Exact solutions for quenches in the Lieb-Liniger Bose gas and Heisenberg spin chains

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Work done in collaboration with (among others):
Plan of the talk

- Equilibrium dynamics
  - Finite temperatures in Lieb-Liniger
  - Spinons in Heisenberg

- Out-of-equilibrium dynamics
  - Quantum dots
  - Interaction quench in Lieb-Liniger
  - Anisotropy quench in XXZ

- Summary & perspectives
Applications of integrability in many-body physics

Quantum magnetism

Ultracold atoms

Quantum dots, NV centers

Atomic nuclei
Models discussed in this talk:

- Interacting Bose gas (Lieb-Liniger)
  \[ \mathcal{H}_N = - \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < l \leq N} \delta(x_j - x_l) \]

- Heisenberg spin-1/2 chain
  \[ H = \sum_{j=1}^{N} \left[ J(S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta S^z_j S^z_{j+1}) - H_z S^z_j \right] \]

- Central spin (Gaudin magnet)
  \[ H_{\text{int}} = B_z S^z_0 + \sum_{j=1}^{N} A_j \vec{S}_0 \cdot \vec{I}_j \]
The Bethe Wavefunction

Michel Gaudin's book *La fonction d'onde de Bethe* is a uniquely influential masterpiece on exactly solvable models of quantum mechanics and statistical physics. Available in English for the first time, this translation brings his classic work to a new generation of graduate students and researchers in physics. It presents a mixture of mathematics interspersed with powerful physical intuition, retaining the author's unmistakably honest tone.

The book begins with the Heisenberg spin chain, starting from the coordinate Bethe Ansatz and culminating in a discussion of its thermodynamic properties. Delta-interacting bosons (the Lieb-Liniger model) are then explored, and extended to exactly solvable models associated with a reflection group. After discussing the continuum limit of spin chains, the book covers six- and eight-vertex models in extensive detail, from their lattice definition to their thermodynamics. Later chapters examine advanced topics such as multicomponent delta-interacting systems, Gaudin magnets and the Toda chain.

MICHEL GAUDIN is recognized as one of the foremost experts in this field, and has worked at Commissariat à l'énergie atomique (CEA) and the Service de Physique Théorique, Saclay. His numerous scientific contributions to the theory of exactly solvable models are well known, including his famous formula for the norm of Bethe wavefunctions.

JEAN-SÉBASTIEN CAUX is a Professor in the theory of low-dimensional quantum condensed matter at the University of Amsterdam. He has made significant contributions to the calculation of experimentally observable dynamical properties of these systems.
The general idea, simply stated:

Start with your favourite quantum state
(expressed in terms of Bethe states)

\[ \mathcal{O} \rightarrow \left| \{\{\lambda}\} \right\rangle \]

Apply some operator on it

Reexpress the result in the basis of Bethe states:

\[ \mathcal{O}\left|\{\{\lambda}\}\right\rangle = \sum_{\{\mu\}} F_{\{\mu\},\{\lambda}\}}^{\mathcal{O}} \left|\{\mu\}\right\rangle \]

using ‘matrix elements’

\[ F_{\{\mu\},\{\lambda}\}}^{\mathcal{O}} = \langle \{\{\mu}\} | \mathcal{O} | \{\{\lambda}\} \rangle \]
Equilibrium dynamics from integrability
Equilibrium dynamics using integrability

Cases which we’ve handled:

- Heisenberg model
- XXZ gapless antiferromagnets
- XXZ gapped antiferromagnets
- Spin-1 chain (Babujan-Takhtajan)
- Lieb-Liniger (repulsive interactions)
- Lieb-Liniger (attractive interactions)
- Richardson model
- Central spin model
- Perturbed integrable models: NRG from BA
Repulsive Lieb-Liniger gas

Density-density (dynamical SF) (J-S C & P Calabrese, PRA 2006)

\[
S(k, \omega) = \frac{2\pi}{L} \sum_\alpha |\langle 0 | \rho_k | \alpha \rangle|^2 \delta(\omega - E_\alpha + E_0)
\]
Correspondence with excitations

