Gaudin-like determinants for overlaps in integrable systems – Quench Action approach for the Néel-to-XXZ quench

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Outline

- Motivation – the quench protocol
- The spin-1/2 XXZ Heisenberg chain (fixing the notation)
- Overlap of the zero-momentum Néel state with XXZ Bethe states – Gaudin-like determinant formula
- Quench action approach: generalized TBA equations
- Analytical solution
- Conclusion and outlook

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In collaboration with... Jean-Sébastien Caux
Davide Fioretto
Jacopo De Nardis
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Bram Wouters

The quench protocol: $|\Psi_0\rangle \longrightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$

Here: Quench to the spin-1/2 XXZ chain starting from the ground state of the Ising model

[B. Wouters, J. De Nardis, MB, D. Fioretto, J.-S. Caux, arXiv:1405.0172, to be published in PRL]

Initial state:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\downarrow\ldots\rangle + |\downarrow\uparrow\downarrow\uparrow\ldots\rangle\right)$$

Hamiltonian: (lattice size N, σ_j^α = Pauli matrices at lattice site j)

$$H = \sum_{j=1}^{N}\Big(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)\Big)$$

PBC's: $\sigma_{N+1}^\alpha = \sigma_1^\alpha$, $\alpha = x, y, z$; anisotropy parameter $\Delta = \mathrm{ch}(\eta) \geq 1$

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Objects of interest: Time evolution of observables [in particular for large t, in the limit $N \to \infty$]

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = \langle\Psi_0|e^{iHt}\mathcal{O}e^{-iHt}|\Psi_0\rangle = \sum_{\lambda,\lambda'}\langle\Psi_0|\lambda\rangle\langle\lambda'|\Psi_0\rangle e^{i(\omega_\lambda-\omega_{\lambda'})t}\langle\lambda|\mathcal{O}|\lambda'\rangle$$

$\to$ Three ingredients: 1) Matrix elements $\langle\lambda|\mathcal{O}|\lambda'\rangle$, 2) Energies ω_λ, 3) Overlaps $\langle\Psi_0|\lambda\rangle$

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Problem: double sum over the Hilbert space $\sum_{\lambda,\lambda'}$

Quench action approach

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Solution: Quench Action approach → talk by J.-S. Caux (this morning)

- Restriction to a certain class of operators (so-called "weak operators" in the thermodynamic limit)
- Applying a saddle-point approximation by minimizing the "quench action"
- Result: Expectation values (not only) for long times after the quench. Here: $t \to \infty$
  But first(!) TD limit $N \to \infty$ with magnetization fixed to zero, denoted by $\lim_{\mathrm{th}}$:

$$\lim_{t\to\infty} \lim{}_{\mathrm{th}} \langle\Psi(t)|\,\mathcal{O}\,|\Psi(t)\rangle = \langle\rho^{\mathrm{sp}}|\,\mathcal{O}\,|\rho^{\mathrm{sp}}\rangle \,.$$

- Generalized TBA equations for the saddle point state:

$$0 = \left.\frac{\delta S_{QA}[\rho]}{\delta\rho_n}\right|_{\rho=\rho^{\mathrm{sp}}} \quad \text{with} \quad S_{QA}[\rho] = 2S[\rho] - S_{YY}[\rho]$$

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ABA for the XXZ spin chain

Algebraic Bethe ansatz for the spin-1/2 XXZ chain

– Yang-Baxter algebra (2×2 monodromy matrix $T(\lambda)$; λ, μ spectral parameter):

$$\check{R}(\lambda-\mu)\,[T(\lambda) \otimes T(\mu)] = [T(\mu) \otimes T(\lambda)]\,\check{R}(\lambda-\mu)$$

with R-matrix of the 6-vertex model

$$\check{R}(\lambda) = \frac{1}{\mathrm{sh}(\lambda+\eta)} \begin{pmatrix} \mathrm{sh}(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \mathrm{sh}(\eta) & \mathrm{sh}(\lambda) & 0 \\ 0 & \mathrm{sh}(\lambda) & \mathrm{sh}(\eta) & 0 \\ 0 & 0 & 0 & \mathrm{sh}(\lambda+\eta) \end{pmatrix}$$

