Gaudin-like determinants for overlaps in integrable systems – Quench Action approach for the Néel-to-XXZ quench

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Recent Advances in Quantum Integrable Systems

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• Motivation – the quench protocol

• The spin-1/2 XXZ Heisenberg chain (fixing the notation)

• Overlap of the zero-momentum Néel state with XXZ Bethe states – Gaudin-like determinant formula

• Quench action approach: generalized TBA equations

• Analytical solution

• Conclusion and outlook
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Quench action approach: generalized TBA equations

Analytical solution

Conclusion and outlook

In collaboration with... Jean-Sébastien Caux Davide Fioretto Jacopo De Nardis Rogier Vlijm Bram Wouters
The quench protocol: $|\Psi_0\rangle \longrightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$

Here: Quench to the spin-1/2 XXZ chain starting from the ground state of the Ising model

Initial state:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\uparrow\downarrow\ldots\rangle + |\downarrow\uparrow\downarrow\uparrow\ldots\rangle)$$

Hamiltonian: (lattice size $N$, $\sigma_j^\alpha = \text{Pauli matrices at lattice site } j$)

$$H = \sum_{j=1}^N \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta (\sigma_j^z \sigma_{j+1}^z - 1) \right)$$

PBC's: $\sigma_{N+1}^\alpha = \sigma_1^\alpha$, $\alpha = x, y, z$; anisotropy parameter $\Delta = \text{ch}(\eta) \geq 1$
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Objects of interest: Time evolution of observables [in particular for large $t$, in the limit $N \to \infty$]

$$\langle \psi(t) | \mathcal{O} | \psi(t) \rangle = \langle \psi_0 | e^{iHt} \mathcal{O} e^{-iHt} | \psi_0 \rangle = \sum_{\lambda, \lambda'} \langle \psi_0 | \lambda \rangle \langle \lambda' | \psi_0 \rangle e^{i(\omega_\lambda - \omega_{\lambda'})t} \langle \lambda | \mathcal{O} | \lambda' \rangle$$

$\rightarrow$ Three ingredients: 1) Matrix elements $\langle \lambda | \mathcal{O} | \lambda' \rangle$, 2) Energies $\omega_\lambda$, 3) Overlaps $\langle \psi_0 | \lambda \rangle$
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Problem: double sum over the Hilbert space $\sum_{\lambda,\lambda'}$
Quench action approach

**Problem:** double sum over the Hilbert space (overlap coefficients $S_\lambda = - \ln \langle \lambda | \Psi_0 \rangle$):

$$\langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle = \sum_{\lambda, \lambda'} e^{-S_\lambda - S_{\lambda'}} e^{i(\omega_\lambda - \omega_{\lambda'})t} \langle \lambda | \mathcal{O} | \lambda' \rangle$$

**Solution: Quench Action approach →** talk by J.-S. Caux (this morning)

- Restriction to a certain class of operators (so-called “weak operators” in the thermodynamic limit)
- Applying a saddle-point approximation by minimizing the “quench action”
- Result: Expectation values (not only) for long times after the quench. Here: $t \to \infty$
  But first(!) TD limit $N \to \infty$ with magnetization fixed to zero, denoted by $\lim_{\text{th}}$:

$$\lim_{t \to \infty} \lim_{\text{th}} \langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle = \langle \rho^{\text{sp}} | \mathcal{O} | \rho^{\text{sp}} \rangle.$$  

- Generalized TBA equations for the saddle point state:

$$0 = \left. \frac{\delta S_{QA}[\rho]}{\delta \rho_n} \right|_{\rho=\rho^{\text{sp}}} \quad \text{with} \quad S_{QA}[\rho] = 2S[\rho] - S_{YY}[\rho]$$
Motivation

Quench action approach

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ABA for the XXZ spin chain
Algebraic Bethe ansatz for the spin-1/2 XXZ chain

- Yang-Baxter algebra (2 × 2 monodromy matrix $T(\lambda)$; $\lambda$, $\mu$ spectral parameter):

$$ \check{R}(\lambda - \mu) [T(\lambda) \otimes T(\mu)] = [T(\mu) \otimes T(\lambda)] \check{R}(\lambda - \mu) $$

with R-matrix of the 6-vertex model

$$ \check{R}(\lambda) = \frac{1}{\text{sh}(\lambda + \eta)} \begin{pmatrix} \text{sh}(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \text{sh}(\eta) & \text{sh}(\lambda) & 0 \\ 0 & \text{sh}(\lambda) & \text{sh}(\eta) & 0 \\ 0 & 0 & 0 & \text{sh}(\lambda + \eta) \end{pmatrix} $$

- Monodromy matrix (product in auxiliary space of $N$ Lax operators):

$$ T(\lambda) = \prod_{n=1}^{N} L_n(\lambda) = L_1(\lambda) \ldots L_N(\lambda) =: \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} $$

with Lax operators on lattice sites $n = 1, \ldots, N$ (2 × 2 matrix in auxiliary space)

$$ L_n(\lambda) = \frac{1}{\text{sh}(\lambda + \eta/2)} \begin{pmatrix} \text{sh} \left( \lambda + \frac{\eta}{2} \sigma_n^z \right) & \text{sh}(\eta)\sigma_n^- \\ \text{sh}(\eta)\sigma_n^+ & \text{sh} \left( \lambda - \frac{\eta}{2} \sigma_n^z \right) \end{pmatrix} $$

with Pauli matrices $\sigma_n^z, \sigma_n^\pm = \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$ acting on lattice site $n$
Algebraic Bethe ansatz for the spin-1/2 XXZ chain