Particle-like

Hole-like

Umklapp
Repulsive Lieb-Liniger gas

Dynamical structure factor at finite $T$

M. Panfil and J-SC, PRA 89 (2014)
Observing elementary excitations of correlated one-dimensional Bose gases

N. Fabbri, M. Panfil, D. Clément, L. Fallani, M. Inguscio, C. Fort and J.-S. Caux

arxiv:1406.2176

Density correlations using Bragg spectroscopy
Cold atoms

Observing elementary excitations of correlated one-dimensional Bose gases
N. Fabbri, M. Panfil, D. Clément, L. Fallani, M. Inguscio, C. Fort and J.-S. Caux
arxiv:1406.2176

Intuitive picture of correlations: from ‘quasiparticles’

weak interactions

strong interactions
Heisenberg spin chain

$S(k, \omega), \quad \Delta = 1, \quad h = 0$
Investigating spin chains using neutrons
Investigating spin chains using neutrons

Effective particles: spinons
Quantum spin chains
Correlations, experiments (INS, RIXS), prefactors, ...

$\text{(C}_5\text{D}_{12}\text{N})_2\text{CuBr}_4$

$\text{Sr}_2\text{CuO}_3$

$\text{KC}_3\text{CuF}_3$

Thielemann, Rüegg, Rønnow, Läuchli, Caux, Normand, Biner, Krämer, Güdel, Stahn, Habicht, Kiefer, Boehm, McMorrow, Mesot, PRL 2009

Lake, Tennant, Caux, Barthel, Schollwöck, Nagler, Frost, PRL 2013

Walters, Perring, Caux, Savici, Gu, Lee, Ku, Zaliznyak, NATURE PHYSICS 2009

Schlappa, Wohlfeld, Zho, Mourigal, Haervkort, Strocov, Hozoi, Monney, Nishimoto, Singh, Revcolevschi, Caux, Patthey Rønnow, van den Brink, Schmitt, NATURE 2012
Counting fractional spinon excitations in the quantum Heisenberg antiferromagnetic chain

Martin Mourigal,¹,²,³, * Mechthild Enderle,¹ Axel Klöpperpieper,⁴ Jean-Sébastien Caux,⁵ Anne Stunault,¹ and Henrik M. Rønnow²

(Nature Physics 2013)

CuSO₄ · 5D₂O

Roger Hiorns, Seizure
Multi-spinon processes: intuitive picture

a) Fully polarized state
\[ \langle S_{i}^{z} \rangle = S \]

b) Zero-magnetic field state
\[ \langle S_{i}^{x,y} \rangle = 0 \quad \text{and} \quad \langle S_{i}^{x,y} \rangle \sim (-1)^{n-1} \]

Ising \( \Delta \rightarrow \infty \)

Heisenberg \( \Delta = 1 \)

c) Energy transfer (meV) vs. Momentum transfer (h/1, 1/2, 1/2)

Energy transfer (meV) per Cu

d) Energy transfer (meV) vs. Momentum transfer (h, -1/2, -1/2)
Experimental data is accurate enough to discriminate between:

- Müller ansatz (off)
- BA: 2 spinons (off)
- BA: 2 + 4 spinons (on!)

Frequency-integrated:
Experiment accurate enough to discriminate between:

- Müller Ansatz
- Field theory
- Bethe Ansatz (best)
Babujan-Takhtajan spin-1 chain

\[ H = J \sum_{j=1}^{N} \left[ S_j \cdot S_{j+1} - (S_j \cdot S_{j+1})^2 - H_z S^z_j \right] \]

- Ground state: correlated gas of 2-strings
- String deformations: must be taken into account
- Correlations: skewed to higher energies (as compared to S=1/2)

R. P. Vlijm and J.-S. C., JSTAT 2014
Out-of-equilibrium dynamics from integrability
The simple pendulum on its head

Kapitsa pendulum, 1951

Pyotr L. Kapitsa
(8/7/1894-8/4/1984)
The Kapitza pendulum
Out-of-equilibrium using integrability