– Monodromy matrix (product in auxiliary space of N Lax operators):

$$T(\lambda) = \prod_{n=1}^{N} L_n(\lambda) = L_1(\lambda) \dots L_N(\lambda) =: \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

with Lax operators on lattice sites $n = 1, \dots, N$ (2×2 matrix in auxiliary space)

$$L_n(\lambda) = \frac{1}{\mathrm{sh}(\lambda+\eta/2)} \begin{pmatrix} \mathrm{sh}\left(\lambda + \frac{\eta}{2}\sigma_n^z\right) & \mathrm{sh}(\eta)\sigma_n^- \\ \mathrm{sh}(\eta)\sigma_n^+ & \mathrm{sh}\left(\lambda - \frac{\eta}{2}\sigma_n^z\right) \end{pmatrix}$$

with Pauli matrices $\sigma_n^z, \sigma_n^\pm = \frac{1}{2}(\sigma_n^x \pm \mathrm{i}\sigma_n^y)$ acting on lattice site n

Algebraic Bethe ansatz for the spin-1/2 XXZ chain

– Transfer matrices $t(\lambda) = \mathrm{tr}_a\big[T(\lambda)\big] = A(\lambda) + D(\lambda)$ build a commutative family: $[t(\lambda), t(\mu)] = 0$

– Conserved charges of the XXZ spin chain:

$$Q_{m+1} = \frac{\partial^m}{\partial\lambda^m} \ln[t(\lambda)]\bigg|_{\lambda=\eta/2}$$

where $H = 2\,\mathrm{sh}(\eta) Q_2$

– Bethe states $|\{\lambda_j\}_{j=1}^M\rangle = \prod_{j=1}^M B(\lambda_j)\,|\uparrow\rangle^{\otimes N}$ (λ_j arbitrary = "off-shell")

Eigenstates of the transfer matrix with eigenvalue

$$\tau(\lambda) = \prod_{k=1}^M \frac{\mathrm{sh}(\lambda-\lambda_k-\eta)}{\mathrm{sh}(\lambda-\lambda_k)} + \left[\frac{\mathrm{sh}(\lambda-\eta/2)}{\mathrm{sh}(\lambda+\eta/2)}\right]^N \prod_{k=1}^M \frac{\mathrm{sh}(\lambda-\lambda_k+\eta)}{\mathrm{sh}(\lambda-\lambda_k)}$$

if the parameters λ_j, $j = 1, \dots, M$, fulfill the Bethe equations ("on-shell")

$$\left[\frac{\mathrm{sh}(\lambda_j+\eta/2)}{\mathrm{sh}(\lambda_j-\eta/2)}\right]^N = -\prod_{k=1}^M \frac{\mathrm{sh}(\lambda_j-\lambda_k+\eta)}{\mathrm{sh}(\lambda_j-\lambda_k-\eta)}, \qquad j = 1, \dots, M$$

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Norm formula

– Norm of an <u>on-shell</u> Bethe state (Gaudin matrix G):

$$\| \, |\{\lambda_j\}_{j=1}^M\rangle \, \| = \sqrt{\langle\{\lambda_j\}_{j=1}^M|\{\lambda_j\}_{j=1}^M\rangle},$$

$$\langle\{\lambda_j\}_{j=1}^M|\{\lambda_j\}_{j=1}^M\rangle = \mathrm{sh}^M(\eta) \prod_{\substack{j,k=1 \\ j\neq k}}^{M} \frac{\mathrm{sh}(\lambda_j - \lambda_k + \eta)}{\mathrm{sh}(\lambda_j - \lambda_k)} \, \mathrm{det}_M(G),$$

$$G_{jk} = \delta_{jk}\left(N K_{\eta/2}(\lambda_j) - \sum_{l=1}^{M} K_\eta(\lambda_j - \lambda_l)\right) + K_\eta(\lambda_j - \lambda_k),$$

where $K_\eta(\lambda) = \mathrm{sh}(2\eta)/[\mathrm{sh}(\lambda+\eta)\,\mathrm{sh}(\lambda-\eta)]$

[first suggested by Gaudin, McCoy, Wu (1981), then rigorously proven by Korepin (1982)]

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– Eigenstates of the magnetization $S^z = \sum_{n=1}^N \sigma_n^z/2$ with eigenvalue $N/2-M$

Sector of fixed magnetization $S^z = N/2-M$; Bethe states with fixed number M of spectral parameters; Here: $M = N/2$

– Bethe state parity invariant if the set of spectral parameters fulfills $\{\lambda_j\}_{j=1}^M = \{-\lambda_j\}_{j=1}^M$

