Recent Advances in Quantum Integrable Systems

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Universal results for two-dimensional percolation

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Based on :

GD, J. Viti, J. Cardy, Universal Amplitude Ratios of Two-Dimensional Percolation from field Theory, J. Phys. A 43 (2010) 152001

GD, J. Viti, On three-point connectivity in two-dimensional percolation, J. Phys. A 44 (2011) 032001

GD, J. Viti, Potts q-color field theory and scaling random cluster model, Nucl. Phys. B 852 (2011) 149

GD, J. Viti, Crossing probability and number of crossing clusters in off-critical percolation, J. Phys. A 45 (2012) 032005

GD, Parafermionic excitations and critical exponents of random cluster and O(n) models, Ann. Phys. 333 (2013) 1

M. Picco, R. Santachiara, J. Viti, GD, Connectivities of Potts Fortuin-Kasteleyn clusters and time-like Liouville correlator, Nucl. Phys. B 875 (2013) 719

percolation easily formulated but theoretically challenging, even in 2D; proper CFT still missing
critical-boundary sector best known [Cardy, ..., S.Smirnov]

This talk

three basic questions for the other sectors of the theory:

- critical, bulk: connectivity constant

- off-critical, bulk: amplitude ratios

- off-critical, boundary: crossing probabilities

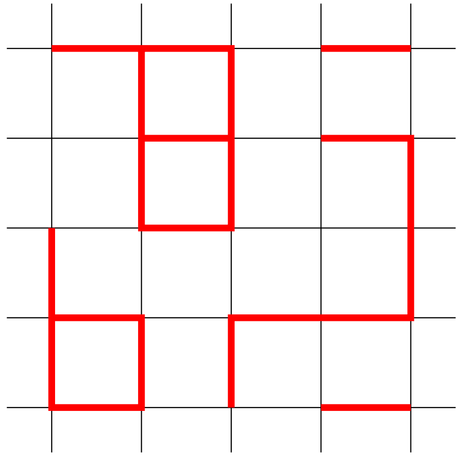
and also:

- critical exponents from field theory

Random percolation

bond present with prob. p

$W = p^{\#\,\text{bonds}}(1-p)^{\#\,\text{absent bonds}}$



for $p > p_c$ there is a probability $P > 0$ that a site belongs to a cluster extending to infinity

- prototype for 'geometrical' phase transitions
- $c = 0$ CFT at p_c in 2D

Percolation as the $q \to 1$ Potts model

Potts model: $\mathcal{H} = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}$, $\quad s(x) = 1, \ldots, q$

$$Z = \sum_{\{s(x)\}} e^{-\mathcal{H}} = \quad \text{[Fortuin, Kasteleyn, '69]} \qquad p = 1 - e^{-J}$$

$$\propto \sum_{\text{bond configs}} p^{\#\,\text{bonds}} (1-p)^{\#\,\text{absent bonds}} \, q^{\#\,\text{clusters}}$$

connectivities: $P_n(x_1, \ldots, x_n) =$ prob. $x_1, ..., x_n$ in same cluster

$$\sigma_\alpha(x) \equiv q\delta_{s(x),\alpha} - 1$$

$$\langle \delta_{s(x_1),\alpha} \delta_{s(x_2),\alpha} \rangle = \frac{P_2(x_1,x_2)}{q} + \frac{1-P_2(x_1,x_2)}{q^2} \quad \Rightarrow \quad \langle \sigma_\alpha \sigma_\alpha \rangle = (q-1)P_2$$

similarly $\quad \langle \sigma_\alpha \sigma_\alpha \sigma_\alpha \rangle = (q-1)(q-2)P_3$

Critical exponents from field theory [GD, '13]

- Potts exponents determined from lattice [Nienhuis, '82]

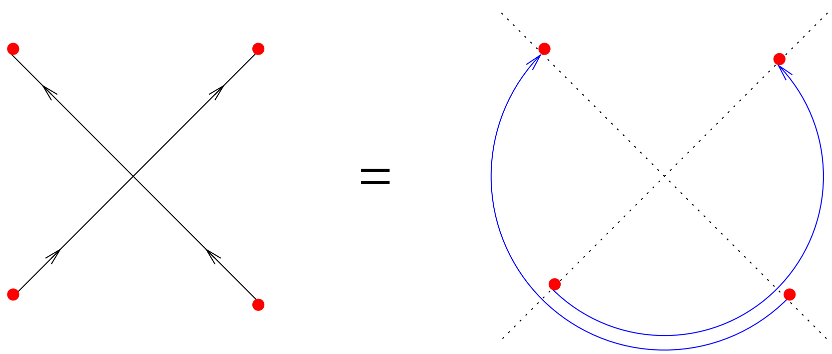
- CFT identification in Dotsenko, Fateev, '84