It’s possible to treat some situations using BA

**Highly excited initial (eigen)states:**
- The super Tonks-Girardeau gas
- Split Fermi sea in Lieb-Liniger
- Interaction quench in Richardson
- Domain wall release in Heisenberg
- Geometric quench
- Interaction turnoff in Lieb-Liniger
- Release of trapped Lieb-Liniger

**Quenched states:**
- BEC to Lieb-Liniger quench
- Néel to XXZ quench

**Driven systems:**
- Spin echo in quantum dots
Driven systems:

Dynamics in quantum dots
Central spin model for quantum dot

Nearby spins: full hyperfine interaction (couplings \( \sim \) el. wavefn)

Spins further away: bath, mean-field dynamics (Ornstein-Uhlenbeck)

\[
H_{\text{int}} = B_z S_0^z + \sum_{j=1}^{N} A_j \vec{S}_0 \cdot \vec{I}_j
\]

\[
A_k = \frac{A}{n_0} |\psi(r_k)|^2 = \frac{A}{N} \exp \left[ -\frac{(k - 1)}{N_0} \right]
\]

\[
V(t) = B(t) S_0^z
\]

\[
\langle B(t)B(0) \rangle = b^2 \exp(-Rt)
\]

R. van den Berg et al., arxiv:1407.6503
Usual spin echo sequence \( \frac{\pi y}{2} - \tau - \pi x - \tau - \frac{\pi y}{2} \)

here focus on \( \tau - \pi x - \tau \) and assume ideal pulses

**Time evolution:** 2nd-order Suzuki-Trotter decomposition

\[
U(t, t + dt) \approx U^{(2)} = e^{-i \frac{dt}{2} H_{\text{int}}} e^{-i dt V(t + \frac{dt}{2})} e^{-i \frac{dt}{2} H_{\text{int}}}
\]

Simulate two types of initial states:

**Néel bath:**

\[
|\psi_0\rangle = \frac{1}{\sqrt{2}} (|\up\rangle + |\down\rangle) \otimes |\up\down\up\down\up\down\ldots\rangle
\]

**Random bath:**

\[
|\psi_0\rangle = \frac{1}{\sqrt{2^{N+1}}} (|\up\rangle + |\down\rangle) \otimes_{k=1}^{N} (|\up\rangle + e^{i\phi_k} |\down\rangle)
\]
Spin echo simulations

$N = 14$, all curves averaged over 100 OU processes
Quantum quenches
Quenches
(more generally: ‘prepare and release’)

The problem: considering a generic initial state, what is the time evolution of the system?

\[ |\Psi(t)\rangle = e^{-iHt} |\Psi(t = 0)\rangle \]

Hamiltonian driving time evolution

initial state is NOT eigenstate of H

Observable expectation values depend on time:

\[ \bar{O}(t) \equiv \frac{\langle \Psi(t)|O|\Psi(t)\rangle}{\langle \Psi(t)|\Psi(t)\rangle} \]
First question: what is the steady state long after the quench?

Fundamental issue: does the system relax? thermalize?

Crucial point: *time evolution in the presence of myriads of constraints (due to integrability) is special*

Conjecture: *steady state is described by a generalized Gibbs ensemble (GGE)*

\[
\lim_{t \to \infty} \bar{O}(t) = \langle \mathcal{O} \rangle_{GGE} = \frac{\text{Tr}\{\mathcal{O} e^{-\sum_n \beta_n Q_n}\}}{\text{Tr}\{e^{-\sum_n \beta_n Q_n}\}}
\]

Rigol, Dunjko, Yurovsky, Olshanii, PRL 2007

see also Jaynes, Phys. Rev. 1957
GGE implementation

Generalized inverse temperatures to be set using the initial conditions on conserved charges

\[ \langle \hat{Q}_m \rangle = \text{Tr} \left\{ \hat{Q}_m e^{-\sum_n \beta_n \hat{Q}_n} \right\} / Z_{GGE} \quad m = 0, 1, 2, \ldots \]

where \( Z_{GGE} = \text{Tr} e^{-\sum_n \beta_n \hat{Q}_n} \)