Overlap of Néel with XXZ Bethe states

Overlap formula – Main result

- Overlap of the (zero-momentum) Néel state with XXZ on-shell Bethe states
  [MB, J. De Nardis, B. Wouters, J.-S. Caux, arXiv:1401.2877]

$$\frac{\langle \Psi_0 | \{\pm\lambda_j\}_{j=1}^{N/4}\rangle}{\| | \{\lambda_j\}_{j=1}^{N/4}\rangle \|} = \sqrt{2}\left[\prod_{j=1}^{N/4} \frac{\sqrt{\mathrm{th}(\lambda_j+\eta/2)\,\mathrm{th}(\lambda_j-\eta/2)}}{2\,\mathrm{sh}(2\lambda_j)}\right] \frac{\det_{N/4}(G^{(1)})}{\sqrt{\det_{N/2}(G^{(0)})}}$$

where $N/2$ even and

$$G_{jk}^{(\sigma)} = \delta_{jk}\left(N K_{\eta/2}(\lambda_j) - \sum_{l=1}^{N/4} K_\eta^{(\sigma)}(\lambda_j,\lambda_l)\right) + K_\eta^{(\sigma)}(\lambda_j,\lambda_k)\,, \quad j,k = 1,\ldots,N/4$$

$$K_\eta^{(\sigma)}(\lambda,\mu) = K_\eta(\lambda-\mu) + \sigma K_\eta(\lambda+\mu)\,, \qquad K_\eta(\lambda) = \frac{\mathrm{sh}(2\eta)}{\mathrm{sh}(\lambda+\eta)\,\mathrm{sh}(\lambda-\eta)}$$

Remarks:

- Bethe roots complex numbers (string solutions)
- Bethe states are parity invariant: $\{\lambda_j\}_{j=1}^{N/2} = \{-\lambda_j\}_{j=1}^{N/2} \equiv \{\pm\lambda_j\}_{j=1}^{N/4}$ (overlaps with non-parity-invariant Bethe states vanish [MB, De Nardis, Wouters, Caux, arXiv:1403.7469])
- $N/2$ odd can be treated similarly
- In the Quench Action approach only thermodynamic leading behavior needed

Overlap formula – Sketch of the proof (Part I)

First step: Getting a determinant formula [Tsuchiya (1998), Pozsgay, Kozlowski (2012)]

Main ideas:

- Consider a 6-vertex model with reflecting ends (reflection equation needed)
- Define partition function that (after a simple transformation) turns into the overlap of a Bethe state with a certain boundary state (= product state of local two-site states)

Result ($\widetilde{\lambda}_j$ arbitrary(!) complex numbers, $s_{x,y} = \mathrm{sh}(x+y)$, $M = N/2$)**:**

$$\langle \Psi_0 | \{\widetilde{\lambda}_j\}_{j=1}^M \rangle = \sqrt{2} \left[\prod_{j=1}^M \frac{s_{\widetilde{\lambda}_j,+\eta/2}}{s_{2\widetilde{\lambda}_j,0}} \frac{s^M_{\widetilde{\lambda}_j,-\eta/2}}{s^M_{\widetilde{\lambda}_j,+\eta/2}} \right] \left[\prod_{j>k=1}^M \frac{s_{\widetilde{\lambda}_j+\widetilde{\lambda}_k,\eta}}{s_{\widetilde{\lambda}_j+\widetilde{\lambda}_k,0}} \right] \det{}_M(1+U)$$

$$U_{jk} = \frac{s_{2\widetilde{\lambda}_k,\eta}\, s_{2\widetilde{\lambda}_k,0}}{s_{\widetilde{\lambda}_j+\widetilde{\lambda}_k,0}\, s_{\widetilde{\lambda}_j-\widetilde{\lambda}_k,\eta}} \left[\prod_{\substack{l=1 \\ l\neq k}}^M \frac{s_{\widetilde{\lambda}_k+\widetilde{\lambda}_l,0}}{s_{\widetilde{\lambda}_k-\widetilde{\lambda}_l,0}} \right] \left[\prod_{l=1}^M \frac{s_{\widetilde{\lambda}_k-\widetilde{\lambda}_l,-\eta}}{s_{\widetilde{\lambda}_k+\widetilde{\lambda}_l,+\eta}} \right] \left(\frac{s_{\widetilde{\lambda}_k,+\eta/2}}{s_{\widetilde{\lambda}_k,-\eta/2}} \right)^N$$

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Remarks:

- expression inconvenient to perform the thermodynamic limit
- singularities in the prefactor + zeroes of the determinant for parity-invariant states
- But: expression valid for <u>off-shell</u> Bethe states
- Idea: perform the limit to parity-invariant states (not necessarily on-shell Bethe states)

Overlap formula – Sketch of the proof (Part II)

Reducing the determinant (off-shell formula):