- problem for self-contained CFT derivation: implementing S_q for q real

scale invariant scattering in 1+1 dimensions:

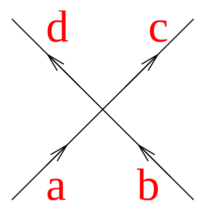
right/left movers created by chiral (parafermionic) fields ψ, $\bar{\psi}$

no dimensionless Lorentz invariant in RL scattering $\Rightarrow$ S=constant



$$S = e^{i\pi(s_{\bar{\psi}} - s_\psi)} = e^{-2i\pi\Delta_\psi}$$

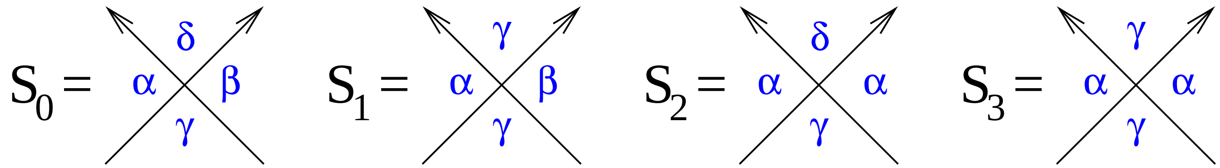
in general S_{ab}^{cd} =



unitarity: $S_{ab}^{ef}\left[S_{ef}^{cd}\right]^* = \delta_a^c \delta_b^d$

crossing: $S_{ab}^{cd} = \left[S_{a\bar{d}}^{c\bar{b}}\right]^*$

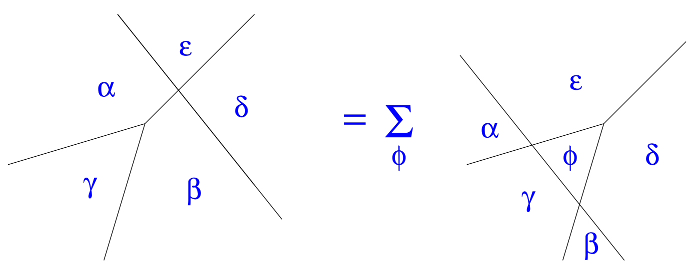
S_q **symmetry :** trajectories as cluster boundaries



$$S_0 = S_0^* \equiv \rho_0 \qquad S_1 = S_2^* \equiv \rho e^{i\varphi} \qquad S_3 = S_3^* \equiv \rho_3$$

$$(q-3)\rho_0^2 + \rho^2 = 1 \qquad (q-4)\rho_0^2 + 2\rho_0\rho\cos\varphi = 0$$
$$(q-2)\rho^2 + \rho_3^2 = 1 \qquad (q-3)\rho^2 + 2\rho\rho_3\cos\varphi = 0$$

Branching:



$S_1 = S_2^2 + (q-3)S_2S_0, \ldots$

$$\Rightarrow \quad \rho_0 = -1 \qquad \rho = \sqrt{4-q} \qquad 2\cos\varphi = -\sqrt{4-q} \qquad \rho_3 = q-3$$

$$S = S_3 + (q-2)S_2 = e^{-4i\varphi}$$

in CFT, requiring Ising as special case, self-duality, locality w.r.t. ε, $\varepsilon \cdot \sigma \sim \sigma$, gives

$$\Delta_\varepsilon = \Delta_{2,1} \qquad \Delta_\sigma = \Delta_{\frac{t-1}{2},\frac{t+1}{2}} \qquad \Delta_\psi = \Delta_{1,3}$$

with $\Delta_{m,n} = \frac{[(t+1)m-tn]^2-1}{4t(t+1)}$ $\qquad c = 1 - \frac{6}{t(t+1)}$

$$e^{-4i\varphi(q)} = e^{-2i\pi\Delta_\psi(t)} \quad \Rightarrow \quad \sqrt{q} = 2\sin\frac{\pi(t-1)}{2(t+1)}$$

(can be repeated for $O(n)$)

- $\Delta_\sigma|_{q=1} = \Delta_{1/2,3/2}$ $\Rightarrow$ no diff. eqs for bulk connectivities, contrary to the boundary ones ($\Delta_\sigma|_{boundary} = \Delta_{1,3}$)