– Transfer matrices $t(\lambda) = \text{tr}_a[T(\lambda)] = A(\lambda) + D(\lambda)$ build a commutative family: $[t(\lambda), t(\mu)] = 0$

– Conserved charges of the XXZ spin chain:

$$Q_{m+1} = \left. \frac{\partial^m}{\partial \lambda^m} \ln[t(\lambda)] \right|_{\lambda = \eta/2}$$

where $H = 2\text{sh}(\eta)Q_2$

– Bethe states $|\{\lambda_j\}_{j=1}^M\rangle = \prod_{j=1}^M B(\lambda_j) |\uparrow\rangle \otimes^N$ ($\lambda_j$ arbitrary = “off-shell”)

Eigenstates of the transfer matrix with eigenvalue

$$\tau(\lambda) = \prod_{k=1}^M \frac{\text{sh}(\lambda - \lambda_k - \eta)}{\text{sh}(\lambda - \lambda_k)} + \left[ \frac{\text{sh}(\lambda - \eta/2)}{\text{sh}(\lambda + \eta/2)} \right]^N \prod_{k=1}^M \frac{\text{sh}(\lambda - \lambda_k + \eta)}{\text{sh}(\lambda - \lambda_k)}$$

if the parameters $\lambda_j, j = 1, \ldots, M$, fulfill the Bethe equations (“on-shell”)

$$\left[ \frac{\text{sh}(\lambda_j + \eta/2)}{\text{sh}(\lambda_j - \eta/2)} \right]^N = -\prod_{k=1}^M \frac{\text{sh}(\lambda_j - \lambda_k + \eta)}{\text{sh}(\lambda_j - \lambda_k - \eta)}, \quad j = 1, \ldots, M$$
The XXZ model

Algebraic Bethe ansatz for the spin-1/2 XXZ chain

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Norm formula

- Norm of an on-shell Bethe state (Gaudin matrix $G$):

$$\| |\{\lambda_j\}_{j=1}^M \| = \sqrt{\langle \{\lambda_j\}_{j=1}^M | \{\lambda_j\}_{j=1}^M \rangle},$$

$$\langle \{\lambda_j\}_{j=1}^M | \{\lambda_j\}_{j=1}^M \rangle = \text{sh}^M(\eta) \prod_{\substack{j, k=1 \\text{to} \ M \\text{\&} \ j \neq k}} \frac{\text{sh}(\lambda_j - \lambda_k + \eta)}{\text{sh}(\lambda_j - \lambda_k)} \det_M(G),$$

$$G_{jk} = \delta_{jk} \left( NK_{\eta/2}(\lambda_j) - \sum_{l=1}^M K_{\eta}(\lambda_j - \lambda_l) \right) + K_{\eta}(\lambda_j - \lambda_k),$$

where $K_{\eta}(\lambda) = \text{sh}(2\eta)/[\text{sh}(\lambda + \eta) \text{sh}(\lambda - \eta)]$

[first suggested by Gaudin, McCoy, Wu (1981), then rigorously proven by Korepin (1982)]
The XXZ model

**Norm formula**

- Norm of an on-shell Bethe state (Gaudin matrix $G$):

$$\parallel \{\lambda_j\}_{j=1}^M \parallel = \sqrt{\langle \{\lambda_j\}_{j=1}^M | \{\lambda_j\}_{j=1}^M \rangle},$$

$$\langle \{\lambda_j\}_{j=1}^M | \{\lambda_j\}_{j=1}^M \rangle = \text{sh}^M(\eta) \prod_{j,k=1}^M \frac{\text{sh}(\lambda_j - \lambda_k + \eta)}{\text{sh}(\lambda_j - \lambda_k)} \det M(G),$$

$$G_{jk} = \delta_{jk} \left( NK_\eta/2(\lambda_j) - \sum_{l=1}^M K_\eta(\lambda_j - \lambda_l) \right) + K_\eta(\lambda_j - \lambda_k),$$

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- Eigenstates of the magnetization $S^z = \sum_{n=1}^N \sigma_n^z/2$ with eigenvalue $N/2 - M$

  Sector of fixed magnetization $S^z = N/2 - M$; Bethe states with fixed number $M$ of spectral parameters; Here: $M = N/2$

- Bethe state parity invariant if the set of spectral parameters fulfills $\{\lambda_j\}_{j=1}^M = \{-\lambda_j\}_{j=1}^M$
Overlap of Néel with XXZ Bethe states
Overlap formula – Main result