- Set $\widetilde{\lambda}_j = \lambda_j + \varepsilon_j$ ($j = 1, \ldots, N/4$) and $\widetilde{\lambda}_j = -\lambda_{j-N/4} + \varepsilon_{j-N/4}$ ($j = N/4+1, \ldots, N/2$)
  λ_j ($j = 1, \ldots, N/4$) still arbitrary(!)
- Main ingredients of the proof:
  - $\varepsilon_j \to 0$, $j = 1, \ldots, N/4$
  - pseudo parity invariance of the set $\{\widetilde{\lambda}_j\}_{j=1}^{N/2} = \{\lambda_j + \varepsilon_j\}_{j=1}^{N/4} \cup \{-\lambda_j + \varepsilon_j\}_{j=1}^{N/4}$
- Bethe equations are less important (only at the very end)

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Simple determinant manipulations and expanding everything carefully in small ε_j:

$\det_{N/2}[1+U] =$

$$\det_{N/2}\begin{pmatrix} \begin{bmatrix} \varepsilon_1 D_1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \varepsilon_2 e_{12} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \varepsilon_3 e_{13} & 0 \\ 0 & 0 \end{bmatrix} & \cdots \\ \begin{bmatrix} \varepsilon_1 e_{21} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \varepsilon_2 D_2 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \varepsilon_3 e_{23} & 0 \\ 0 & 0 \end{bmatrix} & \\ \begin{bmatrix} \varepsilon_1 e_{31} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \varepsilon_2 e_{32} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \varepsilon_3 D_3 & 0 \\ 0 & 1 \end{bmatrix} & \\ \vdots & & & \ddots \end{pmatrix} = \left[\prod_{j=k}^{N/4} \varepsilon_k\right] \det_{N/4}\begin{bmatrix} D_1 & e_{12} & e_{13} & \cdots \\ e_{21} & D_2 & e_{23} & \\ e_{31} & e_{32} & D_3 & \\ \vdots & & & \ddots \end{bmatrix}$$

Overlap formula – Sketch of the proof (Part II)

$$D_k = N s_{0,\eta} K_{\eta/2}(\lambda_k) - \sum_{\substack{l=1\\ l\neq k}}^{N/4} s_{0,\eta} K_\eta^{(1)}(\lambda_k,\lambda_l) + \frac{s_{2\lambda_k,+\eta}}{s_{2\lambda_k,0}}\mathfrak{A}_k + \frac{s_{2\lambda_k,-\eta}}{s_{2\lambda_k,0}}\bar{\mathfrak{A}}_k$$

$$e_{jk} = K_\eta^{(1)}(\lambda_j,\lambda_k) + \mathfrak{A}_k\left(\frac{s_{2\lambda_j,+\eta}s_{0,\eta}}{s_{\lambda_j+\lambda_k,0}s_{\lambda_j-\lambda_k,+\eta}} - \frac{s_{2\lambda_j,-\eta}s_{0,\eta}}{s_{\lambda_j-\lambda_k,0}s_{\lambda_j+\lambda_k,-\eta}}\right)$$

$$+\mathfrak{A}_k\bar{\mathfrak{A}}_j\frac{s_{2\lambda_j,-\eta}s_{0,\eta}}{s_{\lambda_j-\lambda_k,0}s_{\lambda_j+\lambda_k,-\eta}} - \bar{\mathfrak{A}}_j\left(\frac{s_{2\lambda_j,-\eta}s_{0,\eta}}{s_{\lambda_j-\lambda_k,0}s_{\lambda_j+\lambda_k,-\eta}} + \frac{s_{2\lambda_j,-\eta}s_{0,\eta}}{s_{\lambda_j+\lambda_k,0}s_{\lambda_j-\lambda_k,-\eta}}\right)$$

$$\mathfrak{A}_k = 1+\mathfrak{a}_k\,,\quad \bar{\mathfrak{A}}_k = 1+\mathfrak{a}_k^{-1}\,,\quad \mathfrak{a}_k = \mathfrak{a}(\lambda_k) = \left[\prod_{\substack{l=1\\ \sigma=\pm}}^{N/4}\frac{s_{\lambda_k-\sigma\lambda_l,-\eta}}{s_{\lambda_k-\sigma\lambda_l,+\eta}}\right]\left(\frac{s_{\lambda_k,+\eta/2}}{s_{\lambda_k,-\eta/2}}\right)^N$$