In reality, two major difficulties:

- Conserved charges are generically nontrivial
- Self-consistency problem difficult to solve

In practice: implementable for free theories only (charges: momentum occupation modes)

For interacting cases: not understood in general.
Quantum quenches:

BEC to repulsive Lieb-Liniger quench
Interacting Bose gas (Lieb-Liniger)

\[ \mathcal{H}_N = - \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < l \leq N} \delta(x_j - x_l) \]

Exact eigenstates from Bethe Ansatz:

\[ \Psi(x|\lambda) = F_\lambda \sum_{P \in S_N} A_P(x|\lambda) \prod_{j=1}^{N} e^{i \lambda P_j x_j} \]

\[ F_\lambda = \frac{\prod_{j>k=1}^{N} (\lambda_j - \lambda_k)}{\sqrt{N! \prod_{j>k=1}^{N} ((\lambda_j - \lambda_k)^2 + c^2)}} \]

\[ A_P(x|\lambda) = \prod_{j<k=1}^{N} \left( 1 - \frac{ic \text{ sgn}(x_j - x_k)}{\lambda_{P_j} - \lambda_{P_k}} \right) \]
Quench from BEC to repulsive gas

Start from GS of noninteracting theory,

\[ |0_N\rangle \equiv \frac{1}{\sqrt{L^N N!}} \left( \psi_{k=0}^\dagger \right)^N |0\rangle \]

Turn repulsive interactions on from \( t=0 \) onwards:

particles ‘repel away’ from each other, system heats up, momentum distribution broadens, ...
This is a difficult problem to treat...

1) Generalized Gibbs ensemble logic

\( \hat{Q}_n : \hat{Q}_n |\{\lambda\}_N\rangle = Q_n |\{\lambda\}_N\rangle \)

Conserved charges:

\[ Q_n (\{\lambda\}_N) = \sum_{j=1}^{N} \lambda_j^n \]

Davies 1990; Davies and Korepin

GGE inapplicable, charges take infinite values!

J-S C + J. Mossel, unpublished

2) GGE on lattice, q-deformed model

Works, partial results only (using a few charges)

Kormos, Shashi, Chou, JSC, Imambekov, PRA 2014
The ‘quench action’ approach

in pictures...

\[ \mathcal{H}_0 \rightarrow \mathcal{H} \]

in pre-quench Hilbert space basis

in post-quench Hilbert space basis

J-SC & F.H.L. Essler, PRL 2013
The ‘quench action’ approach

J-SC & F.H.L. Essler, PRL 2013

Quench action landscape: determined by overlaps & entropy


\[ S_Q[\rho] \]

\[ \mathcal{H} \]

Variational approach, implemented by a ‘Generalized thermodynamic Bethe Ansatz’

The ‘quench action’ approach

J-SC & F.H.L. Essler, PRL 2013

Saddle-point state: $\rho_{sp}$ determines steady state

States around saddle-point: determine relaxation towards steady state
Consider a generic integrable model, with eigenstates labeled by quantum numbers \( \{ I \} \).

Resolution of identity: 
\[
1 = \sum_{\{ I \}} \langle \{ I \} | \{ I \} \rangle
\]

Arbitrary initial state can be decomposed in this basis:
\[
| \Psi(t = 0) \rangle = \sum_{\{ I \}} e^{-S^\Psi_{\{ I \}}} | \{ I \} \rangle
\]

using overlap coefficients
\[
S^\Psi_{\{ I \}} = - \ln \langle \{ I \} | \Psi(t = 0) \rangle \in \mathbb{C}
\]
Time dependence: trivially written as

$$|\Psi(t)\rangle = \sum_{\{I\}} e^{-S^\psi_{\{I\}} - i\omega_{\{I\}} t} |\{I\}\rangle$$

