Inhomogeneous Multispecies TASEP on a ring

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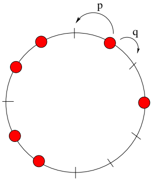
The Asymmetric Simple exclusion Process

The ASEP is a stochastic system of particles hopping on a one dimensional lattice under the constraint that a site of the lattice can be occupied by at most one particle



L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

Since its introduction in the '60 as a biophysical model for protein synthesis of RNA, ASEP has found several very different applications as a (toy) model for traffic flow, formation of shocks, etc... It is fair to say that it plays a fundamental role in our understanding of non-equilibrium processes.

Many approaches has been developed and exact results derived since its introduction and its rich combinatorial structure has been explored.

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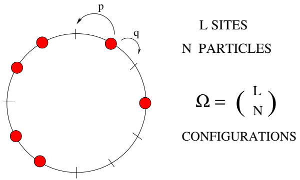

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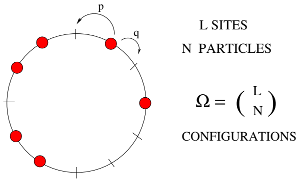

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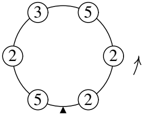
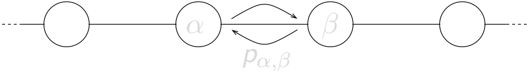
Multispecies ASEP general framework

We consider a periodic lattice $\mathbb{Z}/L\mathbb{Z}$ on which we have for $1 \leq \alpha \leq N$, m_α particles of species α, $\sum_{\alpha=1}^{N} m_\alpha = L$ (conventionally we can think at the particles of type $\alpha = 1$ as empty sites)



The rates $p_{\alpha,\beta}$ for a local exchange $\alpha \leftrightarrow \beta$ depends on the species involved



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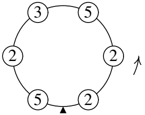

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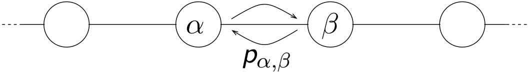

Multispecies ASEP general framework

The master equation for the time evolution of the probability of a configuration w is

$$\frac{d}{dt}P_w(t) = \sum_{w'|w'\to w} \mathcal{M}_{w,w'}P_w(t) - \sum_{w'|w\to w'} \mathcal{M}_{w',w}P_w(t)$$

$$\frac{d}{dt}P(t) = \mathcal{M}P(t) \qquad \mathcal{M} = \sum_{i=1}^{L} p_{\alpha,\beta}M^{(i)}_{\alpha,\beta}$$

In this talk I will focus on the stationary probability

$$\mathcal{M}P = 0$$

for a system on a ring.

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Multispecies ASEP: baxterized form of R-matrix

In order for M to be an integrable "Hamiltonian" we need a $\check{R}$-matrix which satisfies the YBE, the inversion relations and

$$\check{R}(x,x) = \mathbf{1}, \qquad \frac{d}{dx}\check{R}(x,y)|_{x=y=0} \propto \sum_{1\leq\alpha\neq\beta,N} p_{\alpha,\beta} M_{\alpha,\beta}$$

We search it of the "baxterized" form

$$\check{R}(x,y) = \mathbf{1} + \sum_{1\leq\alpha\neq\beta\leq N} g_{\alpha,\beta}(x,y) M_{\alpha,\beta}$$

If we ask that $\forall \alpha \neq \beta$, $p_{\alpha,\beta} \neq 0$ then the only solution (up to permutation of the species) happens for

$$p_{\alpha,\beta} = \begin{cases} p & \text{for} \quad \alpha < \beta \\ q & \text{for} \quad \alpha > \beta \end{cases}$$

in which case the matrices $M^{(i)}$ satisfy (up to rescaling) the Hecke commutation relations.

