Formulas of form factors for the spin-1/2 and integrable spin-s XXZ chains and non-equilibrium dynamics

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The meaning of Ocha-no-Mizu :
Water (``Mizu'') for (``no'') Tea (``Ocha'')

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``Recent Advances in Quantum Integrable Systems 2014'',
Universiyy of Bourgogne ,
Sep 1-5, 2014

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Part 1:

Thermalization and equilibration of isolated quantum systems

Thermalization (equilibration) of isolated quantum systems

- We say that a local operator of an isolated quantum many-body system thermalizes or equilibrates (Cf. M. Rigol et al., (2007, 2008); V.I. Yukalov (2011)) if the expectation value at time *t* approaches a constant value after a long period of time:

$$\langle A(t)\rangle \rightarrow \langle A(\infty)\rangle \qquad (t \gg 1)$$

(i) For non-integrable systems, it is conjectured that the asymptotic value follow the average of the Gibbs distribution

$$\rho_{eq} = \exp(-\beta E)/Z$$

(ii) For integrable systems, it is conjectured that the asymptotic value follow the average of the generalized Gibbs ensembles (GGE)

$$\rho_{eq} = \exp(-\sum \lambda_j I_j)/Z$$

Here I_j (j=1, 2, …, N) denote conserved quantities.
N is the degrees of freedom, the number of sites or particles.

- We recall that the quantum state does no change at all in time.

Quantum quench in integrable systems: Dynamics after a sudden change of a parameter

- Pioneering paper:
  E. Barouch, B.M. McCoy and M. Dresden. (1970)
  asymptotic time evolution of XY spin chain

- Much interest on the dynamics of isolated quantum integrable systems , recently. For instance,

(i) Relaxation behavior of XXZ chain (J.-S. Caux et al., 2009)

(ii) Long time dynamics after a quench in 1D transverse Ising chain
(D. Rossini, S. Suzuki, G. Mussardo, G.E. Santoro, A. Silva, PRA (2010))

(iii) Large time behavior of transverse Ising chain (F. Essler et al. 2011)

(iv) Relaxation to GGE for 1D Bose gas with infinite coupling
(P. Calabrese et al., 2013)

Many talks in this conference

Non-equilibrium dynamics of isolated quantum systems realized in cold atomic experiments

For isolated finite systems of cold atoms

- i) Oscillatory behavior realized in 1D cold atoms

（Non-trivial non-equilibrium dynamics,
Kinoshita et al. Nature **440**, 900 (2006))

- Ii) Relaxation and prethermalization have been observed
(M. Gring et al., Science **337**, 1318 (2012))

Why do local operators of an isolated quantum many-body system equilibrate (relax) in time ? → Typicality of quantum states

- **As a result of typicaiity, we may approximately consider that most of the states | Ψ > of a quantum system are close to be in equilibrium:**

$$|\Psi\rangle\langle\Psi| \cong \rho_{eq},$$

the density matrix of the (micro−)canonical ensemble
(G.G.E. for integrable systems)

- Reference on ``Almost all quantum states are close to be in equilibrium''
H. Tasaki (1998); S. Goldstein et al.(2006) ;
A. Sugita (2006), P. Reimann (2007)
Cf. J. von Neumann (1929)

$$|\phi(t)\rangle = \sum_n c_n \exp(-i\omega_n t)|n\rangle$$

.

- −> The conjecture should be nontrivial , although it looks plausible.
We want to check it with solvable models.
Furthermore, it is not clear how the relaxation dynamics is.

A fundamental inequality of Typicality

due to A. Sugita (2006); (See also P. Reimann, PRL(2007))

$$E\left[\left(\Delta\langle A\rangle\right)^2\right] \leq \frac{|A|_{op}^{\ 2}}{d+1}$$

(1) E[B]: the ensemble average of *B* over states in a energy shell [E- ΔE, E]

(2) $|A|_{op}$ denotes the largest eigenvalue of operator A (operator norm);

(3) $\langle A\rangle = \langle\psi|A|\psi\rangle$:
expectation value of A for a quantum state $|\psi\rangle$;

(4) d: the number of all energy levels in energy shell [E- ΔE, E].

(5) $\Delta \langle A \rangle = \langle A \rangle - \langle A \rangle_{eq}$: deviation from the equilibrium value

The inequality was shown by assuming the Haar measure for the probability on the unit sphere (A. Sugita (2007)):

For a given quantum state

$$|\psi\rangle = \sum_j c_j |E_j\rangle, \quad \text{for } E_j \text{ in energy shell } [E-\Delta E, E],$$

we assume the probability of having $\{c_j\}$ as

$$P(\{c_j\}) = C\,\delta\left(1 - \sum_j |c_j|^2\right).$$

Part 2: Exact dynamics of 1D Bose gas

We observe practically the following:

(1) Relaxation for large systems: N=1000;

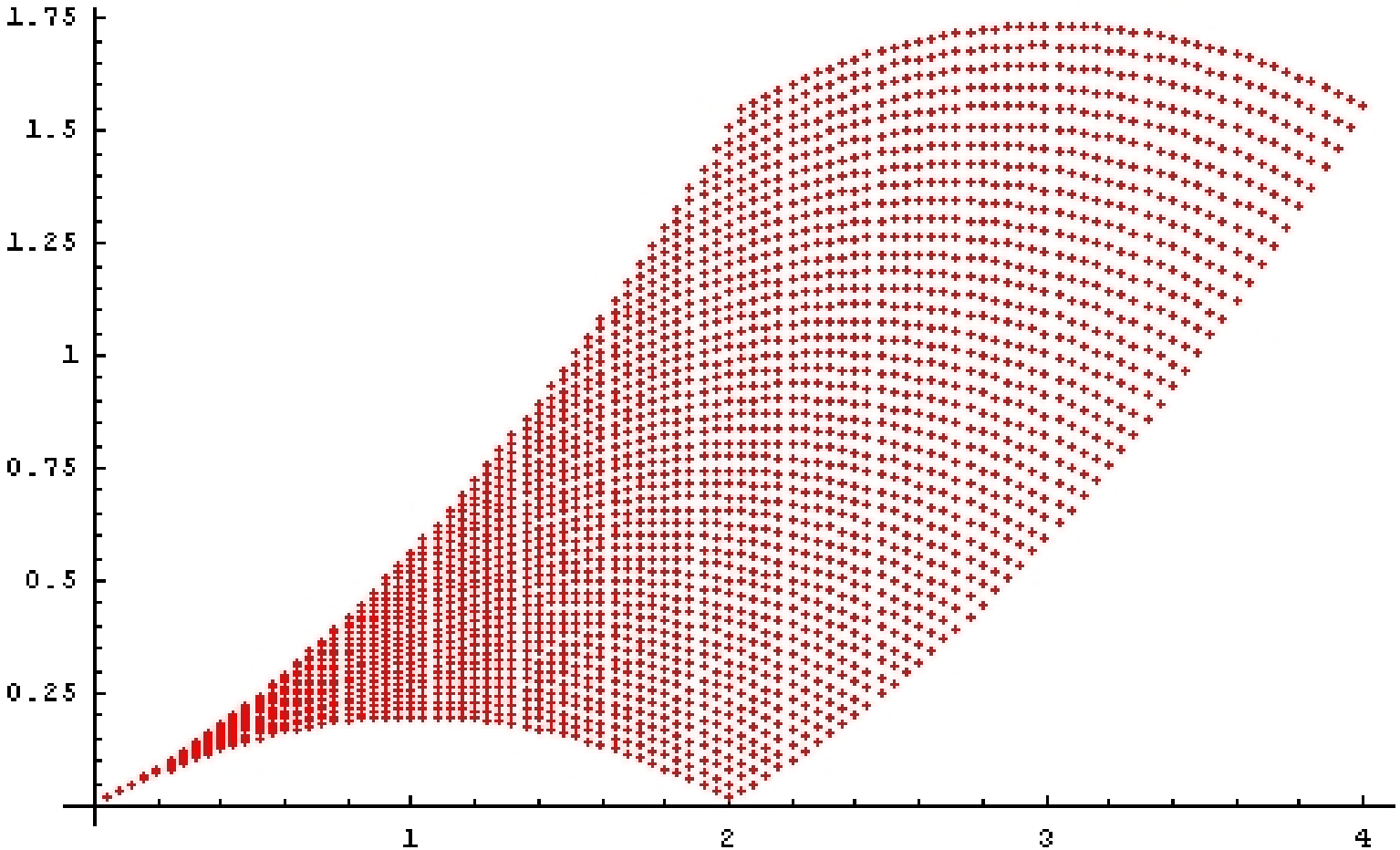
(2) Recurrence for small systems: N=20

Original animations produced by Jun Sato

J. Sato, R. Kanamoto, E. Kaminishi and T.D., PRL vol. 108, 110401 (2012)

Energy -Momentum Spectrum of 1D Bose Gas (c=100)

$$H_{Lieb-Liniger} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^{\,2}} + \sum_{j,k=1}^{N} c\delta(x_j - x_k)$$



Construction of the quantum state of a ``dark soliton'' (J. Sato et al., arXiv:1204.3960)

- We take superposition of excited states with on hole: $(q=0, 1, \cdots, N-1)$

$$|X, N\rangle = \sum_{p=0}^{N-1} \exp(2\pi i pq) |p\rangle$$

Here $|p\rangle$ denotes the Bethe eigenstate with one hole corresponding to momentum $2\pi p/L$, and $q=0, 1, \cdots, N-1$.