The expectation values we’re interested in then become

$$\bar{O}(t) = \sum_{\{I^l\}} \sum_{\{I^r\}} e^{-(S^\psi_{\{I^l\}})^* - S^\psi_{\{I^r\}} + i(\omega_{\{I^l\}} - \omega_{\{I^r\}}) t} \langle \{I^l\} | O | \{I^r\} \rangle \frac{\sum_{\{I\}} e^{-2\Re S^\psi_{\{I\}}}}{\sum_{\{I\}} e^{-2\Re S^\psi_{\{I\}}}}$$

Intractable in general: **double Hilbert space sum**

Need to develop tools to evaluate this...
Start by looking at wavefunction normalization:

$$\langle \Psi(t) | \Psi(t) \rangle = \sum_{\{I\}} e^{-2\Re S^I}$$

In Th. Lim., would like to use the usual functional integral

$$\lim_{Th} \sum_{\{I\}} (...) = \int D\rho \ e^{S^{YY}[\rho]} (...)$$

Including the effective overlaps yields

$$\lim_{Th} \langle \Psi(t) | \Psi(t) \rangle = \int D\rho \ e^{-S^Q[\rho]}$$

with ‘quench action’

$$S^Q[\rho] = S^o[\rho] - S^{YY}[\rho]$$

Saddle-point evaluation:

$$\rho_{sp} : \text{ such that } \left. \frac{\delta S^Q[\rho]}{\delta \rho} \right|_{\rho_{sp}} = 0$$

‘Generalized thermodynamic Bethe Ansatz’
For operator expectation values, we had

$$\bar{O}(t) = \sum_{\{I_l\}} \sum_{\{I_r\}} e^{-(S_{\{I_l\}^*} - S_{\{I_r\}}} + i(\omega_{\{I_l\}} - \omega_{\{I_r\}})t \langle \{I_l\}|O|\{I_r\}\rangle \sum_{\{I\}} e^{-2\Re S_{\{I\}}}$$

Considering operators which are ‘weak’ (creating a non-entropically large nr of excitations when acting on a given state): thermodyn limit is

$$\lim_{Th} \bar{O}(t) = \frac{\int \mathcal{D}\rho e^{-S_Q[\rho]} \lim_{Th} \sum_{\{e\}} e^{-\delta S_{\{e\}}[\rho] - i\omega_{\{e\}}[\rho]t \langle \rho|O|\rho; \{e\}\rangle}}{\int \mathcal{D}\rho e^{-S_Q[\rho]}}$$

relative overlaps
denumerable set of excitations
For operators with non-entropically large matrix elements, can perform a saddle-point evaluation (same saddle-point in numerator and denominator)

\[
\lim_{Th} \tilde{O}(t) = \lim_{Th} \frac{1}{2} \sum_{\{e\}} \left[ e^{-\delta S_{\{e\}}[\rho_{sp}] - i \omega_{\{e\}}[\rho_{sp}]t} \langle \rho_{sp} | O | \rho_{sp}; \{e\} \rangle 
+ e^{-\delta S^{*}_{\{e\}}[\rho_{sp}] + i \omega_{\{e\}}[\rho_{sp}]t} \langle \rho_{sp};\{e\} | O | \rho_{sp} \rangle \right]
\]

Main message: the *full* time dependence is recoverable using a minimal amount of data

- saddle-point distribution (from GTBA)
- excitations in vicinity of sp state (easy)
- differential overlaps
- selected matrix elements
For operators with non-entropically large matrix elements, can perform a saddle-point evaluation (same saddle-point in numerator and denominator)

\[
\lim_{Th} \bar{O}(t) = \lim_{Th} \frac{1}{2} \sum_{\{e\}} \left[ e^{-\delta S_{\{e\}}[\rho_{sp}]} - i\omega_{\{e\}}[\rho_{sp}] t \langle \rho_{sp} | O | \rho_{sp} ; \{e\} \rangle + e^{-\delta S^*_{\{e\}}[\rho_{sp}]} + i\omega_{\{e\}}[\rho_{sp}] t \langle \rho_{sp} ; \{e\} | O | \rho_{sp} \rangle \right]
\]

Main message: the *full* time dependence is recoverable using a minimal amount of data

Large time limit:

\[
\lim_{t \to \infty} \lim_{Th} \bar{O}(t) = \lim_{Th} \langle \rho_{sp} | O | \rho_{sp} \rangle
\]

Single representative state is sufficient to calculate properties of steady state! (think of ETH)

Cassidy, Rigol
PRL 2009
Problem: need to calculate overlaps.

