymptotic results} \]
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What are Schwinger-Dyson equations?

= relations between expectation values from integration by parts

- In the model

\[ Z_N^{-1} e^{-N^2 \beta \mathcal{T}_I(L_N^{(\lambda)})} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N 1_{A}(\lambda_i) d\lambda_i \]

we find for any smooth test function \( h \) and smooth functional \( \mathcal{O} \)

\[ \mathbb{E} \left[ \left( -\beta \sum_i h(\lambda_i) \mathcal{T}_I(L_N^{(\lambda)})[\delta \lambda_i] + \beta \sum_{i<j} \frac{h(\lambda_i) - h(\lambda_j)}{\lambda_i - \lambda_j} + \sum_i h'(\lambda_i) \right) \mathcal{O}(L_N^{(\lambda)}) \right] \\
+ \sum_i \frac{N^{-1} h(\lambda_i) \mathcal{O}'(L_N^{(\lambda)})[\delta \lambda_i]}{\sum_{i=1}^N} + \text{boundary} = 0 \]

Reminder: \[ L_N^{(\lambda)} = \frac{1}{N} \sum_{i=1}^N \delta \lambda_i \]
What are Schwinger-Dyson equations?

- We define the k-points correlators

\[ W_k(x_1, \ldots, x_k) = \text{Cumulant} \left( \int \frac{N \, dL_N^{(\lambda)}(\xi_1)}{x_1 - \xi_1}, \ldots, \int \frac{N \, dL_N^{(\lambda)}(\xi_k)}{x_k - \xi_k} \right) \]

- Choose \( h_z(x) = \frac{1}{z - x} \) and \( \mathcal{O}_{z_2, \ldots, z_k}(L_N^{(\lambda)}) = \prod_{i=2}^{k} \int \frac{dL_N^{(\lambda)}(\xi_i)}{z_i - \xi_i} \)

for \( z, z_i \in \mathbb{C} \setminus A \)

\[ \rightarrow \quad \text{family of functional relations between } W_1, \ldots, W_{r+k-1} \]

indexed by \( k \geq 1 \)

Reminder: \( r \)-body interactions
The master operator

- Decompose \( W_1(z) = N \left( W_{eq}(z) + \delta_{-1} W_1(z) \right) \)
  with \( W_{eq}(z) = \int \frac{d\mu_{eq}(\xi)}{z - \xi} \)

- The Schwinger-Dyson equations can be recast

\[
(\mathcal{K} + \delta \mathcal{K})[\delta_{-1} W_1](z) = A_1 + \text{boundary}
\]

\[
(\mathcal{K} + \delta \mathcal{K})[W_n(\cdot, z_2, \ldots, z_n)](z) = A_n + \text{boundary}
\]

with:

\[
\mathcal{K}[f](z) = 2W_{eq}(z)f(z) - \mathcal{T}_f'(\mu_{eq}) \left[ \frac{f(\lambda)d\lambda}{z - \lambda} \right]
\]

\[
\delta \mathcal{K}[f](z) = 2\delta_{-1} W_1(z)f(z) + N^{-1}(1 - 2/\beta) \partial_z f(z) + \cdots
\]
- Introduce norms $\|f\|_{\Gamma} = \sup_{z \in \text{Ext}(\Gamma)} |f(z)|$

- Large deviations of empirical measure

\[
\|N\delta_{-1}W_1\|_{\Gamma_1} \leq C_1 (N \ln N)^{1/2} \\
\|W_k\|_{\Gamma_k} \leq C_k (N \ln N)^{k/2}
\]