Overlap of the (zero-momentum) Néel state with XXZ on-shell Bethe states

\[ \langle \Psi_0 | \{ \pm \lambda_j \}_{j=1}^{N/4} \rangle = \sqrt{2} \prod_{j=1}^{N/4} \frac{\sqrt{\text{th}(\lambda_j + \eta/2) \text{th}(\lambda_j - \eta/2)}}{2 \text{sh}(2\lambda_j)} \det_{N/4}(G^{(1)}) \]

where \( N/2 \) even and

\[ G^{(\sigma)}_{jk} = \delta_{jk} \left( NK_{\eta/2}(\lambda_j) - \sum_{l=1}^{N/4} K^{(\sigma)}_{\eta}(\lambda_j, \lambda_l) \right) + K^{(\sigma)}_{\eta}(\lambda_j, \lambda_k), \quad j, k = 1, \ldots, N/4 \]

\[ K^{(\sigma)}_{\eta}(\lambda, \mu) = K_{\eta}(\lambda - \mu) + \sigma K_{\eta}(\lambda + \mu), \quad K_{\eta}(\lambda) = \frac{\text{sh}(2\eta)}{\text{sh}(\lambda + \eta) \text{sh}(\lambda - \eta)} \]

Remarks:

- Bethe roots complex numbers (string solutions)
- Bethe states are parity invariant: \( \{ \lambda_j \}_{j=1}^{N/2} = \{-\lambda_j\}_{j=1}^{N/2} = \{ \pm \lambda_j \}_{j=1}^{N/4} \) (overlaps with non-parity-invariant Bethe states vanish)
- \( N/2 \) odd can be treated similarly
- In the Quench Action approach only thermodynamic leading behavior needed
Overlap formula – Sketch of the proof (Part I)

First step: Getting a determinant formula [Tsuchiya (1998), Pozsgay, Kozlowski (2012)]

Main ideas:
- Consider a 6-vertex model with reflecting ends (reflection equation needed)
- Define partition function that (after a simple transformation) turns into the overlap of a Bethe state with a certain boundary state (= product state of local two-site states)

Result ($\tilde{\lambda}_j$ arbitrary(!) complex numbers, $s_{x,y} = \text{sh}(x + y)$, $M = N/2$):

$$
\langle \psi_0 | \{ \tilde{\lambda}_j \}_{j=1}^M \rangle = \sqrt{2} \left[ \prod_{j=1}^M \frac{s_{\tilde{\lambda}_j, + \eta/2}}{s_{2\tilde{\lambda}_j, 0}} \frac{s_{\tilde{\lambda}_j, - \eta/2}}{s_{M_{\tilde{\lambda}_j}, + \eta/2}} \right] \left[ \prod_{j>k=1}^M \frac{s_{\tilde{\lambda}_j + \tilde{\lambda}_k, \eta}}{s_{\tilde{\lambda}_j + \tilde{\lambda}_k, 0}} \right] \det_M (1 + U) 
$$

$$
U_{jk} = \frac{s_{2\tilde{\lambda}_k, \eta}}{s_{\tilde{\lambda}_j + \tilde{\lambda}_k, 0}} \frac{s_{2\tilde{\lambda}_k, 0}}{s_{\tilde{\lambda}_j - \tilde{\lambda}_k, \eta}} \left[ \prod_{l=1}^M \frac{s_{\tilde{\lambda}_k + \tilde{\lambda}_l, 0}}{s_{\tilde{\lambda}_k - \tilde{\lambda}_l, 0}} \right] \left[ \prod_{l=1}^M \frac{s_{\tilde{\lambda}_k - \tilde{\lambda}_l, - \eta}}{s_{\tilde{\lambda}_k + \tilde{\lambda}_l, + \eta}} \right] \left( \frac{s_{\tilde{\lambda}_k, + \eta/2}}{s_{\tilde{\lambda}_k, - \eta/2}} \right)^N
$$
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Result ($\tilde{\lambda}_j$ arbitrary(!) complex numbers, $s_{x,y} = \text{sh}(x + y)$, $M = N/2$):

$$\langle \Psi_0 | \{\tilde{\lambda}_j\}_{j=1}^M \rangle = \sqrt{2} \left[ \prod_{j=1}^M \frac{\tilde{s}_{\lambda_j, + \eta/2}}{s_{2\lambda_j, 0}} \frac{s_{2\lambda_j, 0}}{s_{\lambda_j, + \eta/2}} \right] \left[ \prod_{j>k=1}^M \frac{\tilde{s}_{\lambda_j + \lambda_k, \eta}}{s_{\lambda_j + \lambda_k, 0}} \right] \det_{M}(1 + U)$$

$$U_{jk} = \frac{s_{2\lambda_k, \eta} s_{2\lambda_k, 0}}{s_{\lambda_j + \lambda_k, 0} s_{\lambda_j - \lambda_k, \eta}} \left[ \prod_{l=1}^M \frac{s_{\lambda_k + \lambda_l, 0}}{s_{\lambda_k - \lambda_l, 0}} \right] \left[ \prod_{l \neq k}^M \frac{s_{\lambda_k - \lambda_l, -\eta}}{s_{\lambda_k + \lambda_l, +\eta}} \right] \left( \frac{s_{\lambda_k, + \eta/2}}{s_{\lambda_k, - \eta/2}} \right)^N$$

Remarks:
- expression inconvenient to perform the thermodynamic limit
- singularities in the prefactor + zeroes of the determinant for parity-invariant states
- But: expression valid for off-shell Bethe states
- Idea: perform the limit to parity-invariant states (not necessarily on-shell Bethe states)
Overlap of Néel with XXZ Bethe states

Overlap formula – Sketch of the proof (Part II)

Reducing the determinant (off-shell formula):

Set $\tilde{\lambda}_j = \lambda_j + \varepsilon_j$ ($j = 1, \ldots, N/4$) and $\tilde{\lambda}_j = -\lambda_{j-N/4} + \varepsilon_{j-N/4}$ ($j = N/4 + 1, \ldots, N/2$)

$\lambda_j$ ($j = 1, \ldots, N/4$) still arbitrary(!)