``Delta function'' becomes a ``dark soliton''

Time evolution of the density profile of 1D Bose gas

- The density operator $\rho(x) = \psi(x)^{\dagger}\psi(x)$
- The density profile at time t is calculated with the form factor expansion:

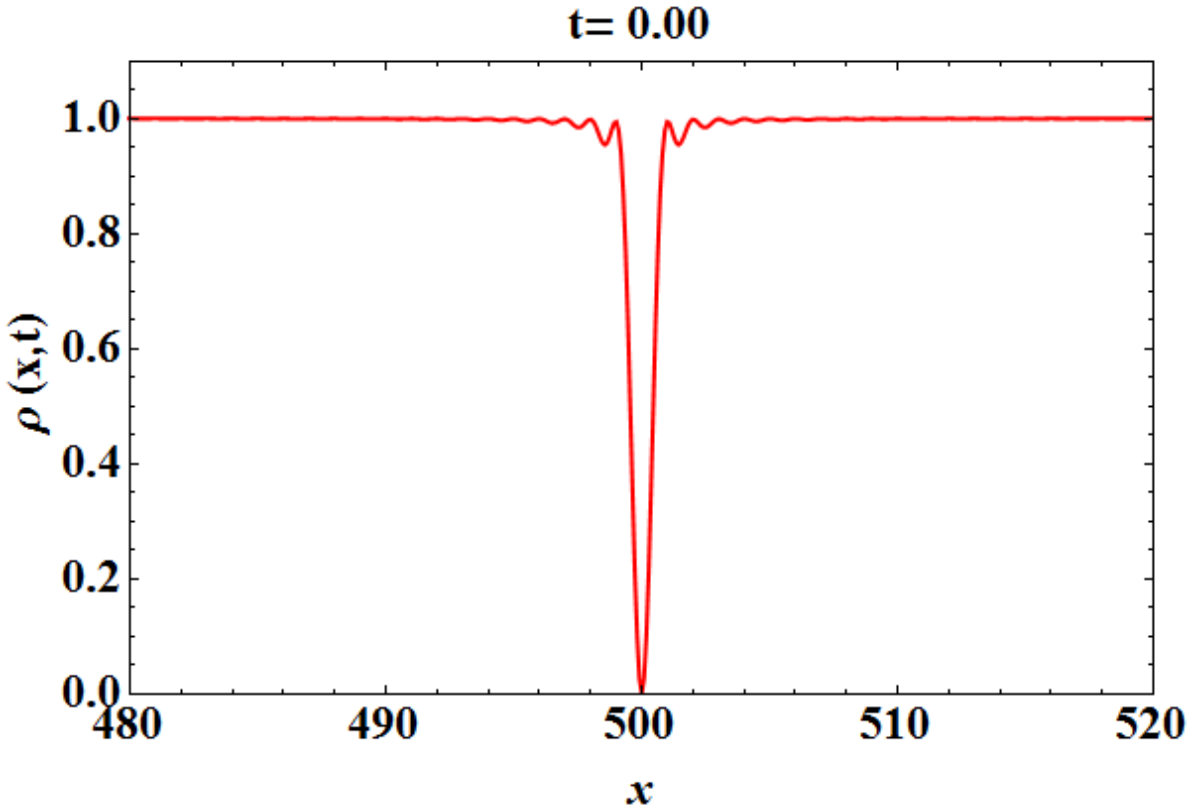
$$< X(t) \mid \rho(x) \mid X(t) >$$
$$= \frac{1}{N^2} \sum_{p,p'=0}^{N-1} \exp(i(P-P')x - i(E_p - E_{p'})t)$$
$$\times < p \mid \rho(0) \mid p' >$$

where $P = 2\pi p / L, P' = 2\pi p' / L$

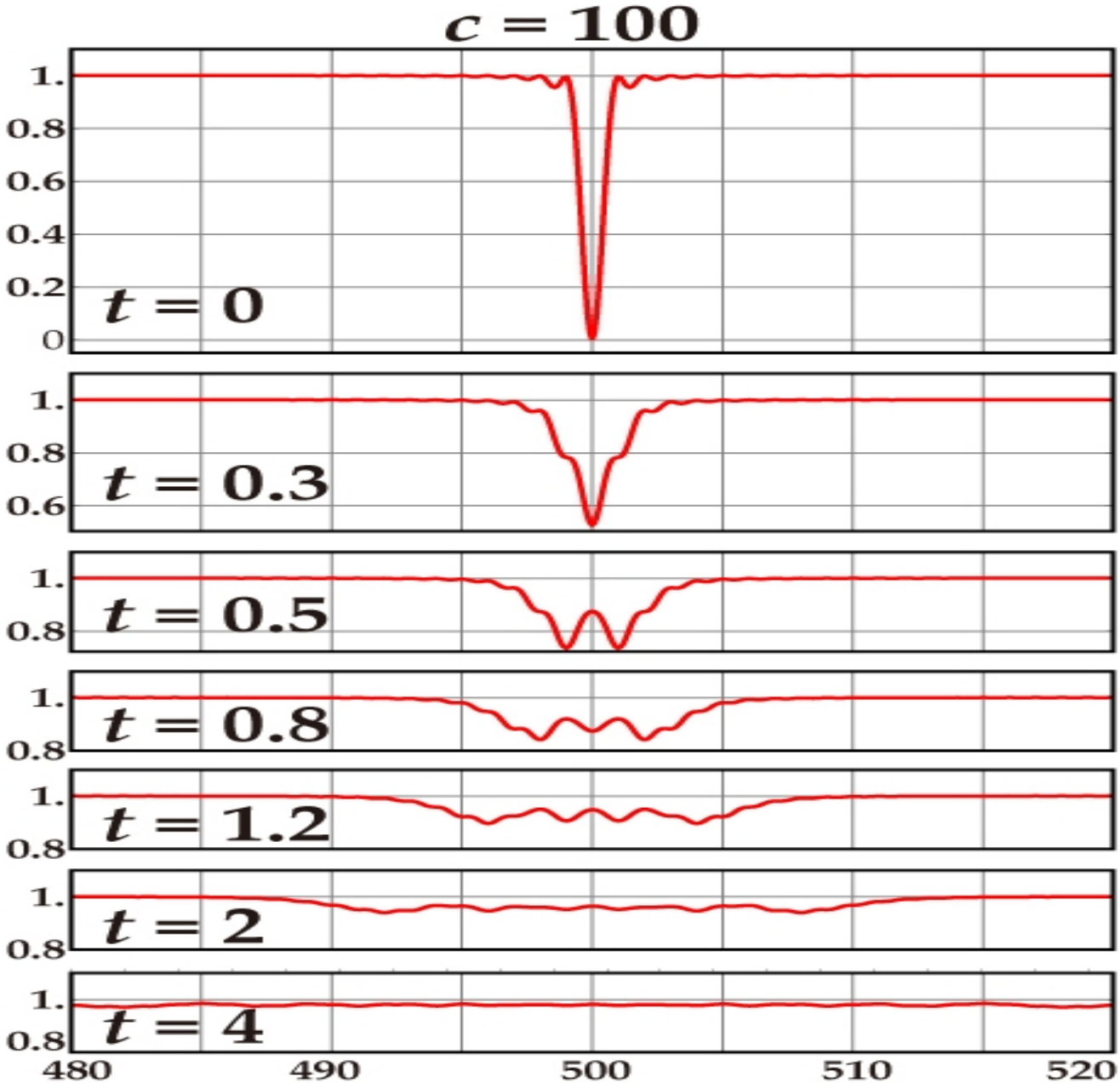
and q=0.

Relaxation for $N = 1000$; $\rho(x,t)$ density profile

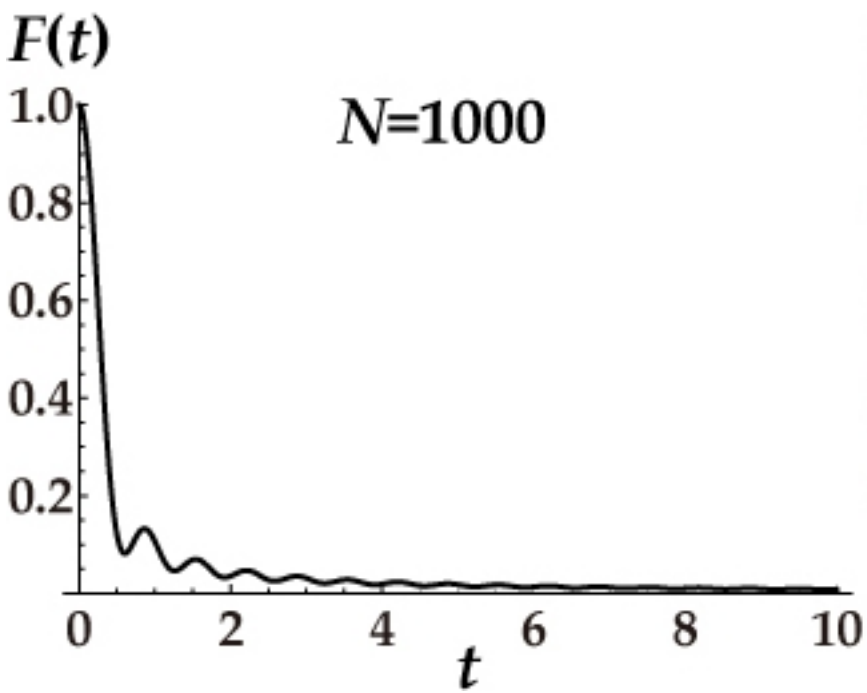
J. Sato, R. Kanamoto, E. Kaminishi and T.D., PRL (2012)