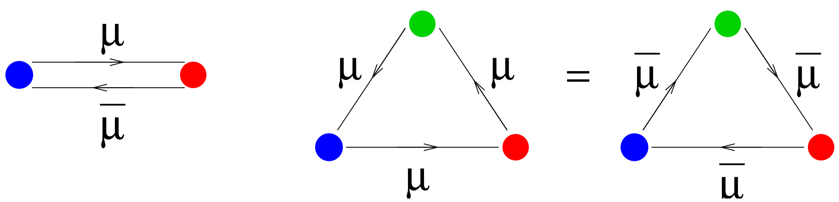
Universal connectivity constant [GD, Viti, '11]

$P_n(x_1,\dots,x_n) \;\equiv\;$ probability $x_1,\dots,x_n$ in same cluster

$P_2 \propto |x_1 - x_2|^{-4\Delta_\sigma}, \qquad P_3 = R\sqrt{P_2(x_1,x_2)P_2(x_1,x_3)P_2(x_2,x_3)}$

$$\begin{cases} P_2 = \frac{\langle\sigma_\alpha\sigma_\alpha\rangle}{q-1} = \langle\mu_{\alpha\beta}\mu_{\beta\alpha}\rangle = \langle\mu\bar{\mu}\rangle = \langle\phi\phi\rangle \\ P_3 = \frac{\langle\sigma_\alpha\sigma_\alpha\sigma_\alpha\rangle}{(q-1)(q-2)} = \langle\mu_{\alpha\beta}\mu_{\beta\gamma}\mu_{\gamma\alpha}\rangle = \langle\mu\mu\mu\rangle = \sqrt{2}\,\langle\phi\phi\phi\rangle \end{cases} \Longrightarrow R = \sqrt{2}\,C_{\phi\phi\phi}$$

$\mu_{\alpha\beta} \;\leftrightarrow\; \mu,\bar{\mu} \quad (n<4):$



$\phi \equiv \frac{\mu+\bar{\mu}}{\sqrt{2}}$

color symmetry factorizes at 3-point level

structure constants in neutral $c<1$ CFT [Al.Zamolodchikov, '05]:

$$\mathcal{C}_{\Delta_1,\Delta_2,\Delta_3} = \frac{A\,\Upsilon(2\beta-\beta^{-1}+a_1+a_2+a_3)}{[\Upsilon(2a_1+\beta)\Upsilon(2a_1+2\beta-\beta^{-1})]^{\frac{1}{2}}}\times$$
$$\times\ \frac{\Upsilon(a_1+a_2-a_3+\beta)\Upsilon(a_2+a_3-a_1+\beta)\Upsilon(a_3+a_1-a_2+\beta)}{[\Upsilon(2a_2+\beta)\Upsilon(2a_2+2\beta-\beta^{-1})\Upsilon(2a_3+\beta)\Upsilon(2a_3+2\beta-\beta^{-1})]^{\frac{1}{2}}}$$

$$c = 1-\frac{6}{t(t+1)} \qquad \beta=\sqrt{t/(t+1)} \qquad \Delta_i = a_i(a_i+\beta-\beta^{-1})$$

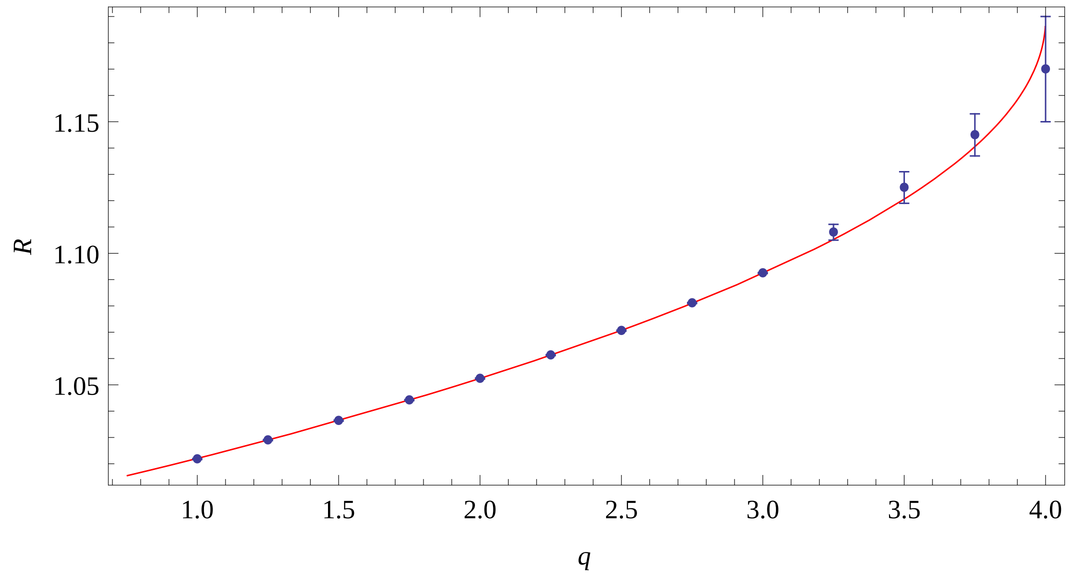
$$A=\frac{\beta^{\beta^{-2}-\beta^{2}-1}[\gamma(\beta^2)\gamma(\beta^{-2}-1)]^{1/2}}{\Upsilon(\beta)} \qquad \gamma(x)\equiv\frac{\Gamma(x)}{\Gamma(1-x)}$$