Main ingredients of the proof:

- $\varepsilon_j \to 0$, $j = 1, \ldots, N/4$
- pseudo parity invariance of the set $\{\tilde{\lambda}_j\}_{j=1}^{N/2} = \{\lambda_j + \varepsilon_j\}_{j=1}^{N/4} \cup \{-\lambda_j + \varepsilon_j\}_{j=1}^{N/4}$

- Bethe equations are less important (only at the very end)
Overlap formula – Sketch of the proof (Part II)

Reducing the determinant (off-shell formula):

- Set $\tilde{\lambda}_j = \lambda_j + \varepsilon_j \ (j = 1, \ldots, N/4)$ and $\tilde{\lambda}_j = -\lambda_{j-N/4} + \varepsilon_{j-N/4} \ (j = N/4 + 1, \ldots, N/2)$
- $\lambda_j \ (j = 1, \ldots, N/4)$ still arbitrary(!)

Main ingredients of the proof:

- $\varepsilon_j \to 0, j = 1, \ldots, N/4$
- pseudo parity invariance of the set $\{\tilde{\lambda}_j\}_{j=1}^{N/2} = \{\lambda_j + \varepsilon_j\}_{j=1}^{N/4} \cup \{-\lambda_j + \varepsilon_j\}_{j=1}^{N/4}$

- Bethe equations are less important (only at the very end)

Simple determinant manipulations and expanding everything carefully in small $\varepsilon_j$:

$$\det_{N/2}[1 + U] =$$

$$\det_{N/2} \left( \begin{array}{cccc} \varepsilon_1 D_1 & 0 & \varepsilon_2 e_{12} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \varepsilon_1 e_{21} & 0 \\
0 & 0 & 0 & \varepsilon_2 D_2 \\
\varepsilon_1 e_{31} & 0 & \varepsilon_2 e_{32} & 0 \\
0 & 0 & 0 & \varepsilon_3 e_{32} \\
\varepsilon_1 e_{13} & 0 & \varepsilon_2 e_{23} & 0 \\
0 & 0 & 0 & \varepsilon_3 D_3 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array} \right) = \prod_{j=k}^{N/4} \varepsilon_k \det_{N/4} \left( \begin{array}{cccc} D_1 & e_{12} & e_{13} & \cdots \\
e_{21} & D_2 & e_{23} & \cdots \\
e_{31} & e_{32} & D_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array} \right)$$
Overlap of Néel with XXZ Bethe states

Overlap formula – Sketch of the proof (Part II)

\[ D_k = N s_0, \eta K_{\eta/2}(\lambda_k) - \sum_{l=1}^{N/4} s_0, \eta K^{(1)}_{\eta}(\lambda_k, \lambda_l) + \frac{s_2 \lambda_k, \eta s_0, \eta}{s_2 \lambda_k, 0} \mathcal{A}_k + \frac{s_2 \lambda_k, -\eta s_0, \eta}{s_2 \lambda_k, 0} \bar{\mathcal{A}}_k \]

\[ e_{jk} = K^{(1)}_{\eta}(\lambda_j, \lambda_k) + \mathcal{A}_k \left( \frac{s_2 \lambda_j, \eta s_0, \eta}{s \lambda_j + \lambda_k, 0 s \lambda_j - \lambda_k, \eta} - \frac{s_2 \lambda_j, -\eta s_0, \eta}{s \lambda_j - \lambda_k, 0 s \lambda_j + \lambda_k, -\eta} \right) \]

\[ + \mathcal{A}_k \bar{\mathcal{A}}_j \frac{s_2 \lambda_j, -\eta s_0, \eta}{s \lambda_j - \lambda_k, 0 s \lambda_j + \lambda_k, -\eta} - \bar{\mathcal{A}}_j \left( \frac{s_2 \lambda_j, -\eta s_0, \eta}{s \lambda_j - \lambda_k, 0 s \lambda_j + \lambda_k, -\eta} + \frac{s_2 \lambda_j, -\eta s_0, \eta}{s \lambda_j + \lambda_k, 0 s \lambda_j - \lambda_k, -\eta} \right) \]

\[ \mathcal{A}_k = 1 + a_k, \quad \bar{\mathcal{A}}_k = 1 + a_k^{-1}, \quad a_k = a(\lambda_k) = \left[ \prod_{l=1}^{N/4} \frac{s \lambda_k - \sigma \lambda_l, -\eta}{s \lambda_k - \sigma \lambda_l, \eta} \right] \left( \frac{s \lambda_k, \eta/2}{s \lambda_k, -\eta/2} \right)^N \]