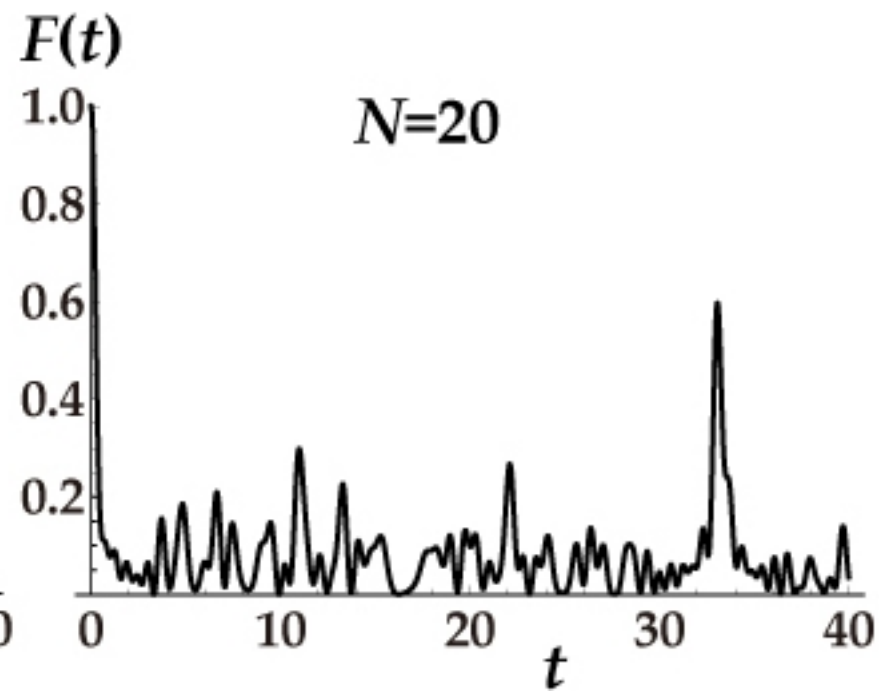


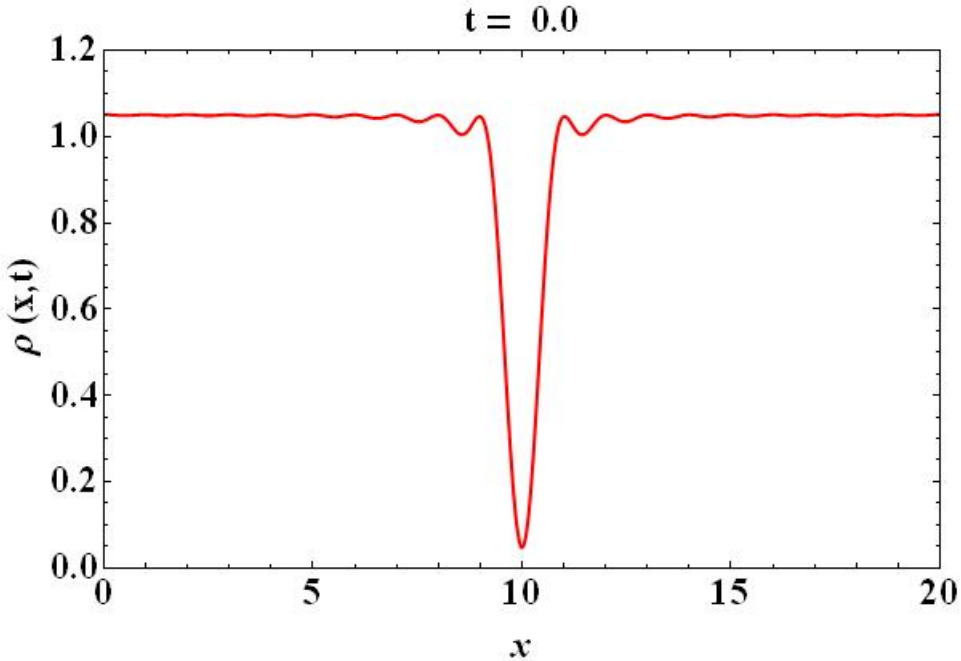
$c = 100$

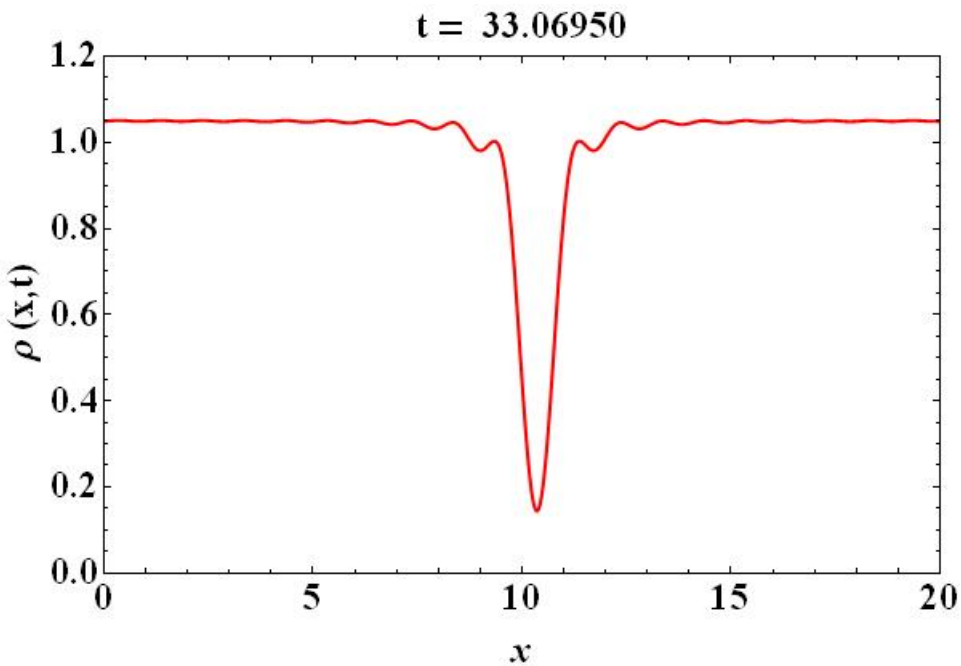

$c = 100$
$t = 0$
$t = 0.3$
$t = 0.5$
$t = 0.8$
$t = 1.2$
$t = 2$
$t = 4$
1.
0.8
0.6
0.4
0.2
0
480
490
500
510
520

Fidelity $F(t) = |\langle \psi(t) | \psi(0) \rangle|^2$
of a ``quantum localized state''
for large system (relaxation) N=1000
and for small system (recurrence) N=20






t = 0.0
1.2
1.0
0.8
0.6
0.4
0.2
0.0
ρ (x,t)
0
5
10
15
20
x


t = 33.06950
1.2
1.0
0.8
0.6
0.4
0.2
0.0
ρ (x,t)
0
5
10
15
20
x

Part 3: Expressions of Form factors of the XXZ chains

The XXZ Hamiltonian: spin-1/2 case

The Hamiltonian of the XXZ spin chain under the periodic boundary conditions (P.B.C.) is given by

$$\mathcal{H}_{\mathrm{XXZ}} = \sum_{j=1}^{L} \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right) . \tag{1}$$

Here σ_j^a $(a = X, Y, Z)$ denote the Pauli matrices defined on the jth site and Δ the anisotropy of the exchange coupling.
The P.B.C. are given by

$$\sigma_{L+1}^a = \sigma_1^a \qquad \text{for} \quad a = X, Y, Z.$$

Here we define

$$\Delta = (q + q^{-1})/2 \,, \qquad (q = \exp \eta, \quad \eta = -i\zeta).$$

$|\Delta| > 1$: massive regime
$|\Delta| < 1$: massless regime (CFT with $c = 1$).

Form factors and the spectral functions

[1] N. Kitanine, J.M. Maillet and V. Terras, Nucl. Phys. B **554** [FS] (1999) 647–678

Form factors: $\langle\mu|\sigma_m^z|\lambda\rangle, \quad \langle\mu|\sigma_m^\pm|\lambda\rangle$

Quantum Inverse Scattering + Scalar product formula

[2] M. Karbach et al., (2002);
J. Sato, M. Shiroishi and M. Takahashi, (2004);
J-.S. Caux, R. Hagemans and J.-M. Maillet (2005)

Algebraic Bethe ansatz: R-matrix and monodromy matrix

The R-matrix of the XXZ spin chain is given by

$$R_{ij}(u) = \varphi(u+\eta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[i,j]} . \tag{2}$$

In terms of $\varphi(x) = \sinh x$ we difine $b(u)$ and $c(u)$ by

$$b(u) = \varphi(u)/\varphi(u+\eta)\,, \quad c(u) = \varphi(\eta)/\varphi(u+\eta). \tag{3}$$

We define the monodromy matix $T(\lambda)$ by

$$T_{0,\,12\cdots L}(\lambda;\,\{w_j\}_L) = a(\lambda) R_{0L}(\lambda - w_L) \cdots R_{02}(\lambda - w_2) R_{01}(\lambda - w_1) \tag{4}$$

We denote the operator-valued matrix elements of the monodormy matrix as follows.

$$T_{0,\,12\cdots L}(\lambda;\,\{w_j\}_L) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} . \tag{5}$$

Let $|0\rangle$ denote the vacuum vector in which all spin are up. We have

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle\,, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle \tag{6}$$

where $w_j = \eta/2 + z_j$ ($\eta = -i\zeta$ and z_j are real for spin 1/2)

$$a(\lambda) = \prod_{j=1}^{L} \varphi(\lambda + \eta/2 - z_j)\,, \quad d(\lambda) = \prod_{j=1}^{L} \varphi(\lambda - \eta/2 - z_j). \tag{7}$$

The Bethe-ansatz equations

$$\frac{a(\lambda_j)}{d(\lambda_j)} = \prod_{k\neq j}^{n} \frac{b(\lambda_k - \lambda_j)}{b(\lambda_j - \lambda_k)} \tag{8}$$

Slavnov's formula for the scalar product

Scalar product: $\langle 0|C(\mu_1)\cdots C(\mu_n)\cdot B(\lambda_1)\cdots B(\lambda_n)|0\rangle$