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Multispecies TASEP: R-matrix

In this talk we want to focus to the Totally Asymmetric case

$$p_{\alpha>\beta} = 0$$

In this case the "baxterized" solutions of the YBE are more interesting, they are parametrized by $2N-2$ parameters $\tau = \{\tau_1, \ldots, \tau_{N-1}\}, \nu = \{\nu_2, \ldots, \nu_N\}$

$$\check{R}(x,y) = \mathbf{1} + \sum_{1\le\alpha<\beta\le N} g_{\alpha,\beta}(x,y) M_{\alpha,\beta}$$

$$g_{\alpha,\beta}(x,y) = \frac{(y-x)(\tau_\alpha + \nu_\beta)}{(\tau_\alpha y - 1)(\nu_\beta x + 1)}$$

and the rates are given therefore by

$$p_{\alpha<\beta} = \tau_\alpha + \nu_\beta$$

Positivity results/conjectures [Lam-Williams, L.C. . .]

Conjectures

One can normalize the unique solution of $\mathcal{M}\psi = 0$ (unnormalized probabilties) in such a way that
1- the components ψ_w are polynomials of τ, ν with positive integer coefficients;
2 For $\nu_\alpha = 0$ (or $\tau_\alpha = 0$), and restricting to the sector with 1 particle per species (w is just a permutation) the polynomials ψ_w are Schubert positive.

Arita and Mallick have provided a Matrix Product solution for ψ_w in the case where $\forall\alpha$, $\nu_\alpha = 0$, which implies point 1.

Open question

Construct a Matrix Product for the general case $\nu_\alpha \neq 0$.

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Spectral parameters

Define as usual $R_{i,j}(x,y) = P_{i,j}\check{R}_{i,j}(x,y)$ and construct the transfer matrix

$$T(t|\mathbf{z}) = \mathrm{tr}_a R_{a,1}(t,z_1) R_{a,2}(t,z_2) \dots R_{a,L}(t,z_L)$$

$$\mathcal{M} = T^{-1}(0|\mathbf{0}) \frac{dT(t|\mathbf{0})}{dt}|_{t=0}$$

On each sector of particle content $\mathbf{m}$ the matrix $T(t|\mathbf{z})$ is stochastic (and irreducible) therefore we shall study the unique solution of

$$T(t|\mathbf{z})\psi(\mathbf{z}) = \psi(\mathbf{z})$$

and one recover the M-TASEP "unnormalized" stationary probability

$$\psi_w = \psi_w(\mathbf{0}).$$

Exchange equations

The eigenvector $\psi(\mathbf{z})$ is a polynomial in the spectral parameters $\mathbf{z}$ and in the rates τ, ν. Moreover, since

$$\check{R}_i(z_i, z_{i+1}) T(t|\mathbf{z}) \check{R}_i(z_{i+1}, z_i) = s_i \circ T(t|\mathbf{z})$$

where s_i acts on function by the exchange $z_i \leftrightarrow z_{i+1}$, one can normalize $\psi(\mathbf{z})$ in such a way that it satisfies the following exchange equations

$$\check{R}_i(z_i, z_{i+1}) \psi(\mathbf{z}) = s_i \circ \psi(\mathbf{z})$$

We shall discuss the minimal degree solution of such equations.

Exchange equations in components

Once expanded in components, the exchange equations read as follows

$$\psi_{\dots,w_i=w_{i+1},\dots}(\mathbf{z}) = s_i \circ \psi_{\dots,w_i=w_{i+1},\dots}(\mathbf{z})$$

$$\psi_{\dots,w_i>w_{i+1},\dots}(\mathbf{z}) = \hat{\pi}_i(w_i, w_{i+1})\psi_{\dots,w_{i+1}<w_i,\dots}(\mathbf{z})$$

and

$$\hat{\pi}_i(\alpha,\beta) = \frac{(\tau_\alpha z_{i+1} - 1)(\nu_\beta z_i + 1)}{\tau_\alpha + \nu_\beta} \frac{1 - s_i}{z_i - z_{i+1}}$$

This system of equation is cyclic: given $\psi_w(\mathbf{z})$ for a word w one can obtain $\psi_{w'}(\mathbf{z})$ for any other w' by acting with the $\hat{\pi}$ operators.