- Large deviation of single eigenvalue: boundary effects $\in o(e^{-cN})$

- Rigidity of SD equations: if $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c \|f\|_{\Gamma_i}$

\[
\begin{cases}
\|N\delta_{-1}W_1\|_{\Gamma_{i_1}} \leq c_1 (\eta_N \kappa_N + 1) \\
\|W_k\|_{\Gamma_{i_k}} \leq c_k (\eta_N^k \kappa_N + N^{2-k})
\end{cases}
\]

\[
\begin{cases}
\|N\delta_{-1}W_1\|_{\Gamma_{i_1+2}} \leq c'_1 (\eta_N(\eta_N/N) \kappa_N + 1) \\
\|W_k\|_{\Gamma_{i_k+2}} \leq c'_k (\eta_N^k (\eta_N/N) \kappa_N + N^{2-k})
\end{cases}
\]
Corollary

If $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c \|f\|_{\Gamma_i}$

we have, for any $M \geq 0$ an asymptotic expansion

$$W_k = \sum_{m=k-2}^{M-1} N^{-m} W_k^{[m]} + O(N^{-M}; \Gamma_{M,k})$$

Remark:

$(g + 1)$ cuts

c = nb. critical conditions

$\dim \ker \mathcal{K} = g + c$
Scheme of the proof

Models with fixed filling fractions

1. Eq. measure and regularity (potential theory)
2. Invertibility of $\mathcal{K}$ (functional + cx analysis)
3. Expansion of correlators
4. Expansion of partition fn.
5. Expansion of partition fn.

Initial model (multi-cut regime)

- same large deviations estimates
- same Schwinger-Dyson equations

1. Eq. measure and regularity (potential theory)
From large deviations on single eigenvalue: up to $o(e^{-cN})$, we can choose

$$A = \bigcup_{h=0}^{g} A_h$$

We will study $\mu^{(A_0,\ldots,A_g)}_{(N_0,\ldots,N_g)} = \mu^A_N$ conditioned to have \[\begin{cases} N_0 \text{ first } \lambda' \text{'s in } A_0 \\ N_1 \text{ next } \lambda' \text{'s in } A_1 \\ \text{etc.} \end{cases}\]

The partition function decomposes

$$Z_N^A = \sum_{N_0+\cdots+N_g=N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{(N_0,\ldots,N_g)}^{(A_0,\ldots,A_g)}$$

$\epsilon_h = N_h/N$ are the filling fractions
The return of the master operator

- The correlators $W_k$ in the initial model
- $W_{k;\epsilon}$ in the conditioned model

satisfy the same Schwinger-Dyson equations

We have

$$\int_{A_{h_1}} \cdots \int_{A_{h_k}} W_{k;\epsilon}(z_1, \ldots, z_k) \prod_{i=1}^{k} \frac{dz_i}{2i\pi} = \delta_{k,1} N \epsilon_{h_1}$$

$\implies$ we need the restriction $\mathcal{K}_{0;\epsilon}$ of $\mathcal{K}_\epsilon$ to the codim. = g subspace

$$\{ f, \forall h, \int_{A_h} f(z) dz = 0 \}$$

**Lemma**

For $\epsilon$ close enough to $\epsilon_{eq}$

$\mathcal{K}_{0;\epsilon}$ is continuously invertible, and $\mathcal{K}_{0;\epsilon}^{-1}$ depends smoothly on $\epsilon$
Asymptotic expansion of correlators in the conditioned model

Corollary

For $\epsilon$ close enough to $\epsilon_{eq}$

we have, for any $M \geq 0$, an asymptotic expansion

$$W_{k;\epsilon} = \sum_{m=k-2}^{M-1} N^{-m} W_{k;\epsilon}^{[m]} + O(N^{-M}; \Gamma_{M,k})$$

depending smoothly on $\epsilon$, with remainder uniform in $\epsilon$
Partition function of the conditioned model

\[
\frac{Z^{(T_1)}_{N;\epsilon}}{Z^{(T_0)}_{N;\epsilon}} = \exp \left( -\frac{\beta N^{2-r}}{r!} \int \partial_t T_t(x_1, \ldots, x_r) \prod_{i=1}^r dL_{N}^{(\lambda)}(x_i) \right)
\]

can be expressed in terms of \( W_{j;\epsilon}^{T_t} \) for the model with interaction \( T_t \)