Let us assume that μ_α satisfy the Bethe ansatz equations. We have

$$\begin{aligned}&\langle 0|C(\mu_1)\cdots C(\mu_n)\cdot B(\lambda_1)\cdots B(\lambda_n)|0\rangle\\ =\ &\frac{\prod_a^n d(\lambda_a)\det\hat{H}((\mu),(\lambda))}{\prod_{a<b}\varphi(\mu_a-\mu_b)\prod_{a<b}\varphi(\lambda_b-\lambda_a)}\end{aligned} \tag{9}$$

where

$$\hat{H}((\mu),(\lambda)) = \frac{\varphi(\eta)}{\varphi(\mu_a-\lambda_b)}$$
$$\times\left(a(\lambda_b)\prod_{a=1,a\neq\alpha}^{n}\varphi(\mu_a-\lambda_b+\eta)-d(\lambda_b)\prod_{a=1,a\neq\alpha}^{n}\varphi(\mu_a-\lambda_b-\eta)\right)$$

Spin-1/2 local operators

We consider spin-1/2 elementary operators $e^{\varepsilon',\varepsilon}$ for $\varepsilon',\varepsilon=0,1$, as follows.

$$e^{0,1}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad e^{1,0}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad e^{1,1}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

Let $e^{\varepsilon',\varepsilon}$ be 2 x 2 matrices with only nonzero element 1 at entry $(\varepsilon',\varepsilon)$ for $\varepsilon',\varepsilon=0,1$. The matrix elements (a,b) for $a,b=0,1$ are given by

$$\left(e^{\varepsilon',\varepsilon}\right)_{a,b}=\delta_{\varepsilon',a}\delta_{\varepsilon,b}\,. \tag{10}$$

We define $e_k^{\varepsilon',\varepsilon}$ on the kth site of the L-site chain for $k=1,2,\ldots,L$.

$$e_k^{\varepsilon',\varepsilon}=I^{\otimes(k-1)}\otimes e^{\varepsilon',\varepsilon}\otimes I^{\otimes(L-k)}\,. \tag{11}$$

We consider the mth product of elementary matrices

$$e_1^{\varepsilon_1',\varepsilon_1}\cdots e_m^{\varepsilon_m',\varepsilon_m}\,. \tag{12}$$

For given sets $\{\varepsilon'_\alpha\}$ and $\{\varepsilon'_\beta\}$, we introduce sets $\boldsymbol{\alpha}^\pm$ as follows.

$$\begin{aligned} \boldsymbol{\alpha}^- &= \{\alpha;\, \varepsilon'_\alpha = 1,\, 1 \le \alpha \le m\}, \\ \boldsymbol{\alpha}^+ &= \{\beta;\, \varepsilon_\beta = 0,\, 1 \le \beta \le m\}\,. \end{aligned} \tag{13}$$

We denote by r and r'
$r = \boldsymbol{\alpha}^-$ and $r' = \boldsymbol{\alpha}^+$

Let us denote by Σ_m the set of integers $1, 2, \ldots, m$, i.e.
$\Sigma_m = \{1, 2, \ldots, m\}$.

In terms of $\boldsymbol{\alpha}^\pm$ we express the product of elementary operators as

$$e_1^{\varepsilon'_1,\varepsilon_1} \cdots e_m^{\varepsilon'_m,\varepsilon_m} = \prod_{\alpha\in\boldsymbol{\alpha}^-} \sigma_\alpha^- |0\rangle\langle 0| \prod_{\beta\in\Sigma_m\setminus\boldsymbol{\alpha}^+} \sigma_\beta^+\,. \tag{14}$$

Result 1

Form factors (matrix elements) between Bethe states

Suppose that $\{\lambda_\alpha\}_M$ and $\{\mu_\alpha\}_N$ satisfy the Bethe-ansatz equations For a given mth product of spin-1/2 elementary operators $e_1^{\varepsilon_1^{'},\varepsilon_1}\cdots e_m^{\varepsilon_1^{'},\varepsilon_m}$ we define the matrix element $F_m^{(1,p)}$ between the co-vector $\langle\{\mu_\alpha\}_N|$ and the Bethe eigenvector $|\{\lambda_\beta\}_M\rangle$ by

$$F_m^{(1\,p)}(\{\varepsilon_\alpha^{'}\}_m;\{\varepsilon_\beta\}_m) = \langle 0|\prod_{\alpha=1}^{N} C(\mu_\alpha)\, e_1^{\varepsilon_1^{'},\varepsilon_1}\cdots e_m^{\varepsilon_1^{'},\varepsilon_m}\prod_{\beta=1}^{M} B(\lambda_\beta)|0\rangle \quad (15)$$

Here $N = M + P$.

We can show rigorously the following expression of $F_m^{(1\,p)}(\{\varepsilon_\alpha^{'}\}_m;\{\varepsilon_\beta\}_m)$.

$$
\begin{aligned}
&\prod_{1\le j<k\le m}\frac{1}{\varphi(w_j-w_k)}\prod_{1\le\alpha<\beta\le M}\frac{1}{\varphi(\lambda_\beta-\lambda_\alpha)}\cdot\frac{\hat{\Lambda}_m^{-1}(\{\lambda_\gamma\}_M)}{\hat{\Lambda}_m^{-1}(\{\mu_\gamma\}_N)}\\
&\times\prod_{j\in\boldsymbol{\alpha}^-}\left(\sum_{a_j=1}^{N}\right)\sum_{t'=0}^{r'}\sum_{\boldsymbol{\alpha}_J^+\subset\boldsymbol{\alpha}^+}^{|\boldsymbol{\alpha}_J^+|=t'}\prod_{j\in\boldsymbol{\alpha}}\left(\sum_{a_j'=1}^{N}\right)\prod_{j\in\boldsymbol{\alpha}_K^+}\left(\sum_{a_j'=N+1}^{N+j}\right)\\
&\times\frac{\prod_{j\in\boldsymbol{\alpha}^-}\left(\prod_{k=1}^{j-1}\varphi(\mu_{a_j}-w_k+\eta)\prod_{k=j+1}^{m}\varphi(\mu_{a_j}-w_k)\right)}{\prod_{1\le j<k\le m+P}\varphi(\mu_{b_k}-\mu_{b_j}+\eta)}\\
&\times\prod_{j\in\boldsymbol{\alpha}^+}\left(\prod_{k=1}^{j-1}\varphi(\mu_{a_j'}-w_k-\eta)\prod_{k=j+1}^{m}\varphi(\mu_{a_j'}-w_k)\right)\\
&\times(-1)^{r-t}\,\sigma(P_1P_2P_3)\,\frac{\prod_{k=1}^{m+P}\prod_{\alpha=1}^{N}\varphi(\mu_\alpha-\mu_{b_k}+\eta)}{\prod_{j=1}^{m}\prod_{\alpha=1}^{N}\varphi(\mu_\alpha-w_j+\eta)}\\
&\times\frac{\det\widehat{H}_M((\mu_{z(\alpha)})_{N-t}\#(w_{\kappa_j})_{t-P},\,\{\lambda_\gamma\}_M)}{\prod_{1\le\alpha<\beta\le N}\varphi(\mu_{z(\alpha)}-\mu_{z(\beta)})}\prod_{k=1}^{m+P}a(\mu_{b_k}).
\end{aligned}
\tag{16}
$$

Here $t = r + t'$ and $\sigma(P_j)$ for $j = 1, 2, 3$ are given by

$$\begin{aligned}
\sigma(P_1) &= \prod_{j \in \boldsymbol{\alpha}_J^+} \prod_{k \in \boldsymbol{\alpha}_K^+; k<j} (-1) \\
\sigma(P_2) &= \prod_{j,k \in \boldsymbol{\alpha}_K^+, a_j' < a_k'} (-1) \\
\sigma(P_3) &= \prod_{j \in \boldsymbol{\alpha}_K^+} \prod_{\kappa \in \widehat{W}; \kappa + N < a_j'} (-1)
\end{aligned} \tag{17}$$

We define sequence $(b_k)_{r+r'}$ by

$$(b_1, b_2, \ldots, b_{r+r'}) = (a'_{j'_{\max}}, \ldots, a'_{j'_{\min}}, a_{j_{\min}}, \ldots, a_{j_{\max}}) .$$

We define set Z by

$$Z = \Sigma_N \setminus a(\boldsymbol{\alpha}_J^+) \cup a(\boldsymbol{\alpha}^-). \tag{18}$$

We introduce set $\widehat{a}(\boldsymbol{\alpha}_K^+)$ by $\widehat{a}(\boldsymbol{\alpha}_K^+) = \{a_k' - N; k \in \boldsymbol{\alpha}_K^+\}$. We define set $\widehat{W}$ by

$$\widehat{W} = \Sigma_m \setminus \widehat{a}(\boldsymbol{\alpha}_K^+)\,. \tag{19}$$

We define a sequence $(z(\alpha))_{N-t}$ by putting the elements of Z in increasing order: $z(1) < z(2) < \cdots < z(N-t)$ where $Z = \{z(i); i = 1, 2, \ldots, N-t\}$. We define a sequence $(\kappa_j)_t$ by putting the elements of $\widehat{W}$ in increasing order: $\kappa_1 < \kappa_2 < \cdots < \kappa_{t-P}$ where $\widehat{W} = \{\kappa_j;\, j = 1, 2, \ldots, t-P\}$.

$\widehat{\Lambda}_m(\{\lambda_\gamma\})$ denotes the m th product of the eigenvalues of the transfer matrix.

Some more details will be given in
T. Deguchi, in preparation.