After further determinant manipulations,... Off-shell overlap formula:

\[ \langle \psi_0 | \{ \pm \lambda_j \}_{j=1}^{N/4} \rangle = \langle \psi_0 | \{ \lambda_j + \epsilon_j \}_{j=1}^{N/4} \cup \{ -\lambda_j + \epsilon_j \}_{j=1}^{N/4} \rangle \bigg|_{\{ \epsilon_j \to 0 \}_{j=1}^{N/4}} = \gamma \det_{N/4}(G^{(1)}) \]
Overlap of Néel with XXZ Bethe states

Overlap formula – Sketch of the proof (Part II)

\[
D_k = N s_0, \eta K_{\eta/2}(\lambda_k) - \sum_{l=1}^{N/4} s_0, \eta K_{\eta}^{(1)}(\lambda_k, \lambda_l) + \frac{s_{2\lambda_k, + \eta}}{s_{2\lambda_k, 0}} A_k + \frac{s_{2\lambda_k, - \eta}}{s_{2\lambda_k, 0}} \bar{A}_k
\]

\[
e_{jk} = K_{\eta}^{(1)}(\lambda_j, \lambda_k) + A_k \left( \frac{s_{2\lambda_j, + \eta} s_0, \eta}{s_{\lambda_j, 0} s_{\lambda_j, + \eta}} - \frac{s_{2\lambda_j, - \eta} s_0, \eta}{s_{\lambda_j, 0} s_{\lambda_j, - \eta}} \right) + A_k \bar{A}_j \left( \frac{s_{2\lambda_j, - \eta} s_0, \eta}{s_{\lambda_j, - \eta} s_{\lambda_j, - \eta}} - \bar{A}_j \left( \frac{s_{2\lambda_j, - \eta} s_0, \eta}{s_{\lambda_j, - \eta} s_{\lambda_j, - \eta}} + \frac{s_{2\lambda_j, - \eta} s_0, \eta}{s_{\lambda_j, + \eta} s_{\lambda_j, - \eta}} \right) \right)
\]

\[
A_k = 1 + a_k, \quad \bar{A}_k = 1 + a^{-1}_k, \quad a_k = a(\lambda_k) = \prod_{\sigma = \pm}^{N/4} \frac{s_{\lambda_k - \sigma \lambda_k, - \eta}}{s_{\lambda_k - \sigma \lambda_k, + \eta}} \left( \frac{s_{\lambda_k, + \eta/2}}{s_{\lambda_k, - \eta/2}} \right)^N
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After further determinant manipulations,... Off-shell overlap formula:

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\langle \psi_0 | \{ \pm \lambda_j \}_{j=1}^{N/4} \rangle = \langle \psi_0 | \{ \lambda_j + \varepsilon_j \}_{j=1}^{N/4} \cup \{-\lambda_j + \varepsilon_j\}_{j=1}^{N/4} \rangle \bigg|_{\{\varepsilon_j \to 0\}_{j=1}^{N/4}} = \gamma \text{det}_{N/4}(G^{(1)})
\]
On-shell overlap formula

After inserting Bethe equations ($\mathcal{A}_k = 0$), etc... dividing by the norm, finally...

$$\frac{\langle \Psi_0 | \{\pm \lambda_j \}_{j=1}^{N/4} \rangle}{\| \{\pm \lambda_j \}_{j=1}^{N/4} \|} = \sqrt{2} \left[ \prod_{j=1}^{N/4} \frac{\sqrt{\text{th}(\lambda_j + \eta/2) \text{th}(\lambda_j - \eta/2)}}{2 \text{sh}(2\lambda_j)} \right] \frac{\det_{N/4}(G^{(1)})}{\sqrt{\det_{N/2}(G^{(0)})}}$$
On-shell overlap formula

After inserting Bethe equations ($\mathcal{U}_k = 0$), etc... dividing by the norm, finally...

$$\frac{\langle \Psi_0 | \{ \pm \lambda_j \}_{j=1}^{N/4} \rangle}{\| \{ \pm \lambda_j \}_{j=1}^{N/4} \|} = \sqrt{2} \left[ \prod_{j=1}^{N/4} \frac{\sqrt{\text{th}(\lambda_j + \eta/2) \text{th}(\lambda_j - \eta/2)}}{2 \text{sh}(2\lambda_j)} \right] \frac{\det_{N/4}(G^{(1)})}{\sqrt{\det_{N/2}(G^{(0)})}}$$

Thermodynamic limit:

- Ratio of determinants $\left( \frac{\det_{N/4}(G^{(1)})}{\sqrt{\det_{N/2}(G^{(0)})}} = \sqrt{\frac{\det_{N/4}(G^{(+1)})}{\det_{N/4}(G^{(-1)})}} \right)$ subleading
- Leading part in the TD limit

$$2S_\lambda = -2 \ln(\langle \Psi_0 | \lambda \rangle) \sim \sum_{j=1}^{N/4} \ln \left[ \frac{4 \text{sh}^2(2\lambda_j)}{\text{th}(\lambda_j + \eta/2) \text{th}(\lambda_j - \eta/2)} \right] \to N \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} g_n(\lambda) \rho_n(\lambda) d\lambda$$

directly translates into the “driving terms” of the GTBA equations

- String hypothesis: $|\lambda\rangle \to |\{ \rho_n \}_{n=1}^{\infty} \rangle$, $\frac{1}{N} \sum_{j=1}^{N/4} (\ldots) \to \sum_{n=1}^{\infty} \int_{0}^{\pi/2} (\ldots) \rho_n(\lambda) d\lambda$
Overlap of Néel with XXZ Bethe states