- We can find a interpolating family \((T_t)_{t \in [0,1]}\)
  - respecting uniformly our assumptions
  - for which \( Z^{(T_0)}_{N;\epsilon} \) is known (Selberg integrals)

so we deduce an expansion

\[
Z^{(T_1)}_{N;\epsilon} = Z^{(T_0)}_{N;\epsilon} \times \exp \left( \sum_{m=-2}^{M-1} N^{-m} F^{[m]}_{\epsilon} + O(N^{-M}) \right)
\]

with remainder uniform in \( \epsilon \) close enough to \( \epsilon_{eq} \)
Sums and interferences - 1/3

We initially wanted to compute

\[ Z_N = \sum_{N_0 + \cdots + N_g = N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{(N_0, \ldots, N_g)} \]

- From large deviations of empirical measures:

\[ Z_N = \left( \sum_{|N - N_{\mathbb{E}*}| \leq \ln N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{(N_0, \ldots, N_g)} \right) (1 + O(e^{-cN})) \]

- For \( N - N \approx o(N) \), we just proved, with \( \epsilon = (N_h/N)_{1 \leq h \leq g} \)

\[ \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{N\epsilon} = N^{\gamma N + \gamma'} \exp \left[ \sum_{m=-2}^{M-1} N^{-m} F_{\epsilon}^{[m]} + O(N^{-M}) \right] \]

where \( F_{\epsilon}^{[m]} \) depend smoothly on \( \epsilon \approx \epsilon_{eq} \)

- Extra lemma: \( (\nabla_{\epsilon} F^{[-2]})_{\epsilon_{eq}} = 0 \) and \( (\nabla_{\epsilon} \nabla_{\epsilon} F^{[-2]})_{\epsilon_{eq}} < 0 \)
We plug the asymptotic formula and use a Taylor expansion at $\epsilon \approx \epsilon_{eq}$

- E.g. up to $o(1)$:

$$Z_N = N^{\gamma N + \gamma'} e^{N^2 F_{eq}[-2]} + N F_{eq}[-1] + F_{eq}^{[0]}$$

$$\times \left( \sum_{|N - N \epsilon_{eq}| \leq \ln N} e^{\frac{1}{2} (\nabla \otimes^2 F[-2])_{eq} \cdot (N - N \epsilon_{eq}) \otimes^2 + (\nabla F[-1])_{eq} \cdot (N - N \epsilon_{eq})} \right) (1 + O(e^{-c'(\ln N)^3/N}))$$

It is the general term of a super-exponentially fast converging series:

$$Z_N = N^{\gamma N + \gamma'} e^{N^2 F_{eq}[-2]} + N F_{eq}[-1] + F_{eq}^{[0]}$$

$$\times \left( \sum_{N \in \mathbb{Z}^g} e^{\frac{1}{2} (\nabla \otimes^2 F[-2])_{eq} \cdot (N - N \epsilon_{eq}) \otimes^2 + (\nabla F[-1])_{eq} \cdot (N - N \epsilon_{eq})} \right) (1 + O(e^{-c''(\ln N)^3/N}))$$

- We recognize $\Theta - N \epsilon_{eq} \left( (\nabla F[-1])_{eq} \parallel (\nabla \otimes^2 F[-2])_{eq} \right)$
Including higher orders yields terms of the form

$$\sum_{N \in \mathbb{Z}^g} \frac{1}{p!} \left( \prod_{i=1}^{p} \frac{(\nabla \otimes \ell_i F[m_i])_{eq}}{\ell_i!} \right) \cdot (N - N_{\epsilon_{eq}}) \otimes (\sum_i \ell_i) \cdot e^{-\frac{1}{2} Q \cdot (N - N_{\epsilon_{eq}}) \otimes^2 + w \cdot (N - N_{\epsilon_{eq}})}$$