Possible physical applications

(i) Evaluation of the reduced density matrices

(ii) Non-equilibrium dynamics of the XXX and XXZ spin chains

Result 2

Spin-$\ell/2$ form factors (matrix elements) between Bethe states through the fusion method

Suppose that $\{\lambda_\alpha\}_M$ and $\{\mu_\alpha\}_N$ satisfy the BAEs. For a given product of spin-$\ell/2$ elementary operators $\widehat{E}_1^{a,\,b\,(\ell\,w)}$ we define the matrix element $F_{a,b}^{(\ell,\,w)}$ between the Bethe eigenvectors $\langle\{\mu_\alpha\}_N|$ and $|\{\lambda_\beta\}_M\rangle$ by

$$
\begin{aligned}
& F_m^{(\ell\,w)}(\{a_j\},\{b_j\}) \\
= \quad & \langle 0| \prod_{\alpha=1}^{N} C^{(\ell\,w)}(\mu_\alpha)\, \widehat{E}_1^{a_1,\,b_1\,(\ell\,w)} \cdots \widehat{E}_m^{a_m,\,b_m\,(\ell\,w)} \prod_{\beta=1}^{M} B^{(\ell\,w)}(\lambda_\beta)|0\rangle
\end{aligned}
$$

They are consistent with the multiple-integral rep of correlation functions
[1] T. D. and C. Matsui, Nucl. Phys. B **831**[FS] (2010) 359–407.
[2] T. D., JSTAT (2012) P04001

Details will be given in : T. Deguchi, in preparation.

Part 4: A numerical study on the relaxation dynamics of the XXZ chains

Toward an ultimate goal:

Checking Equilibration to GGE

(1) Time evolution of fidelity

(2) Time evolution of local magnetization $< \sigma^z_m >$

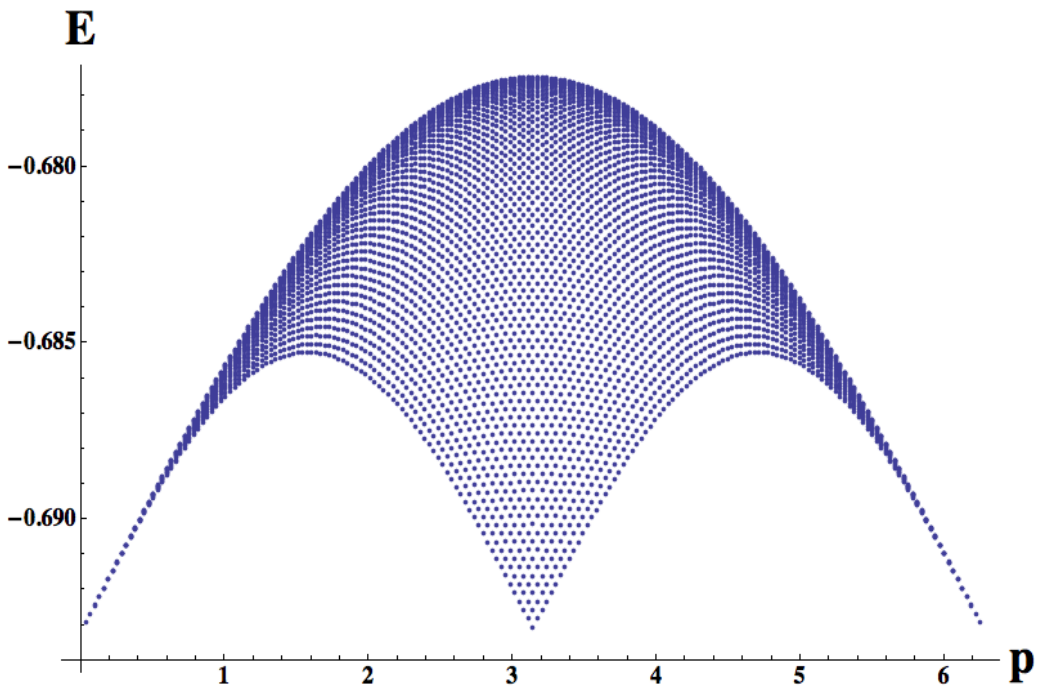
(1) Time evolution of fidelity for real and complex solutions of BAE for the spin-1/2 XXX chain (M=N/2-1)

M: the number of down spins

- Spinons; kinks, lowest excitations of spin-½ XXX chain (N^2 states)
- We consider quantum states with the sum of

(i) all spinons with equal weight: all-spinon state;
all spinons with random weights

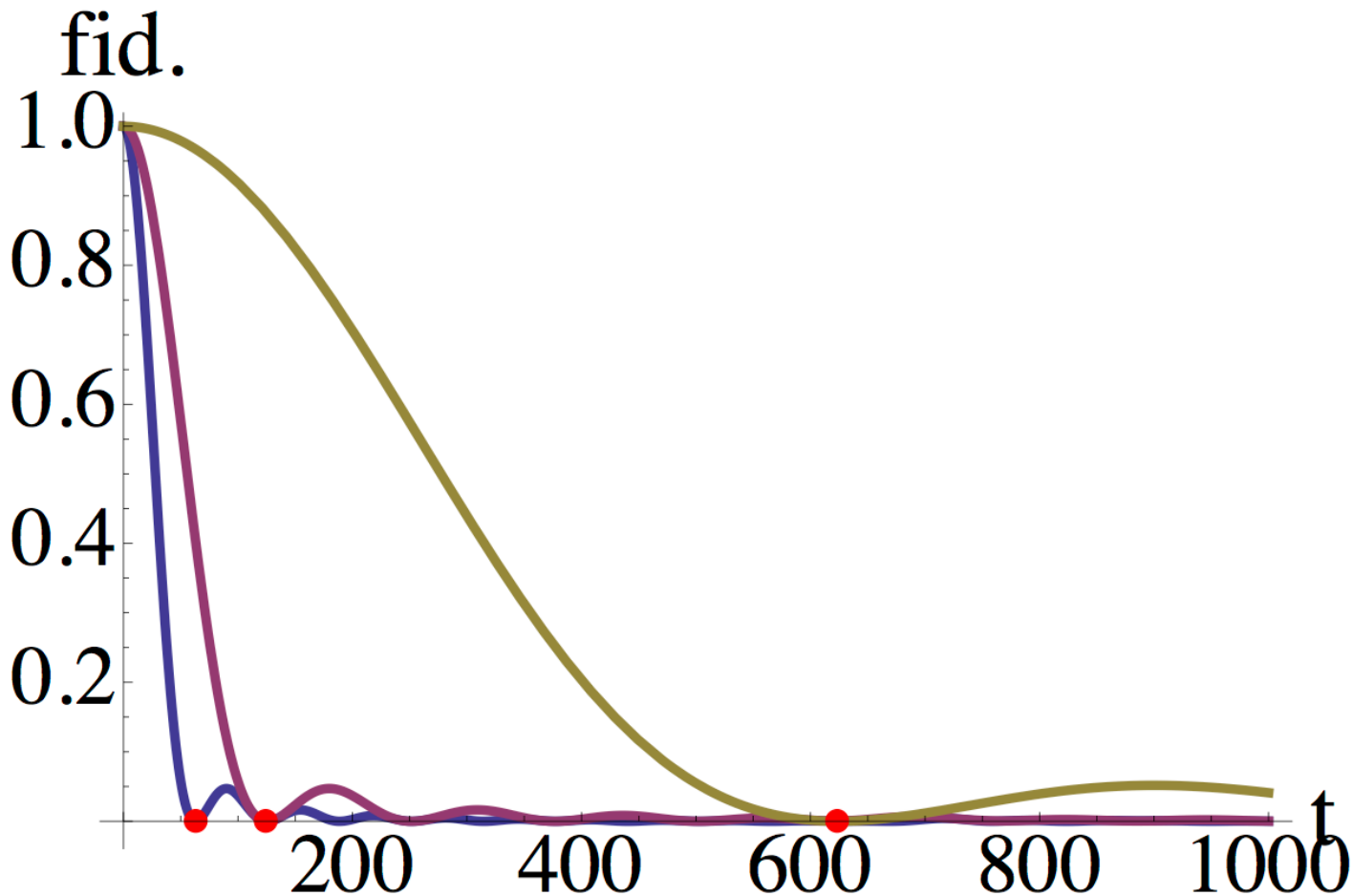
(ii) 2-string solutions (bound states)



Time evolution of the fidelity for spinons of the spin-1/2 XXX chain: N=1000 and M=N/2-1=499. ΔE＝0.01, 0.05, 0.1