$$\Upsilon(x)=\exp\left\{\int_0^\infty\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t}-\frac{\sinh^2\left[\left(\frac{Q}{2}-x\right)\frac{t}{2}\right]}{\sinh\frac{\beta t}{2}\sinh\frac{t}{2\beta}}\right]\right\} \qquad Q=\beta+\beta^{-1}$$

Relation with Liouville and minimal model structure constants: Kostov, Petkova, '06; Harlow, Maltz, Witten, '11; Giribet, '12; Schomerus, Suchanek, '12; Picco, Santachiara, Viti, GD, '13

$$R = \sqrt{2}\, C_{\phi\phi\phi} = \sqrt{2}\, \mathcal{C}_{\Delta_\sigma \Delta_\sigma \Delta_\sigma}$$

Monte Carlo verification [Picco, Santachiara, Viti, GD, '13; Ziff, Simmons, Kleban, '11 for q=1] :



q	1	2	3	4
R (theory)	1.022013..	1.052447..	1.092355..	1.1892..
R (numerical)	1.0218(2)	1.0524(2)	1.0925(2)	1.17(2)

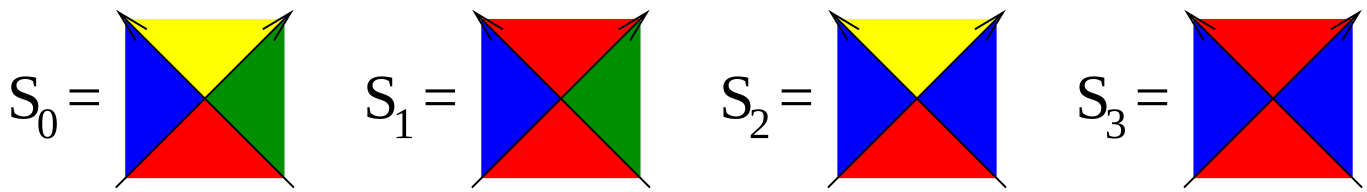
- $q = 2$ sheds light on Zamolodchikov's "mysterious numbers"

Away from criticality

$$\mathcal{A}_{\text{scaling Potts}} = \mathcal{A}_{CFT} + (J - J_c) \int d^2x\, \varepsilon(x)$$

$$\Delta_\varepsilon = \Delta_{2,1} \quad \Longrightarrow \quad \text{integrability} \quad \Longrightarrow \quad \text{exact } S\text{-matrix}$$

kink-kink scattering amplitudes [Chim, A.Zamolodchikov, '92]:



$$S_0(\theta) = \frac{\sinh\lambda\theta \, \sinh\lambda(\theta - i\pi)}{\sinh\lambda\left(\theta - \frac{2\pi i}{3}\right) \sinh\lambda\left(\theta - \frac{i\pi}{3}\right)} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \qquad S_1(\theta) = \frac{\sin\frac{2\pi\lambda}{3} \, \sinh\lambda(\theta - i\pi)}{\sin\frac{\pi\lambda}{3} \, \sinh\lambda\left(\theta - \frac{2i\pi}{3}\right)} \Pi\left(\frac{\lambda\theta}{i\pi}\right)$$

$$S_2(\theta) = \frac{\sin\frac{2\pi\lambda}{3} \, \sinh\lambda\theta}{\sin\frac{\pi\lambda}{3} \, \sinh\lambda\left(\theta - \frac{i\pi}{3}\right)} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \qquad S_3(\theta) = \frac{\sin\lambda\pi}{\sin\frac{\pi\lambda}{3}} \Pi\left(\frac{\lambda\theta}{i\pi}\right)$$