Affine 0-Hecke algebra with spectral parameters

The operators $\hat{\pi}_i(\alpha, \beta)$ satisfy a spectral parameter deformation (not baxterization!) of the 0-Hecke algebra (recovered for t_α and ν_α independent of α)

$$\hat{\pi}_i^2(\alpha, \beta) = -\hat{\pi}_i(\alpha, \beta)$$

$$\hat{\pi}_i(\beta, \gamma)\hat{\pi}_{i+1}(\alpha, \gamma)\hat{\pi}_i(\alpha, \beta) = \hat{\pi}_{i+1}(\alpha, \beta)\hat{\pi}_i(\alpha, \gamma)\hat{\pi}_{i+1}(\beta, \gamma)$$

$$[\hat{\pi}_i(\alpha, \beta), \hat{\pi}_j(\gamma, \delta)] = 0 \qquad |i - j| > 2$$

Some consequences of the exchange equations

- If w^* is such that $w^*_i \le w^*_{i+1}$ (and rotated) then

$$\psi_{w^*}(\mathbf{z}) = \prod_{\alpha<\beta} (\tau_\alpha + \nu_\beta)^{m_\alpha m_\beta (\beta-\alpha-1)} \prod_{i=1}^{L} \left(\prod_{\alpha < w^*_i} (1 - \tau_\alpha z_i) \prod_{\beta > w^*_i} (1 + \nu_\beta z_i) \right)$$

- In particular the solution of the exchange equation of minimal degree in the sector $\mathbf{m}$ has degree

$$\deg_{z_i} \psi^{(\mathbf{m})}(\mathbf{z}) = \#\{\alpha | m_\alpha \neq 0\} - 1$$

- Other components can be explicitely computed.

Recursions

• Setting $z_L = \tau^{-1}_{\min(\mathbf{m})}$ or $z_L = -\nu^{-1}_{\max(\mathbf{m})}$ kills all the components for which $w_L \neq \min(\mathbf{m})$ or $w_L \neq \max(\mathbf{m})$ and the other satisfy two remarkable recurrence relation

$$\psi_{w_1,\dots,w_{L-1},w_L=\min(\mathbf{m})}(\mathbf{z})|_{z_L=\tau^{-1}_{\min(\mathbf{m})}} = K^-(\mathbf{z}\setminus z_L)\psi_{w_1,\dots,w_{L-1}}(\mathbf{z}\setminus z_L)$$

$$\psi_{w_1,\dots,w_{L-1},w_L=\max(\mathbf{m})}(\mathbf{z})|_{z_L=-\nu^{-1}_{\max(\mathbf{m})}} = K^+(\mathbf{z}\setminus z_L)\psi_{w_1,\dots,w_{L-1}}(\mathbf{z}\setminus z_L)$$

where the factors $K^{\pm}(\mathbf{z}\setminus z_L)$ can be easily computed by inspection of $\psi_{w^*}(\mathbf{z})$.

Simplest non trivial component: the building block

Let $w^{(\alpha)}$ such that for $i \leq j \leq L - m_\alpha$

$$w_i \neq \alpha \qquad \text{and} \qquad w_i \leq w_j$$

For example

$$w^{(6)} = 2\ 2\ 3\ 5\ 5\ 5\ 7\ 9\ 9\ 6\ 6\ 6$$

Then

$$\psi^{(\mathbf{m})}_{w^{(\alpha)}}(\mathbf{z}) = (\text{Trivial Factors}) \times \phi^{(\mathbf{m})}_\alpha(z_1, \ldots, z_{L-m_\alpha})$$

where $\phi^{(\mathbf{m})}_\alpha(z_1, \ldots, z_{L-m_\alpha})$ is a symmetric polynomial in $z_1, \ldots, z_{L-m_\alpha}$ of degree 1 in each variable separately.

- These polynomials turn out to be the building blocks of more general components
- Thanks to the recursion relations they can be computed explicitly

Simplest non trivial component: the building block

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$$\phi^{(\mathbf{m})}_{\alpha}(z_1, \ldots, z_{L-m_\alpha}) = \oint_{\tau^{-1}} \frac{dw}{2\pi i} \frac{\prod_{i=1}^{L-m_\alpha}(w - z_i)}{\prod_{\beta \leq \alpha}(\tau_\beta w - 1)^{m_\beta} \prod_{\beta \geq \alpha}(\nu_\beta w + 1)^{m_\beta}}$$