Bethe and GTBA equations

- Bethe equations in the TD limit [Takahashi (1999)]:

\[
\rho_n(\lambda) \left[ 1 + \eta_n(\lambda) \right] = s \ast [\eta_{n-1}\rho_{n-1} + \eta_{n+1}\rho_{n+1}](\lambda), \quad n \geq 1
\]

\[
\eta_n = \rho_{n,h}/\rho_n, \quad n \geq 1, \quad \eta_0(\lambda) = 1 \text{ and } \rho_0(\lambda) = \delta(\lambda); \quad (f \ast g)(\lambda) = \int_{-\pi/2}^{\pi/2} f(\lambda - \mu)g(\mu)d\mu
\]

- Kernel: \( s(\lambda) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-2ik\lambda} \)

- Saddle point state via

\[
0 = \left. \frac{\delta S_{QA}[\{\rho_n\}]}{\delta \rho_n} \right|_{\{\rho_n\}=\{\rho_n\}_{sp}} \quad \text{with} \quad S_{QA}[\{\rho_n\}] = 2S[\{\rho_n\}] - \frac{1}{2} S_{YY}[\{\rho_n\}]
\]

- Yang-Yang entropy

\[
\frac{S_{YY}[\{\rho_n\}]}{N} = \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} (\rho_n(\lambda) \ln[1 + \eta_n(\lambda)] + \rho_{n,h}(\lambda) \ln[1 + \eta^{-1}_n(\lambda)]) d\lambda
\]

- This + TDL of overlap coefficient ⇒ Partially decoupled form of the GTBA equations:

\[
\ln[\eta_n(\lambda)] = (-1)^n \log \left[ \frac{\varphi_4^2(\lambda)}{\varphi_1^2(\lambda)} \right] + \log \left[ \frac{\varphi_2^2(\lambda)}{\varphi_3^2(\lambda)} \right] + s \ast \left[ \ln(1 + \eta_{n-1}) + \ln(1 + \eta_{n+1}) \right](\lambda),
\]
Bethe equations:

$$\rho_n(\lambda) [1 + \eta_n(\lambda)] = s \ast [\eta_{n-1}\rho_{n-1} + \eta_{n+1}\rho_{n+1}](\lambda), \quad n \geq 1$$

GTBA equations:

$$\ln[\eta_n(\lambda)] = (-1)^n \ln \left[ \frac{\vartheta_4^2(\lambda)}{\vartheta_1^2(\lambda)} \right] + \ln \left[ \frac{\vartheta_2^2(\lambda)}{\vartheta_3^2(\lambda)} \right] + s \ast \left[ \ln(1 + \eta_{n-1}) + \ln(1 + \eta_{n+1}) \right](\lambda),$$

⇒ Solving this gives the steady state described by $$\{\rho_n\}_{n=1}^{\infty}$$
Bethe equations:

\[ \rho_n(\lambda) [1 + \eta_n(\lambda)] = s \ast [\eta_{n-1}\rho_{n-1} + \eta_{n+1}\rho_{n+1}] (\lambda), \quad n \geq 1 \]

GTBA equations:

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⇒ Solving this gives the steady state described by \( \{\rho_n\}_{n=1}^{\infty} \)

Limit to XXX (\( \Delta = 1 \)):

\[ \ln[\eta_n(\lambda)] = (-1)^{n+1} \ln \left[ \text{th}^2 \left( \frac{\pi\lambda}{2} \right) \right] + s \ast \left[ \ln(1 + \eta_{n-1}) + \ln(1 + \eta_{n+1}) \right] (\lambda), \]
Analytical solution of the GTBA equations
Program:

- Mapping GTBA Eqs to well-known functional equations: Y- and T-system
  ⇒ Explicit expressions for all $\eta_n$

- Combining with an explicit expression for $\rho_{1,h}$ (independent of any quench)
  ⇒ Bethe Eqs can be solved analytically ⇒ Explicit expressions for all $\rho_{n \geq 1}$
Analytical solution

Y-system

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  ⇒ Bethe Eqs can be solved analytically ⇒ Explicit expressions for all $\rho_{n\geq1}$

Y-system: [Takahashi; Klümper, Pearce (1992); Suzuki (1999)]

\[
y_n(x + \eta/2)y_n(x - \eta/2) = [1 + y_{n-1}(x)][1 + y_{n+1}(x)], \quad n \geq 1, \quad y_0(x) = 0
\]

Fixing the analyticity properties of the $y$-functions in the physical strip ($\pi$-periodicity in imaginary direction)

\[
PS = \{x \in \mathbb{C} \mid -\eta/2 < \text{Re}(x) < \eta/2, -\pi/2 < \text{Im}(x) \leq \pi/2\}
\]

⇒ Y-system is equivalent to non-linear integral equations (NLIEs)

\[
\ln[y_n(x)] = d_n(x) + s \ast [\ln(Y_{n-1}) + \ln(Y_{n+1})](x), \quad n \geq 1
\]