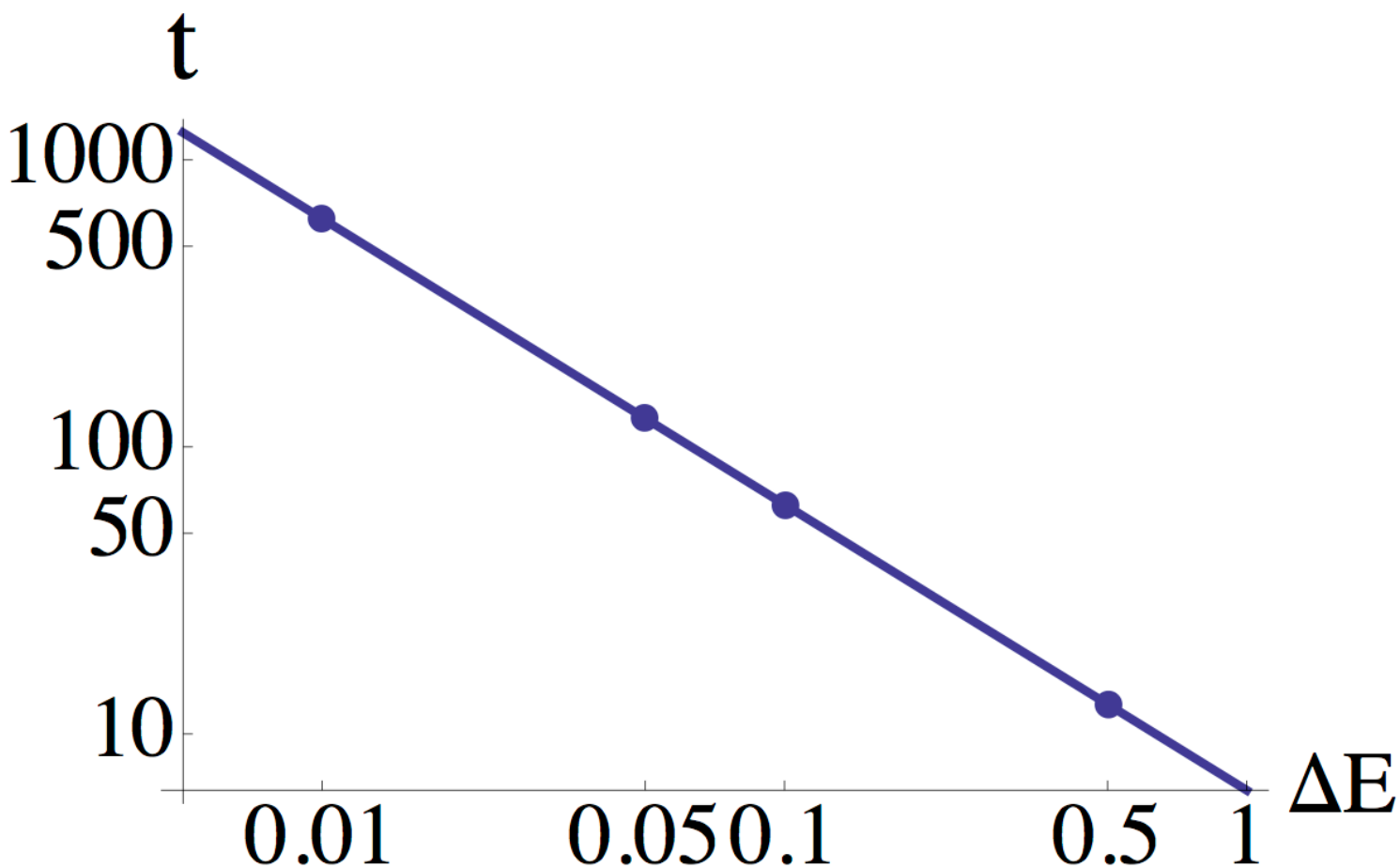
We can fit it with Monnai' s approximate formula of fidelity

(Cf. T. Monnai, arXiv: 1403.6578)

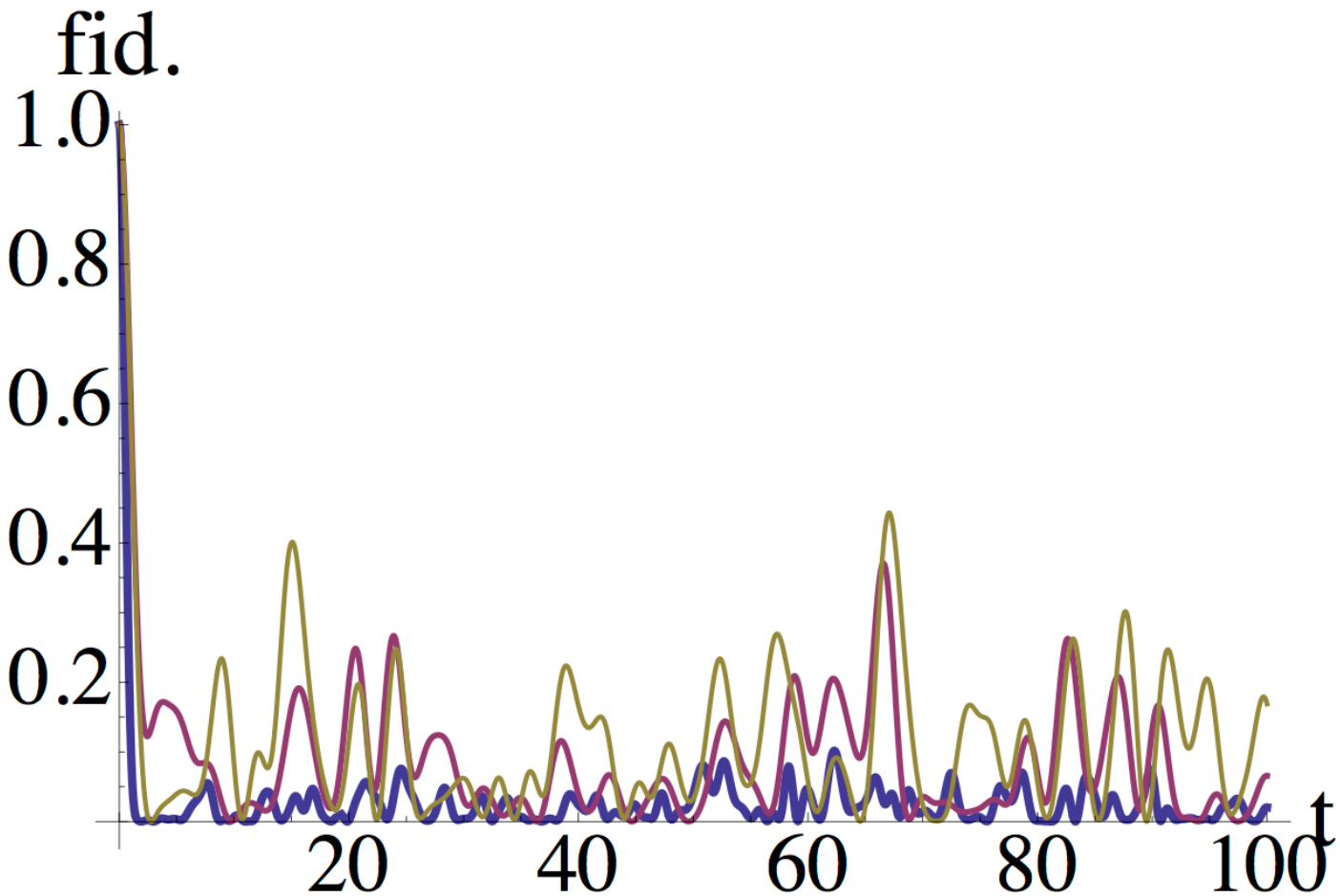


Relaxation time of fidelity versus energy width $\triangle E$ for real BAE solutions（by Ryoko Hatakeyama, Ochanomizu Univ., B4）:

It is given by the Boltzmann time , consistent with rigorous study for relaxation of generic systems: S. Goldstein, T. Hara, and H. Tasaki, arXiv:1402.0324 : $T_R \cong h/\triangle E$



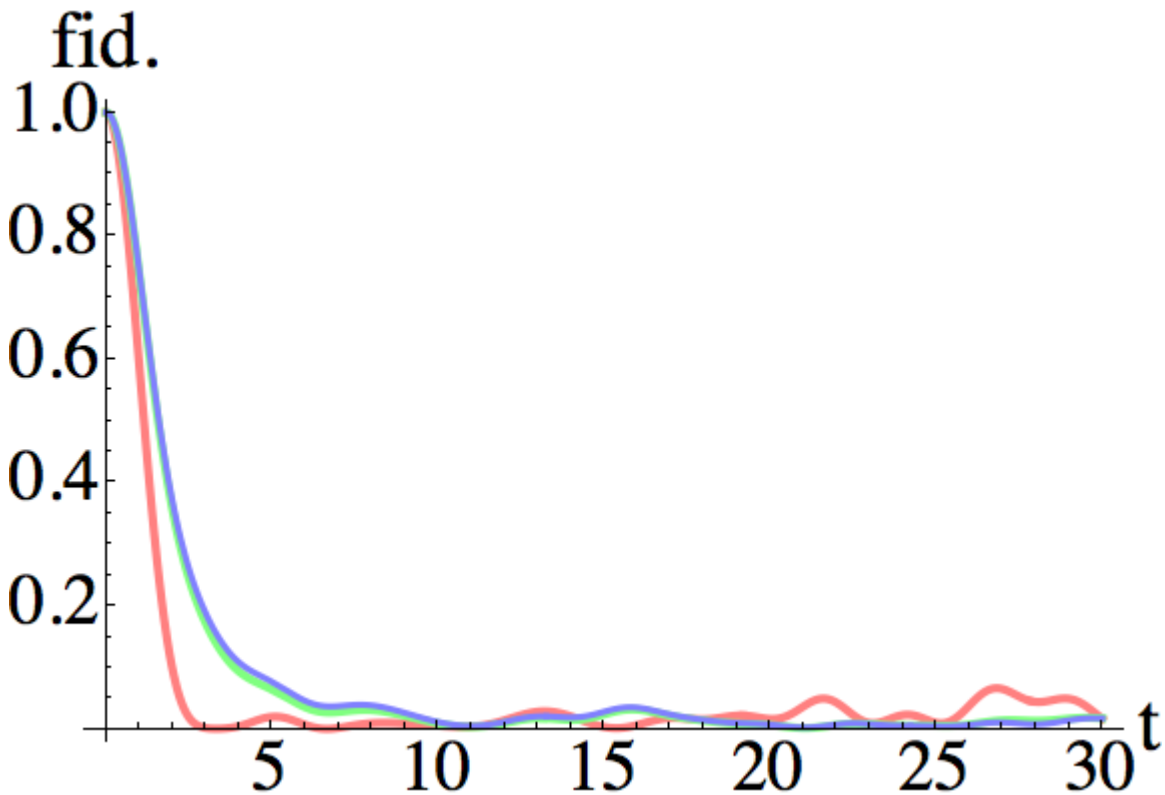
Fidelity of N=10 XXX chain: (M=N/2-1) all solutions (purple); only real solutions (yellow); all string solutions in the same range as real ones (red) (R. Hatakeyama: BAE solutions by R. Hagemans and J.-S. Caux (2007) are checked by P.R. Giri (See also T.D and P.R. Giri, arXiv:1408.7030)