Factorization of components with least ascending

The probability of the words $\tilde{w}$ which have minimal number of ascent

$$\text{for example} \qquad \tilde{w} = 9\ 9\ 7\ 6\ 6\ 6\ 5\ 5\ 5\ 3\ 2\ 2$$

is given by a product of $\phi_\alpha^{(\mathbf{m})}$ polynomials

Conjecture

Calling $\mathbf{z}_\alpha = \{z_i | w_i = \alpha\}$

$$\psi_{\tilde{w}} = (\text{Trivial Normalization}) \prod_\alpha \phi_\alpha^{(\mathbf{m})}(\mathbf{z} \setminus \mathbf{z}_\alpha)$$

The "Trivial Normalization" is $\mathbf{z}$ independent.

Factorization of components with least ascending: corollaries

Corollary I

The formula for the least ascending component implies and generalizes a formula conjectured by Lam and Williams which expresses $\psi_{\tilde{w}}$ in the case $\mathbf{m} = \{\dots, 0, 1, 1 \dots, 1, 0 \dots\}$ and $\nu_\alpha = 0$ as a product of Schubert Polynomials of τ

$$\psi_{L,L-1,\dots,1} = \mathfrak{S}_{1,2,3\dots,L}\mathfrak{S}_{1,3,4\dots,L,2}\mathfrak{S}_{1,4,5,\dots,L,2,3}\mathfrak{S}_{1,L,2,3\dots,L-1}$$

Corollary II

Suppose that we condition w to split as $w^{(k)}w^{(k-1)}\dots w^{(2)}w^{(1)}$, with $w^{(j)}$ of fixed length L_j (possibly 0) and

$$w_i^{(s+1)} > w_j^{(s)}$$

then the events $w^{(j)}$ are independent.

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Normalization

In order to compute actual probabilities we need the normalization

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \sum_{w|\mathbf{m}(w)=\mathbf{m}} \psi_w(\mathbf{z})$$

Thanks to the exchange relations this polynomial turns out to be symmetric in $\mathbf{z}$ and satisfies the recursion relation induced by $\psi(\mathbf{z})$ itself.

Unfortunately in the general case we are not able to provide a formula for $\mathcal{Z}^{(\mathbf{m})}(\mathbf{z})$.

What we can solve is the case $\nu_\alpha = \nu$ for $\alpha \leq \gamma$, $\tau_\alpha = \tau$ for $\alpha \geq \gamma$ for some γ.

Factorized sum rule

If for some γ we have

$$\begin{aligned} \nu_\alpha &= \nu \quad \text{for} \quad \alpha \le \gamma \\ \tau_\alpha &= \tau \quad \text{for} \quad \alpha \ge \gamma \end{aligned}$$

(and $m_\alpha > 0$ for $\min(\mathbf{m}) \le \alpha \le \max(\mathbf{m})$), by projecting "downward" from $\max(\mathbf{m})$ and "upward" from $\min(\mathbf{m})$ until γ

$$\mathcal{Z}^{(\mathbf{m})}(\mathbf{z}) = \prod_{\alpha=\min(\mathbf{m})}^{\gamma-1} \phi_\alpha^{(\mathbf{m}_\alpha^\uparrow)}(\mathbf{z}) \prod_{\alpha=\max(\mathbf{m})}^{\gamma+1} \phi_\alpha^{(\mathbf{m}_\alpha^\downarrow)}(\mathbf{z})$$

where

$$\mathbf{m}_\alpha^\downarrow = \Pi_{\alpha,\alpha+1,\dots}^{\alpha-1}\mathbf{m}, \qquad \mathbf{m}_\alpha^\uparrow = \Pi_{\dots,\alpha-1,\alpha}^{\alpha+1}\mathbf{m}$$

Some open questions

- Correlation functions, currents, etc.
- Do the components $\psi_w(\mathbf{z})$ have a combinatorial expression?
- What is the "right" context for the 0-Hecke algebra with spectra parameters?
  The operators $\hat{\pi}(\alpha, \beta)$ can be used for example to define a family of deformed Grothendick polynomials which depends on the parameters τ, ν. Do they have any geometric meaning?
- Deal with others Weyl groups.

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