– Kernel function $s$ as before

– Driving terms $d_n$ determined by the analytical behavior of $y_n$ inside PS
GTBA Eqs are NLIEs of the form of the Y-system
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Driving terms come from the following analytical behavior:

\[ \eta_n(\lambda) \sim \text{sh}^2(2\lambda), \text{ for small } \lambda \text{ and } n \text{ odd}, \]
\[ \eta_n(\lambda) \sim \text{coth}^2(\lambda), \text{ for small } \lambda \text{ and } n \text{ even}, \]

and there are no further roots or poles for all \( \lambda \in PS\setminus\{0\} \)

Fourier transforms of the logarithmic derivatives:

\[ FT[\ln'(\text{sh}^2(2\lambda))](k) = -4\pi i \text{sh}(\eta k)(1 + (-1)^k), \]
\[ FT[\ln'(\text{coth}^2(\lambda))](k) = 4\pi i \text{sh}(\eta k)(1 - (-1)^k) \]

Dividing by \( \text{ch}(\eta k) \), taking inverse Fourier transform, integrating over \( x \) yields the driving terms of the GTBA Eqs

\[ \Rightarrow \text{Solution of the GTBA Eqs is given by solution of the Y-system} \]
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– Dividing by \( \text{ch}(\eta k) \), taking inverse Fourier transform, integrating over \( x \) yields the driving terms of the GTBA Eqs

\[ \Rightarrow \text{Solution of the GTBA Eqs is given by solution of the Y-system with this analyticity properties} \]
T-system and explicit expressions for $\eta_n$

– Rewriting the y's in terms of T's:

$$y_n(x) = T_{n-1}(x)T_{n+1}(x)/f_n(x), \quad n \geq 1$$

– Y-System $\Leftrightarrow$ T-System [Klümper, Pearce (1992); Suzuki (1999)]

$$T_n(x - \eta/2)T_n(x + \eta/2) = T_{n-1}(x)T_{n+1}(x) + f_n(x), \quad n \geq 1, \quad T_0(x) = 1$$

– Writing $T_1(x) = T_1^{(1)}(x) + T_1^{(2)}(x)$ and defining $a(x) = T_1^{(1)}(x)/T_2^{(1)}(x)$

$\Rightarrow y_1$ is completely determined by auxiliary function $a$:

$$y_1(x) = a(x + \eta/2) + a^{-1}(x - \eta/2) + a(x + \eta/2)a^{-1}(x - \eta/2)$$

– $y_0(x) = 0$ and $y_1(x) = \ldots$, plus Y-system (recursion relation) $\Rightarrow$ all $y_n$'s via $a$
Analytical solution

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\[ a(\lambda) = \frac{\text{sh}(\lambda + \eta) \text{sh}(2\lambda - \eta)}{\text{sh}(\lambda - \eta) \text{sh}(2\lambda + \eta)} \]
T-system and explicit expressions for $\eta_n$

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First function:

$$\eta_1(\lambda) = \frac{\text{sh}^2(2\lambda)\left[\text{ch}(\eta) + 2\text{ch}(3\eta) - 3\text{ch}(2\lambda)\right]}{2\text{sh}(\lambda - \eta/2)\text{sh}(\lambda + \eta/2)\text{sh}(2\lambda + 2\eta)\text{sh}(2\lambda - 2\eta)}$$
Analytical solution for $\rho_{1,h}$

Expectation values of the conserved charges on the Néel state \[ \text{[Essler, Fagotti (2013)]} \]

\[
\lim_{\text{th}} \frac{\langle \psi_0 | Q_{m+1} | \psi_0 \rangle}{N} = -\frac{\Delta}{2} \frac{\partial^{m-1}}{\partial x^{m-1}} \left( \frac{1 - \Delta^2}{\mathrm{ch} \left( \sqrt{1 - \Delta^2 x} \right) - \Delta^2} \right) \bigg|_{x=0}
\]

... and on a Bethe state: (e.g. the steady state)

\[
\lim_{\text{th}} \langle \lambda | \frac{Q_{m+1}}{N} | \lambda \rangle = \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} \rho_n(\lambda) \frac{\partial^m}{\partial \lambda^m} \ln \left[ \frac{\mathrm{sh}(\lambda + n\eta/2)}{\mathrm{sh}(\lambda - n\eta/2)} \right] d\lambda, \quad m \geq 0
\]

To see this, note that an $n$-string with string center $\lambda_{n\alpha}$ contributes a factor

\[
\frac{\mathrm{sh}[\lambda - \lambda_{n\alpha} + \frac{n}{2}(n+1)]}{\mathrm{sh}[\lambda - \lambda_{n\alpha} + \frac{n}{2}(n-1)]}
\]
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Expectation values of the conserved charges on the Néel state \[\text{[Essler, Fagotti (2013)]}\]

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To see this, note that an $n$-string with string center $\lambda_n^\alpha$ contributes a factor

$$
\frac{\text{sh}[\lambda - \lambda^\alpha_n - \frac{n}{2}(n+1)]}{\text{sh}[\lambda - \lambda^\alpha_n + \frac{n}{2}(n-1)]}
$$

Combining with the Bethe equations, eventually leads to

$$
\rho_{1,h}^{\text{sp}}(\lambda) = a_1(\lambda) \left( 1 - \frac{\text{ch}^2(\eta)}{a_1^2(\lambda) \text{sh}^2(2\lambda) + \text{ch}^2(\eta)} \right), \quad a_1(\lambda) = \frac{\text{sh}(\eta)}{\text{ch}(\eta) - \text{ch}(2\lambda)}
$$