We recognize

$$\sum_{N \in \mathbb{Z}^g} \frac{1}{p!} \left( \prod_{i=1}^{p} \frac{(\nabla \otimes \ell_i F[m_i])_{eq}}{\ell_i!} \right) \cdot (\nabla^w (\sum_i \ell_i) \Theta_{-N_{\epsilon_{eq}}}) (w \mid Q)$$

Here

$$Q = (\nabla \otimes^2 F[-2])_{eq} \quad \text{and} \quad w = (\nabla F[-1])_{eq}$$

We justified step by step the heuristics of

Bonnet, David, Eynard ’00, Eynard ’07
Summary: the \((g + 1)\)-cuts regime

- Oscillatory asymptotic expansion

\[
Z_N = N^{\gamma N + \gamma'} (D_N \Theta_{-N\epsilon_{\text{eq}}}) ((\nabla F^{-1})_{\text{eq}} | (\nabla \otimes^2 F^{-2})_{\text{eq}}) \exp \left[ \sum_{m \geq -2} N^{-m} F^m + O(N^{-\infty}) \right]
\]

where \( D_N = \sum_{p \geq 0} \frac{1}{p!} \sum_{\ell_1, \ldots, \ell_p \geq 1 \atop m_1, \ldots, m_p \geq -2 \atop \sum_i (m_i + \ell_i) > 0} N^{-\sum_i (m_i + \ell_i)} \prod_{i=1}^p \frac{(\nabla \otimes \ell_i F^m)_{\text{eq}} \cdot \nabla \otimes \ell_i}{\ell_i!}
\]

acts as a differential operator on the Siegel theta function

\[
\Theta_\mu(w|Q) = \sum_{m \in \mathbb{Z}^g} e^{w \cdot (m+\mu) + \frac{1}{2} (m+\mu) \cdot Q \cdot (m+\mu)}
\]

- Moving characteristics

\[
\mu = -N\epsilon_{\text{eq}} \mod \mathbb{Z}^g
\]

- Quadratic form

\[
Q = -\text{Hessian}_{\epsilon = \epsilon_{\text{eq}}}[T(\mu_{\text{eq}}; \epsilon)]
\]
All order asymptotics for $\beta$-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices

2. Applications to orthogonal polynomials

3. Sketch of the proof of the main result

4. Conclusion
A general method ... in progress with Guionnet, Kozlowski

- Large deviation estimates + rigidity of Schwinger-Dyson equations

- $1$-cut = $1/N$ expansion  multi-cut = oscillatory expansion

- concentration of measure at rate $N^2$ + particle tunneling
  $\rightarrow$ Siegel theta function as “interference function”

with currently some limitations ...

- require positive real weight at leading order

- require off-criticality
  $\rightarrow$ cannot address universality, ortho poly in the bulk, ...
  $\rightarrow$ cannot address (dis)appearance of singularities at $N = \infty$

- require analytic interactions (smooth should be enough)
\[ Z_N = \int \left\{ \prod_{i<j} \sinh[N^\alpha \omega_1(\lambda_i - \lambda_j)] \sinh[N^\alpha \omega_2(\lambda_i - \lambda_j)] \right\} \prod_i e^{T_1(\lambda_i; N)} d\lambda_i \]

- Scale where repulsion balanced by interactions: \( T_1(x; N) = N^{1+\alpha} T(x) \)
- Concentration of measure at rate \( N^{2+\alpha} \) but multi-scale effects
- Strategy: define \( \mu^{(P)}_{eq}, K^{(P)} \) depending on \( P = N^\alpha \)
  + fine asymptotic analysis of \( (K^{(P)})^{-1} \) when \( P \to \infty \)
  (auxiliary 2x2 Riemann-Hilbert problem)

Outcome: assuming \( \left\{ \begin{array}{l} T(x) \text{ smooth, strictly concave, confining, sub-linear} \vspace{1em} \\ \alpha \in [0, 1/6[ \end{array} \right. \)

\[ \to \text{ formula for } \ln Z_N \text{ up to } o(1) \]
\[ \to \text{ asymptotic expansion in } 1/N \text{ and } 1/N^\alpha \]