Remark: only parity-invariant states contribute since

\[ 0 = \langle 0 | \hat{Q} \hat{Q} | I \rangle = \langle 0 | I \rangle \sum_{j=1}^{N} \chi_j^{2m+1} \]

Known large c limit: Gritsev, Rostunov & Demler, JSTAT 2010

\[ \langle \{ \lambda_j \}_{j=1}^{N/2}, \{-\lambda_j\}_{j=1}^{N/2} | 0 \rangle \propto \prod_{\lambda_j > 0} \frac{1}{\lambda_j} \]

Possible for generic interaction?
M. Brockmann JPA 2014

\[
\left\langle \{\lambda_j\}_{j=1}^{N/2}, \{-\lambda_j\}_{j=1}^{N/2} | 0 \right\rangle = \sqrt{\frac{(cL)^{-N}N!}{\det_{j,k=1}^N G_{j,k}}} \cdot \frac{\det_{j,k=1}^{N/2} G_{j,k}^Q}{\prod_{j=1}^{N/2} \lambda_j \frac{\lambda_j^2}{c^2} + \frac{1}{4}}
\]

(reminiscent of Gaudin formula)

with matrix

\[
G_{j,k}^Q = \delta_{j,k} \left( L + \sum_{l=1}^{N/2} K^Q(l_j, l_l) \right) - K^Q(l_j, l_k)
\]

\[
K^Q(\lambda, \mu) = K(\lambda - \mu) + K(\lambda + \mu) \quad K(\lambda) = \frac{2c}{\lambda^2 + c^2}
\]
Quench action approach to BEC-LL quench

We are now in position to apply the quench action logic!

Need thermodynamic limit form of overlaps:

\[
\lim_{Th} \langle \lambda, -\lambda | 0 \rangle = \exp \left( -\frac{L}{2} n \left( \log \frac{c}{n} + 1 \right) \right) \\
\times \exp \left\{ -\frac{L}{2} \int_{0}^{\infty} d\lambda \rho(\lambda) \log \left[ \frac{\lambda^2}{c^2} \left( \frac{\lambda^2}{c^2} + \frac{1}{4} \right) \right] + O(L^0) \right\}
\]

Quench action now defined, saddle-point solution via generalized thermodynamic Bethe ansatz
It is in fact possible to give a closed form solution of the GTBA for the saddle-point state, for any value of the interaction:

\[
\rho(\lambda) = -\frac{\gamma}{2\pi} \frac{\partial a(\lambda)}{\partial \gamma} (1 + a(\lambda))^{-1}
\]

\[
a(\lambda) = \frac{2\pi/\gamma}{\frac{\lambda}{c} \sinh \left(\frac{2\pi\lambda}{c}\right)} I_{1-2i\frac{\lambda}{c}} \left(\frac{4}{\sqrt{\gamma}}\right) I_{1+2i\frac{\lambda}{c}} \left(\frac{4}{\sqrt{\gamma}}\right)
\]
Quench action solution to BEC-LL quench


Subplot: scaled fn
\[ \rho_s(x) = \sqrt{\gamma} \rho(c\sqrt{\gamma}x/2)/2 \]

Large c:
\[ \rho(\lambda) = \frac{1}{2\pi} \frac{4n^2}{\lambda^2 + 4n^2} \]

Small c: semicircle
\[ \rho(\lambda) \sim \frac{1}{\pi\sqrt{\gamma}} \sqrt{1 - \frac{\lambda^2}{4\gamma n^2}} \]

Asymptotics as from q-bosons:
\[ 2\pi \rho(\lambda) \sim \frac{n^4\gamma^2}{\lambda^4} + \frac{n^6\gamma^3(24 - \gamma)}{4\lambda^6} + \ldots \]

Tail explains divergences of evals of conserved charges
Quantum quenches:

Néel to XXZ quench
Quench from Néel to XXZ

Start from Néel state:

From $t=0$ onwards, evolve with XXZ Hamiltonian

\[ H = \sum_{j=1}^{N} \left[ J (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) - H_z S_j^z \right] \]

Can one treat this problem exactly?
“Particle content” of XXZ: nontrivial

Solution of Bethe equations: rapidities + strings

Single particle (bound state of magnons)

\[ \lambda^{j,a}_{\alpha} = \lambda^{j}_{\alpha} + i \frac{\zeta}{2} (n_j + 1 - 2a) + i \delta^{j,a}_{\alpha} O(e^{-(cst)N}) \]

Classification of strings: Bethe, Takahashi, Suzuki, ...
Quench action approach to Néel-XXZ quench

First step: exact overlaps of Néel state with XXZ eigenstates

\[
\frac{\langle \Psi_0 | \{\pm \lambda_j \}_{j=1}^{M/2} \rangle}{\| \{\pm \lambda_j \}_{j=1}^{M/2} \|} = \sqrt{2} \left[ \frac{\prod_{j=1}^{M/2} \sqrt{\tan(\lambda_j + i\eta/2) \tan(\lambda_j - i\eta/2)}}{2 \sin(2\lambda_j)} \right] \sqrt{\frac{\det_{M/2}(G_{jk}^+)}{\det_{M/2}(G_{jk}^-)}}
\]

\[G_{jk}^\pm = \delta_{jk} \left( NK_{\eta/2}(\lambda_j) - \sum_{l=1}^{M/2} K_{\eta}^+(\lambda_j, \lambda_l) \right) + K_{\eta}^\pm(\lambda_j, \lambda_k)\]

\[K_{\eta}^\pm(\lambda, \mu) = K_{\eta}(\lambda - \mu) \pm K_{\eta}(\lambda + \mu)\]

\[K_{\eta}(\lambda) = \frac{\sinh(2\eta)}{\sin(\lambda + i\eta) \sin(\lambda - i\eta)}\]
Quench action approach to Néel-XXZ quench

Second step: generalized TBA


\[ \ln \eta_n(\lambda) = -2 h n - \ln W_n(\lambda) + \sum_{m=1}^{\infty} a_{nm} \ast \ln \left(1 + \eta_m^{-1}\right)(\lambda) \]

where

\[ \eta_n(\lambda) \equiv \rho_{n,h}(\lambda)/\rho_n(\lambda) \quad a_n(\lambda) = \frac{1}{\pi} \frac{\sin n\eta}{\cosh n\eta - \cos 2\lambda} \]

and the effective driving terms (pseudo-energies) are

\[ W_n(\lambda) = \begin{cases} 
\frac{1}{2^n+1} \frac{\cosh n\eta - \cos 2\lambda}{\sin^2 2\lambda} \prod_{j=1}^{n-1} \left( \frac{\cosh(2j-1)\eta - \cos 2\lambda}{(\cosh(2j-1)\eta + \cos 2\lambda)(\cosh 4\eta j - \cos 4\lambda)} \right)^2 & \text{if } n \text{ odd}, \\
\frac{\tan^2 \lambda \cosh n\eta - \cos 2\lambda}{2^n \cosh n\eta + \cos 2\lambda} \prod_{j=1}^{n-2} \frac{1}{(\cosh 2(2j-1)\eta - \cos 4\lambda)^2} \prod_{j=1}^{n-2} \left( \frac{\cosh 2j\eta - \cos 2\lambda}{\cosh 2j\eta + \cos 2\lambda} \right)^2 & \text{if } n \text{ even.} 
\end{cases} \]

Solution of this GTBA gives steady-state

(analytically! See M. Brockmann’s talk)
Quench action approach to Néel-XXZ quench

Equivalent form of generalized TBA:

\[ \ln(\eta_n) = d_n + s \times \left[ \ln(1 + \eta_{n-1}) + \ln(1 + \eta_{n+1}) \right] \]

with driving terms

\[ d_n(\lambda) = \sum_{k \in \mathbb{Z}} e^{-2ik\lambda} \frac{\tanh(\eta k)}{k} \left((-1)^n - (-1)^k\right) \]