Alternative: Use “generating function” for the Néel state \[\text{[Essler, Fagotti (2013)]}\]

$$
\Omega_{\psi_0}(\lambda) = -\frac{\text{sh}(2\eta)}{\text{ch}(2\eta) + 1 - 2\text{ch}(2\lambda)}, \quad \rho_{\psi_0}^\psi(\lambda) = a_1(\lambda) + \frac{1}{2\pi} \left[ \Omega_{\psi_0}(\lambda + \frac{\eta}{2}) + \Omega_{\psi_0}(\lambda - \frac{\eta}{2}) \right]
$$
Explicit expressions for $\rho_n$

- Bethe equations (as functional equations):

$$
\rho_{n+1,h}(\lambda) = \rho_{n,t}(\lambda + \eta/2) + \rho_{n,t}(\lambda - \eta/2) - \rho_{n-1,h}(\lambda), \quad n \geq 1,
$$

where

$$
\rho_{n,t}(\lambda) = \rho_{n,h}(\lambda) \left( 1 + \eta_n^{-1}(\lambda) \right)
$$

- $\rho_n(\lambda) = \rho_{n,h}(\lambda)/\eta_n(\lambda)$ for $n \geq 1 \Rightarrow \text{all } \rho_n \text{ explicitly}$
Explicit expressions for $\rho_n$

- Bethe equations (as functional equations):

$$\rho_{n+1,h}(\lambda) = \rho_{n,t}(\lambda + \eta/2) + \rho_{n,t}(\lambda - \eta/2) - \rho_{n-1,h}(\lambda), \quad n \geq 1, \quad \rho_{0,h}(\lambda) \equiv 0$$

where $\rho_{n,t}(\lambda) = \rho_{n,h}(\lambda) \left(1 + \eta_n^{-1}(\lambda)\right)$

- $\rho_n(\lambda) = \rho_{n,h}(\lambda)/\eta_n(\lambda)$ for $n \geq 1$ \Rightarrow all $\rho_n$ explicitly

For example:

$$\rho_1(\lambda) = \frac{\text{sh}^3(\eta) \text{sh}(2\lambda + 2\eta) \text{sh}(2\lambda + 2\eta)}{\pi f(\lambda - \frac{\eta}{2}) f(\lambda + \frac{\eta}{2}) g(\lambda)}$$

$$\rho_2(\lambda) = \frac{8 \text{sh}^2(\lambda) \text{sh}^3(\eta) \text{ch}(\eta)[3 \text{sh}^2(\lambda) + \text{sh}^2(\eta)][\text{ch}(6\eta) - \text{ch}(4\lambda)]}{\pi f(\lambda) g(\lambda + \frac{\eta}{2}) g(\lambda - \frac{\eta}{2}) h(\lambda)}$$

... 

where $f(\lambda) = \text{ch}^2(\eta) - \text{ch}(2\lambda)$, $g(\lambda) = \text{ch}(\eta) + 2 \text{ch}(3\eta) - 3 \text{ch}(2\lambda)$, and

$$h(\lambda) = 2 \text{ch}(4\lambda) + 2 \text{ch}^2(2\eta)[2 + \text{ch}(2\eta)] - \text{ch}(2\lambda)[3 + 2 \text{ch}(2\eta) + 3 \text{ch}(4\eta)].$$
Remarks about the interpretation of the auxiliary function $\alpha$

- Function $\alpha$ can be interpreted as auxiliary function corresponding to a (spin-1/2) quantum transfer matrix.

- Using standard contour $C$, which encircles the only pole of $1/(1+\alpha(\omega))$ at $\omega = i\pi/2$, one can compute $G$ by explicitly performing the contour integral.

- Nontrivial relation between $G$, $\alpha$ and generating function $\Omega_{\psi_0}$ [Essler, Fagotti (2013)] fulfilled.

- Unfortunately, this explicit $G$ function does not give the correct values of short-range correlation functions (due to the presence of higher nontrivial driving terms, $d_{n\geq2} \neq 0$, in the GTBA equations).
Summary and outlook

Summary

- Overlaps of Néel with XXZ Bethe states (\(\Delta\) arbitrary)
- Quench action approach \(\Rightarrow\) GTBA equations (for the steady state)
- Analytical solution of the GTBA equations
  \(\Rightarrow\) Connection to Y- and T-systems + Explicit expressions for \(p\)’s

Outlook

- Correlation functions using the analytical approach for solving the GTBA equations
- Applications to the Loschmidt echo [Pozsgay, arXiv:1308.3087]
- Overlaps and QAA also for different initial states (e.g. dimer, \(q\)-dimer,...)
- Complete understanding of the structure of GTBA equations (\(\leftrightarrow\) explicit solutions for different initial states)

Quenches from \(\Delta' \neq \infty\) to \(\Delta\) (XXZ) \(\rightarrow\) determinant expression for the overlaps needed!

Thank you for your attention!

Michael Brockmann (UvA)
Néel-to-XXZ quench
Dijon, September 2014
Conclusion

Summary and outlook

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