After further determinant manipulations,... Off-shell overlap formula:

$$\langle\Psi_0|\{\pm\lambda_j\}_{j=1}^{N/4}\rangle = \langle\Psi_0|\{\lambda_j+\varepsilon_j\}_{j=1}^{N/4}\cup\{-\lambda_j+\varepsilon_j\}_{j=1}^{N/4}\rangle\Big|_{\{\varepsilon_j\to 0\}_{j=1}^{N/4}} = \gamma\det{}_{N/4}(G^{(1)})$$

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$$e_{jk} = K_\eta^{(1)}(\lambda_j,\lambda_k) + \mathfrak{A}_k \left(\frac{s_{2\lambda_j,+\eta} s_{0,\eta}}{s_{\lambda_j+\lambda_k,0} s_{\lambda_j-\lambda_k,+\eta}} - \frac{s_{2\lambda_j,-\eta} s_{0,\eta}}{s_{\lambda_j-\lambda_k,0} s_{\lambda_j+\lambda_k,-\eta}} \right)$$
$$+ \mathfrak{A}_k \bar{\mathfrak{A}}_j \frac{s_{2\lambda_j,-\eta} s_{0,\eta}}{s_{\lambda_j-\lambda_k,0} s_{\lambda_j+\lambda_k,-\eta}} - \bar{\mathfrak{A}}_j \left(\frac{s_{2\lambda_j,-\eta} s_{0,\eta}}{s_{\lambda_j-\lambda_k,0} s_{\lambda_j+\lambda_k,-\eta}} + \frac{s_{2\lambda_j,-\eta} s_{0,\eta}}{s_{\lambda_j+\lambda_k,0} s_{\lambda_j-\lambda_k,-\eta}} \right)$$

$$\mathfrak{A}_k = 1 + \mathfrak{a}_k\,, \quad \bar{\mathfrak{A}}_k = 1 + \mathfrak{a}_k^{-1}\,, \quad \mathfrak{a}_k = \mathfrak{a}(\lambda_k) = \left[\prod_{\substack{l=1\\ \sigma=\pm}}^{N/4} \frac{s_{\lambda_k-\sigma\lambda_l,-\eta}}{s_{\lambda_k-\sigma\lambda_l,+\eta}} \right] \left(\frac{s_{\lambda_k,+\eta/2}}{s_{\lambda_k,-\eta/2}} \right)^N$$

After further determinant manipulations,... Off-shell overlap formula:

$$\langle \Psi_0 | \{\pm\lambda_j\}_{j=1}^{N/4} \rangle = \langle \Psi_0 | \{\lambda_j + \varepsilon_j\}_{j=1}^{N/4} \cup \{-\lambda_j + \varepsilon_j\}_{j=1}^{N/4} \rangle \Big|_{\{\varepsilon_j \to 0\}_{j=1}^{N/4}} = \gamma \det{}_{N/4}(G^{(1)})$$

On-shell overlap formula

After inserting Bethe equations ($\mathfrak{A}_k = 0$), etc... dividing by the norm, finally...

$$\frac{\langle \Psi_0 | \{\pm\lambda_j\}_{j=1}^{N/4}\rangle}{\| | \{\pm\lambda_j\}_{j=1}^{N/4}\rangle \|} = \sqrt{2}\left[\prod_{j=1}^{N/4} \frac{\sqrt{\mathrm{th}(\lambda_j+\eta/2)\,\mathrm{th}(\lambda_j-\eta/2)}}{2\,\mathrm{sh}(2\lambda_j)}\right] \frac{\det_{N/4}(G^{(1)})}{\sqrt{\det_{N/2}(G^{(0)})}}$$

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Thermodynamic limit:

- Ratio of determinants $\left(\frac{\det_{N/4}(G^{(1)})}{\sqrt{\det_{N/2}(G^{(0)})}} = \sqrt{\frac{\det_{N/4}(G^{(+1)})}{\det_{N/4}(G^{(-1)})}} \right)$ subleading
- Leading part in the TD limit

$$2S_\lambda = -2\ln(\langle \Psi_0 | \lambda \rangle) \sim \sum_{j=1}^{N/4} \ln \left[\frac{4\,\mathrm{sh}^2(2\lambda_j)}{\mathrm{th}(\lambda_j + \eta/2)\,\mathrm{th}(\lambda_j - \eta/2)} \right] \to N \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} g_n(\lambda) \rho_n(\lambda) d\lambda$$

directly translates into the “driving terms” of the GTBA equations

- String hypothesis: $|\lambda\rangle \to |\{\rho_n\}_{n=1}^{\infty}\rangle, \quad \frac{1}{N} \sum_{j=1}^{N/4} (\ldots) \to \sum_{n=1}^{\infty} \int_0^{\pi/2} (\ldots) \rho_n(\lambda) d\lambda$