Fidelity of N=20 XXX chain

only real solutions (pink); real & real + 2-string solutions in the same range as real ones (green); real & real + 2-string & real + 3 string & real + 2 x 2-string in the same range as real ones (blue) (M=N/2-1)

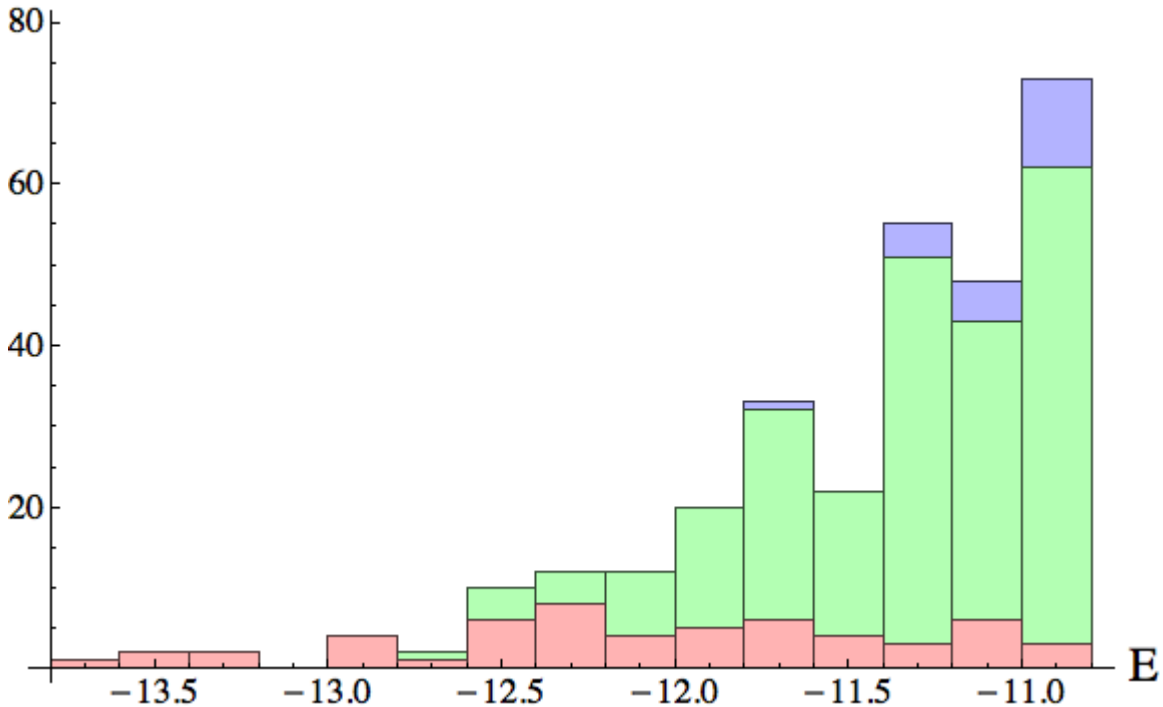
n-strings are approximated by Takahashi-strings (R. Hatakeyama)



Histgram of solutions of BAE for N=20:

Only real solutions (pink); real & real + 2-string in the same range as real ones (green); real & real + 2-string & real + 3 string & real + 2 x 2-string in the same range as real ones (blue)

n-strings are approximated by Takahashi-strings



(2) Time evolution of local magnetization $< \sigma^z_m >$

We transform the form factor of σ^z_m by factoring out the Cauchy determinant (in the XXZ spin chain)

- $< \mu | \sigma^z_m | \lambda >$
  $=$ "Cauchy det $(\mu - \lambda)$" $*$ det$(I + U)$

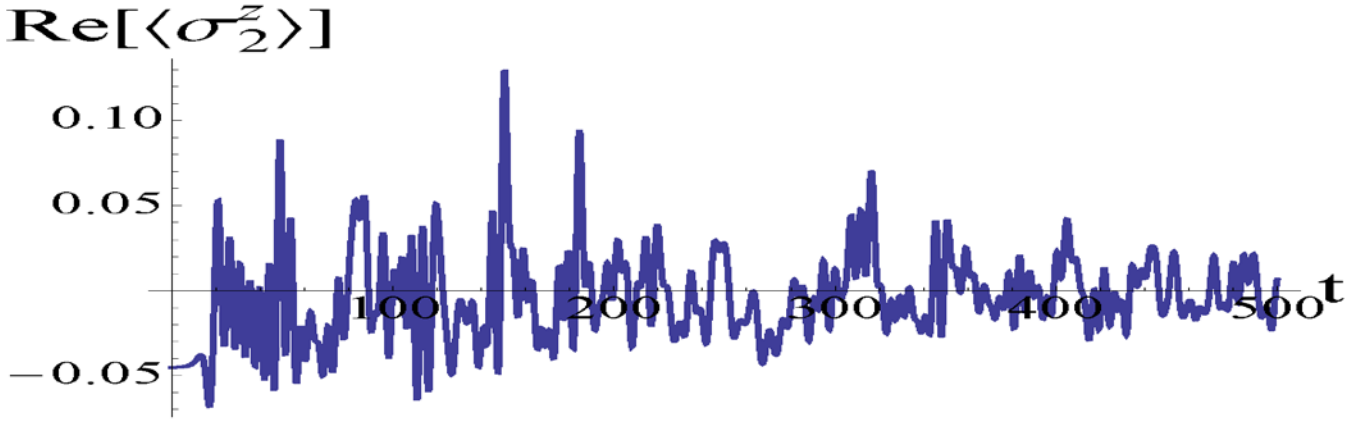
Here $| \mu >$ and $| \lambda >$ are Bethe eigenstates of the XXZ spin chain.

The above formula holds for real and complex rapidities. It is practically useful.

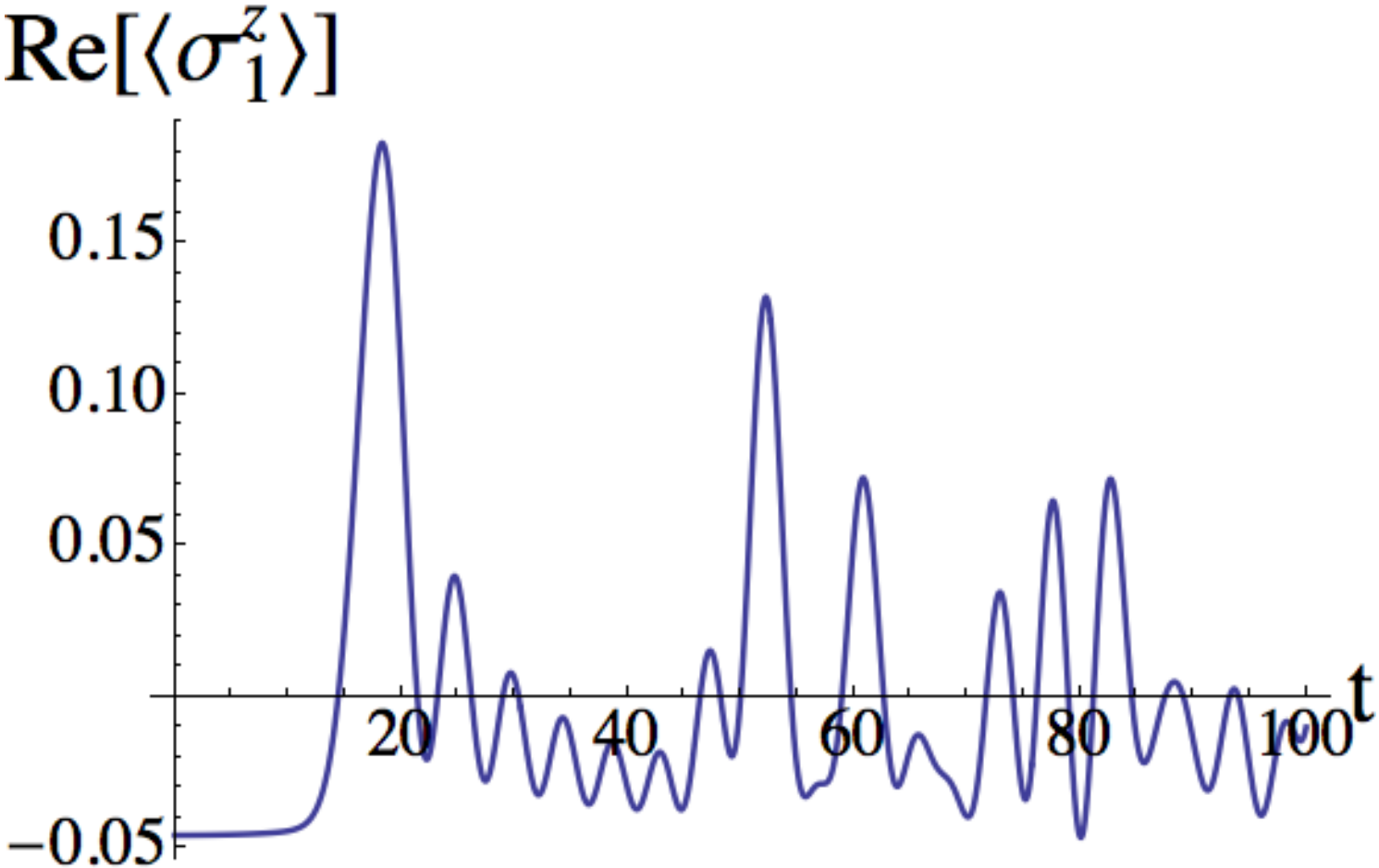
The det (I + U) leads to a Fredholm determinant in the large *N* limit .