GGE with local charges: same form of coupled equations, but driving term only for \( n=1 \):

\[ d_1(\lambda) = -\frac{1}{\pi} \sum_{m=1}^{\infty} \beta_{2m} \sum_{k \in \mathbb{Z}} e^{-2ik\lambda} \frac{k^{2m-2}}{\cosh k\eta} \]

unknown
Néel-XXZ quench: conserved charges

Initial expectation value of local charges:  
\[
\lim_{N \to \infty} \frac{1}{N} \langle \text{Néel} | Q_{n+1} | \text{Néel} \rangle = -\Delta \frac{\partial^{n-1}}{2 \partial x^{n-1}} \left. \frac{1 - \Delta^2}{\cosh[\sqrt{1 - \Delta^2 x}] - \Delta^2} \right|_{x=0}
\]

In 1-to-1 correspondence with 1-string hole density:  

\[
\sum_{k \in \mathbb{Z}} k^{2m-2} \left( \frac{e^{-|k|\eta} - \hat{\rho}_1^h(k)}{2 \cosh k\eta} \right) = \langle Q_{2m} \rangle \quad m \in \mathbb{N}
\]

which fixes

\[
\rho_{\text{Néel}}^1(h)(\lambda) = \frac{\pi^2 a_1^3(\lambda) \sin^2(2\lambda)}{\pi^2 a_1^2(\lambda) \sin^2(2\lambda) + \cosh^2(\eta)}
\]

Quench action nontrivially reproduces this; GGE also of course, but only by definition.
The steady state: Néel to XXZ

Solid lines: quench action

Dashed lines: GGE (local charges)

QA and GGE have different saddle-point densities

Large Delta expansion:

\[
\rho_1^{GGE} - \rho_1^{sp} = \frac{1}{4\pi\Delta^2} + O(\Delta^{-3}),
\]

\[
\rho_2^{GGE} - \rho_2^{sp} = \frac{1 - 3\sin^2(\lambda)}{3\pi\Delta^2} + O(\Delta^{-3}).
\]
Difference in distribution: impact on correlations

Large Delta expansions:

\[
\langle \sigma_1^z \sigma_2^z \rangle_{QA} = -1 + \frac{2}{\Delta^2} - \frac{7}{2\Delta^4} + \frac{77}{16\Delta^6} + \ldots
\]

\[
\langle \sigma_1^z \sigma_2^z \rangle_{GGE} = -1 + \frac{2}{\Delta^2} - \frac{7}{2\Delta^4} + \frac{43}{8\Delta^6} + \ldots
\]

Numerical verification using NLCE (M. Rigol)
Not convinced?

Look at other results by Budapest group
B. Pozsgay, M. Mestyán, M. A. Werner, M. Kormos, G. Zaránd, G. Takács, arxiv 1405.2843

- reobtain our Néel results
- also consider initial dimer state
- obtain numerical (iTEBD) evidence for correlations being different in dimer case

There remains little doubt about the correctness of the quench action results

See also recent results by G. Goldstein & N. Andrei
What’s going on?

Néel to XXZ: current situation

- quench action solution gives correct expectation value for all conserved charges, directly from microscopics
- quench action and (local) GGE steady state distributions do not coincide
- these different distributions lead to different observable expectation values

Possible explanations of this mismatch:

- GGE converges to QA once all (nonlocal*) charges are added
- exceptional states invalidate the QA calculation ruled out

* of which there are exponentially many more than local ones!
Summary & perspectives

Integrability for dynamics

- prototypical strong correlations under control
- detailed experimental correspondence in spin chains
- …and now also in cold atomic Bose gases
- useful tests and lessons for field theory/numerical methods

Quench action logic

- new approach to out-of-equilibrium problems
- gives access to full time evolution with minimal data
- BEC to LL: exact solution (inaccessible to GGE)
- Néel to XXZ: exact solution
- GGE with local charges gives different steady state!