Bethe and GTBA equations

- Bethe equations in the TD limit [Takahashi (1999)]:

$$\rho_n(\lambda)\,[1+\eta_n(\lambda)] = s*[\eta_{n-1}\rho_{n-1}+\eta_{n+1}\rho_{n+1}]\,(\lambda)\,, \qquad n \geq 1$$

$\eta_n = \rho_{n,h}/\rho_n$, $n \geq 1$, $\eta_0(\lambda) = 1$ and $\rho_0(\lambda) = \delta(\lambda)$; $(f*g)(\lambda) = \int_{-\pi/2}^{\pi/2} f(\lambda-\mu)g(\mu)d\mu$

- Kernel: $s(\lambda) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \frac{e^{-2ik\lambda}}{\mathrm{ch}(k\eta)}$
- Saddle point state via

$$0 = \left.\frac{\delta S_{QA}[\{\rho_n\}]}{\delta\rho_n}\right|_{\{\rho_n\}=\{\rho_n\}^{\mathrm{sp}}} \quad \text{with} \quad S_{QA}[\{\rho_n\}] = 2S[\{\rho_n\}] - \frac{1}{2}S_{YY}[\{\rho_n\}]$$

- Yang-Yang entropy

$$\frac{S_{YY}\,[\{\rho_n\}]}{N} = \sum_{n=1}^{\infty}\int_{-\pi/2}^{\pi/2}\left(\rho_n(\lambda)\ln[1+\eta_n(\lambda)] + \rho_{n,h}(\lambda)\ln[1+\eta_n^{-1}(\lambda)]\right)d\lambda$$

- This + TDL of overlap coefficient $\Rightarrow$ Partially decoupled form of the GTBA equations:

$$\ln[\eta_n(\lambda)] = (-1)^n\log\left[\frac{\vartheta_4^2(\lambda)}{\vartheta_1^2(\lambda)}\right] + \log\left[\frac{\vartheta_2^2(\lambda)}{\vartheta_3^2(\lambda)}\right] + s*\left[\ln(1+\eta_{n-1})+\ln(1+\eta_{n+1})\right](\lambda)\,,$$

Bethe equations in the TDL and GTBA equations

Bethe equations:

$$\rho_n(\lambda)\,[1+\eta_n(\lambda)] = s * [\eta_{n-1}\rho_{n-1} + \eta_{n+1}\rho_{n+1}]\,(\lambda)\,, \qquad n \geq 1$$

GTBA equations:

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$\Rightarrow$ Solving this gives the steady state described by $\{\rho_n\}_{n=1}^{\infty}$

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Limit to XXX ($\Delta = 1$):

$$\ln[\eta_n(\lambda)] = (-1)^{n+1} \ln\left[\mathrm{th}^2\left(\frac{\pi\lambda}{2}\right)\right] + s * \big[\ln(1+\eta_{n-1}) + \ln(1+\eta_{n+1})\big](\lambda),$$

Analytical solution of the GTBA equations

Y-system

Program:

- Mapping GTBA Eqs to well-known functional equations: Y- and T-system
  $\Rightarrow$ Explicit expressions for all η_n
- Combining with an explicit expression for $\rho_{1,h}$ (independent of any quench)
  $\Rightarrow$ Bethe Eqs can be solved analytically $\Rightarrow$ Explicit expressions for all $\rho_{n\geq 1}$

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Y-system: [Takahashi; Klümper, Pearce (1992); Suzuki (1999)]

$$y_n(x+\eta/2)y_n(x-\eta/2) = [1+y_{n-1}(x)][1+y_{n+1}(x)], \quad n \geq 1, \qquad y_0(x) = 0$$

Fixing the analyticity properties of the y-functions in the physical strip (π-periodicity in imaginary direction)

$$PS = \{x \in \mathbb{C} | \ -\eta/2 < \mathrm{Re}(x) < \eta/2, \ -\pi/2 < \mathrm{Im}(x) \leq \pi/2\}$$

$\Rightarrow$ Y-system is equivalent to non-linear integral equations (NLIEs)

$$\ln[y_n(x)] = d_n(x) + s * [\ln(Y_{n-1}) + \ln(Y_{n+1})](x), \quad n \geq 1$$

- Kernel function s as before
- Driving terms d_n determined by the analytical behavior of y_n inside PS

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$$\eta_n(\lambda) \sim \coth^2(\lambda)\,, \quad \text{for small } \lambda \text{ and } n \text{ even}\,,$$
$$\text{and there are no further roots or poles for all } \lambda \in PS \backslash \{0\}$$