Time evolution of local magnetization: Re$\langle \sigma^z_m \rangle$ for m=2 (N=50, T=500)

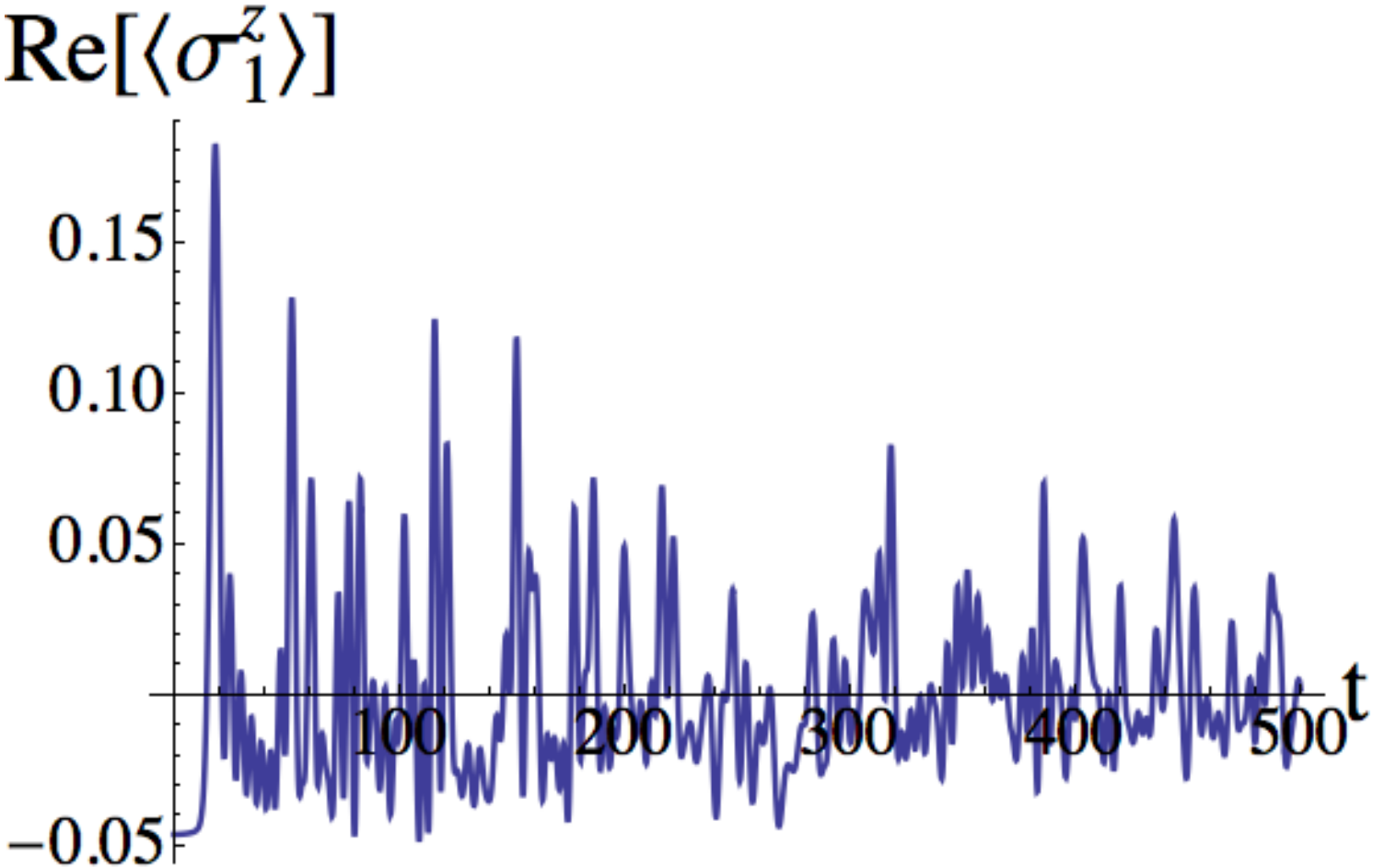
Oscillations do not vanish (by R. Hatakeyama)



Re< σ^z_m> with m=1 (N=50) for the all-spinon state



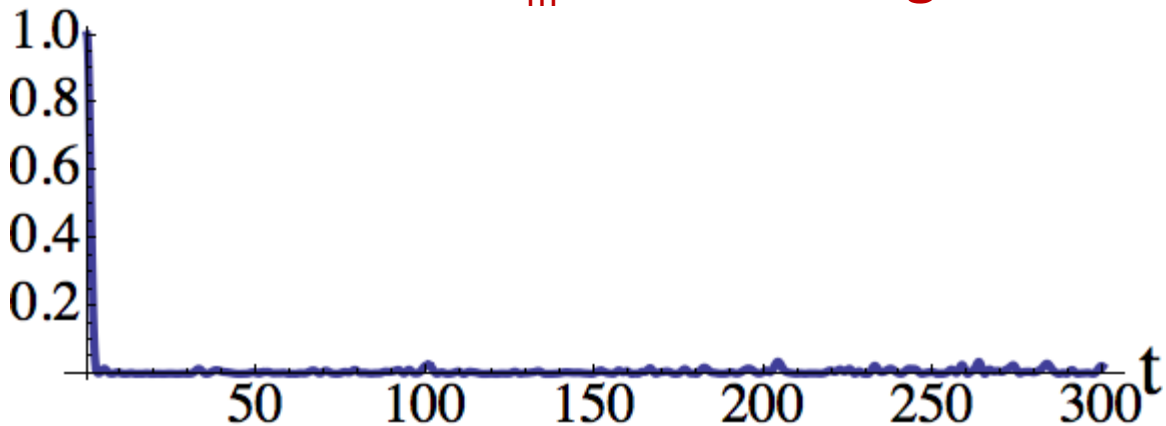
Relaxation of local magnetization: Re< σ^z_m> with m=1 (N=50) for the all-spinon state: Thermal fluctuations of the order of $1/\sqrt{N}$ remain after relaxation (by Ryoko Hatakeyama) T=500

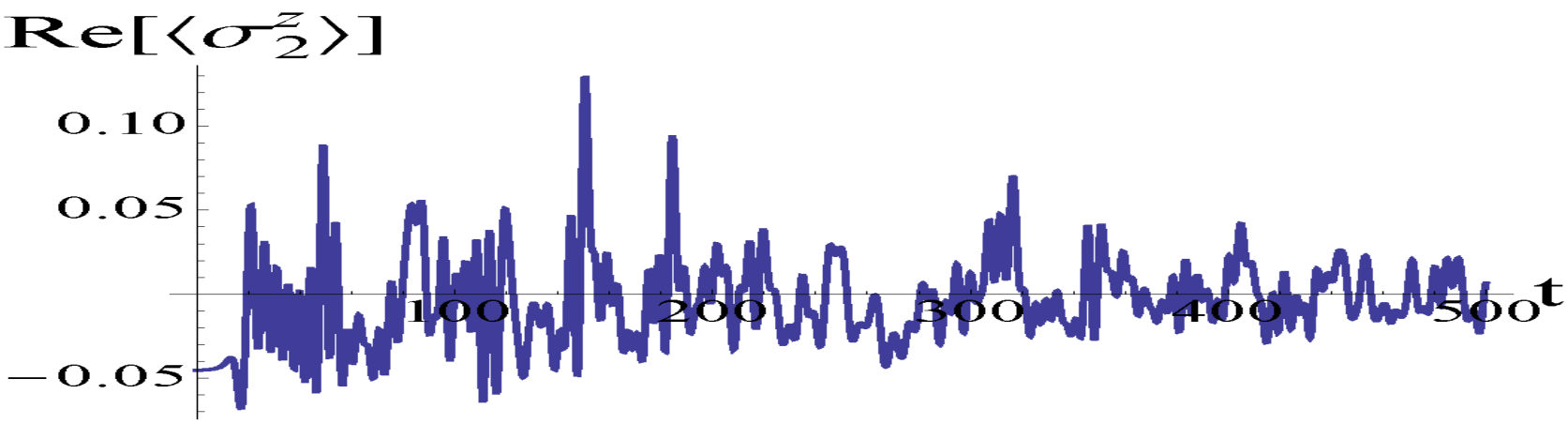


Fidelity versus local magnetization for the all-spinon state:

Re< σ^z_m> for m=2 (N=50, T=500) (by Ryoko Hatakeyama)

Relaxation time of Re< σ^z_m> is much longer than that of fidelity.





Conclusions

(1) Relaxation in quantum systems is associated with **typicality** .
We study it in quantum integrable models with tools such as scalar products.

(2-1) Exact relaxation (equilibration) dynamics of the density profile for 1D Bose gas in an initially localized quantum state with ***N*=1000**.

(2-2) Recurrence may appear for 1D Bose gas with ***N*=20**

(3) Formulas for form factors and the reduced density matrices

(4-1) Time evolution of fidelity may depend on the initial state:
However, we have similar results for real and complex solutions in XXX chain.

(4-2) Relaxation time of fidelity of the XXX chain (for spinons and n-strings)
$T_R \cong h/\triangle E$: the **Boltzmann time** -> it is very fast, consistent with
[Ref]. S. Goldstein, T. Hara, and H. Tasaki, arXiv:1402.0324 .

(4-3) Time evolution of local magnetization for the XXX spin chain has been explicitly observed with an **improved expression of the form factor.**
The relaxation time of $<\sigma^z_m>$ is much longer than that of fidelity.

Thank you