- Fourier transforms of the logarithmic derivatives:

$$FT[\ln'(\mathrm{sh}^2(2\lambda))](k) = -4\pi i\,\mathrm{sh}(\eta k)(1+(-1)^k)\,,$$
$$FT[\ln'(\coth^2(\lambda))](k) = \quad 4\pi i\,\mathrm{sh}(\eta k)(1-(-1)^k)$$

- Dividing by $\mathrm{ch}(\eta k)$, taking inverse Fourier transform, integrating over x yields the driving terms of the GTBA Eqs

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T-system and explicit expressions for η_n

– Rewriting the y's in terms of T's:

$$y_n(x) = T_{n-1}(x)T_{n+1}(x)/f_n(x)\,, \quad n \geq 1$$

– Y-System $\Leftrightarrow$ T-System [Klümper, Pearce (1992); Suzuki (1999)]

$$T_n(x-\eta/2)T_n(x+\eta/2) = T_{n-1}(x)T_{n+1}(x) + f_n(x)\,, \quad n \geq 1\,, \qquad T_0(x) = 1$$

– Writing $T_1(x) = T_1^{(1)}(x) + T_1^{(2)}(x)$ and defining $\mathfrak{a}(x) = T_1^{(1)}(x)/T_2^{(1)}(x)$

$\Rightarrow y_1$ is completely determined by auxiliary function $\mathfrak{a}$:

$$y_1(x) = \mathfrak{a}(x+\eta/2) + \mathfrak{a}^{-1}(x-\eta/2) + \mathfrak{a}(x+\eta/2)\mathfrak{a}^{-1}(x-\eta/2)$$

– $y_0(x) = 0$ and $y_1(x) = \ldots$, plus Y-system (recursion relation) $\Rightarrow$ $\boxed{\text{all } y_n\text{'s via } \mathfrak{a}}$

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$$\mathfrak{a}(\lambda) = \frac{\mathrm{sh}(\lambda+\eta)}{\mathrm{sh}(\lambda-\eta)}\,\frac{\mathrm{sh}(2\lambda-\eta)}{\mathrm{sh}(2\lambda+\eta)}$$

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First function:

$$\eta_1(\lambda) = \frac{\mathrm{sh}^2(2\lambda)\,[\mathrm{ch}(\eta) + 2\,\mathrm{ch}(3\eta) - 3\,\mathrm{ch}(2\lambda)]}{2\,\mathrm{sh}(\lambda-\eta/2)\,\mathrm{sh}(\lambda+\eta/2)\,\mathrm{sh}(2\lambda+2\eta)\,\mathrm{sh}(2\lambda-2\eta)}$$

Analytical solution for $\rho_{1,h}$

Expectation values of the conserved charges on the Néel state [Essler, Fagotti (2013)]

$$\lim_{\mathrm{th}} \frac{\langle\Psi_0|\, Q_{m+1}\,|\Psi_0\rangle}{N} = -\frac{\Delta}{2}\,\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\frac{1-\Delta^2}{\mathrm{ch}\left(\sqrt{1-\Delta^2}x\right)-\Delta^2}\right)\Bigg|_{x=0}$$

... and on a Bethe state: (e.g. the steady state)

$$\lim_{\mathrm{th}}\langle\lambda|\frac{Q_{m+1}}{N}|\lambda\rangle = \sum_{n=1}^{\infty}\int_{-\pi/2}^{\pi/2}\rho_n(\lambda)\frac{\partial^m}{\partial\lambda^m}\ln\left[\frac{\mathrm{sh}(\lambda+n\eta/2)}{\mathrm{sh}(\lambda-n\eta/2)}\right]d\lambda\,,\quad m\geq 0$$

To see this, note that an n-string with string center λ_α^n contributes a factor $\frac{\mathrm{sh}[\lambda-\lambda_\alpha^n-\frac{\eta}{2}(n+1)]}{\mathrm{sh}[\lambda-\lambda_\alpha^n+\frac{\eta}{2}(n-1)]}$

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$$\lim_{\text{th}} \langle \lambda | \frac{Q_{m+1}}{N} | \lambda \rangle = \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} \rho_n(\lambda) \frac{\partial^m}{\partial \lambda^m} \ln \left[\frac{\mathrm{sh}(\lambda + n\eta/2)}{\mathrm{sh}(\lambda - n\eta/2)} \right] d\lambda, \quad m \geq 0$$

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Combining with the Bethe equations, eventually leads to

$$\rho_{1,h}^{\mathrm{sp}}(\lambda) = a_1(\lambda) \left(1 - \frac{\mathrm{ch}^2(\eta)}{a_1^2(\lambda)\, \mathrm{sh}^2(2\lambda) + \mathrm{ch}^2(\eta)} \right), \quad a_1(\lambda) = \frac{\mathrm{sh}(\eta)}{\mathrm{ch}(\eta) - \mathrm{ch}(2\lambda)}$$

Alternative: Use “generating function” for the Néel state [Essler, Fagotti (2013)]

$$\Omega_{\Psi_0}(\lambda) = -\frac{\mathrm{sh}(2\eta)}{\mathrm{ch}(2\eta) + 1 - 2\,\mathrm{ch}(2\lambda)}, \quad \rho_{1,h}^{\Psi_0}(\lambda) = a_1(\lambda) + \frac{1}{2\pi} \left[\Omega_{\Psi_0}\left(\lambda + \tfrac{\eta}{2}\right) + \Omega_{\Psi_0}\left(\lambda - \tfrac{\eta}{2}\right) \right]$$

Explicit expressions for ρ_n

– Bethe equations (as functional equations):

$$\rho_{n+1,h}(\lambda) = \rho_{n,t}(\lambda+\eta/2) + \rho_{n,t}(\lambda-\eta/2) - \rho_{n-1,h}(\lambda)\,, \quad n \geq 1\,, \qquad \rho_{0,h}(\lambda) \equiv 0$$

where $\rho_{n,t}(\lambda) = \rho_{n,h}(\lambda)\left(1+\eta_n^{-1}(\lambda)\right)$

– $\rho_n(\lambda) = \rho_{n,h}(\lambda)/\eta_n(\lambda)$ for $n \geq 1 \Rightarrow$ $\boxed{\text{all } \rho_n \text{ explicitly}}$

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For example:

$$\rho_1(\lambda) = \frac{\mathrm{sh}^3(\eta)\,\mathrm{sh}(2\lambda + 2\eta)\,\mathrm{sh}(2\lambda + 2\eta)}{\pi f(\lambda - \frac{\eta}{2}) f(\lambda + \frac{\eta}{2}) g(\lambda)}$$

$$\rho_2(\lambda) = \frac{8\,\mathrm{sh}^2(\lambda)\,\mathrm{sh}^3(\eta)\,\mathrm{ch}(\eta)[3\,\mathrm{sh}^2(\lambda) + \mathrm{sh}^2(\eta)][\mathrm{ch}(6\eta) - \mathrm{ch}(4\lambda)]}{\pi f(\lambda) g(\lambda + \frac{\eta}{2}) g(\lambda - \frac{\eta}{2}) h(\lambda)}$$

$$\vdots$$

where $f(\lambda) = \mathrm{ch}^2(\eta) - \mathrm{ch}(2\lambda)$, $g(\lambda) = \mathrm{ch}(\eta) + 2\,\mathrm{ch}(3\eta) - 3\,\mathrm{ch}(2\lambda)$, and

$$h(\lambda) = 2\,\mathrm{ch}(4\lambda) + 2\,\mathrm{ch}^2(2\eta)[2 + \mathrm{ch}(2\eta)] - \mathrm{ch}(2\lambda)[3 + 2\,\mathrm{ch}(2\eta) + 3\,\mathrm{ch}(4\eta)]\,.$$

Remarks about the interpretation of the auxiliary function $\mathfrak{a}$

- Function $\mathfrak{a}$ can be interpreted as auxiliary function corresponding to a (spin-1/2) quantum transfer matrix
- Using standard contour $\mathcal{C}$, which encircles the only pole of $1/(1+\mathfrak{a}(\omega))$ at $\omega = i\pi/2$, one can compute G by explicitly performing the contour integral.
- Nontrivial relation between G, $\mathfrak{a}$ and generating function Ω_{Ψ_0} [Essler, Fagotti (2013)] fulfilled
- Unfortunately, this explicit G function does not give the correct values of short-range correlation functions (due to the presence of higher nontrivial driving terms, $d_{n\geq 2} \neq 0$, in the GTBA equations)

Summary and outlook

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- Quench action approach $\Rightarrow$ GTBA equations (for the steady state)
- Analytical solution of the GTBA equations
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Outlook

- Correlation functions using the analytical approach for solving the GTBA equations
- Applications to the Loschmidt echo [Pozsgay, arXiv:1308.3087]
- Overlaps and QAA also for different initial states (e.g. dimer, q-dimer,...)
- Complete understanding of the structure of GTBA equations ($\leftrightarrow$ explicit solutions for different initial states)
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Thank you for your attention!