$$\sqrt{q} = 2\sin\frac{\pi\lambda}{3} \qquad \Pi\left(\frac{\lambda\theta}{i\pi}\right) = \frac{\sinh\lambda\left(\theta + i\frac{\pi}{3}\right)}{\sinh\lambda(\theta - i\pi)} \exp\left(\int_0^\infty \frac{dx}{x} \frac{\sinh\frac{x}{2}\left(1 - \frac{1}{\lambda}\right) - \sinh\frac{x}{2}\left(\frac{1}{\lambda} - \frac{5}{3}\right)}{\sinh\frac{x}{2\lambda}\cosh\frac{x}{2}} \sinh\frac{x\theta}{i\pi}\right)$$

center of mass energy $= 2m\cosh\theta/2$ $\qquad$ $m =$ kink mass

Amplitude ratios for percolation [GD, Viti, Cardy, '10]

critical behavior as $p \to p_c^{\pm}$:

density of the infinite cluster	$P \simeq B\,(p-p_c)^{\beta}$
mean cluster size	$S \simeq \Gamma^{\pm}\,\lvert p-p_c\rvert^{-\gamma}$
correlation length	$\xi \simeq f^{\pm}\,\lvert p-p_c\rvert^{-\nu}$
mean cluster number	$\bar{N}_c \simeq A^{\pm}\,\lvert p-p_c\rvert^{2-\alpha}$

universal amplitude combinations :

$$\frac{\Gamma^-}{\Gamma^+} = \frac{\int d^2x\, P_2^-(x,0)}{\int d^2x\, P_2^+(x,0)}, \qquad \frac{f^-}{f^+}, \qquad \frac{A^-}{A^+}, \qquad \frac{(B\,f^+)^2}{\Gamma^+}, \qquad \sqrt{A^+}\, f^+$$

from scattering to correlations :

$$\langle\Phi(x_1)\Phi(x_2)\rangle = \sum_k |\langle 0|\Phi(0)|k\rangle|^2 \, e^{-r_{12}E_k}$$

$|k\rangle$ = k-particle state

- form factors $\langle 0|\Phi(0)|k\rangle$ exactly computable
- spectral series rapidly convergent

example: magnetic susceptibility amplitude ratio:

q	field theory[a]	lattice
2	37.699	37.6936..[b]
3	13.85	13.83(8)[c,d]
4	4.01	3.9(1)[e]

[a] GD, Cardy, '98 ($k \leq 2$)

[b] Wu, McCoy, Tracy, Barouch, '76

[c] Enting, Guttmann, '03

[d] Shchur, Berche, Butera, '08

[e] Shchur, Janke, '10 (Baxter-Wu model)

connectivities in random percolation

$$P_2^-(x_1,x_2) = \frac{F_\mu^2}{\pi} K_0(mr)+O(e^{-2mr}) \qquad\qquad r \equiv |x_1-x_2|$$

$$P_2^+(x_1,x_2) = \frac{F_\sigma}{\pi^2}\int_0^\infty d\theta\, |f(2\theta)\Omega(2\theta)|^2\, K_0\,(2mr\cosh\theta)+O(e^{-3mr})$$

$$f(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{-2\int_0^\infty \frac{dx}{x}\,\frac{\sinh\frac{x}{3}\cosh\frac{x}{6}}{\sinh^2 x\cosh\frac{x}{2}}\,\sin^2\frac{(i\pi-\theta)x}{2\pi}\right\}$$

$$\Omega(\theta) = \mathcal{C}\int_{-\infty}^{+\infty}\frac{dx}{2\pi}\,W\left(-x-\frac{\theta}{2}+i\pi\right)W\left(-x+\frac{\theta}{2}+i\pi\right)e^{-x/6}$$

$$\mathcal{C} = \exp\left[\int_0^\infty \frac{dt}{t}\,\frac{4\sinh^2\frac{t}{4}\sinh\frac{t}{2}}{\sinh^2 t}\right], \qquad W(\theta) = \frac{1}{\cosh\theta}\,\exp\left[\int_0^\infty \frac{dt}{t}\,\frac{2\sinh\frac{t}{2}}{\sinh^2 t}\sin^2\left(\frac{t}{2\pi}(i\pi-\theta)\right)\right]$$

$$F_\mu^2 = F_\sigma \lim_{\theta\to\infty}|f(\theta)\Omega(\theta)|$$

amplitude ratio	field theory[a]	lattice
A^-/A^+	1	1^b
f^-/f^+	2	-
f^-_{2nd}/f^-	1.001	-
f^-_{2nd}/f^+_{2nd}	3.73	4.0 ± 0.5^d
$4B^2(f^-_{2nd})^2/\Gamma^-$	2.22	2.23 ± 0.10^e
$(-80/27\, A^-)^{1/2} f^-_{2nd}$	0.926	$\approx 0.93^{b+c}$
Γ^-/Γ^+	160.2	162.5 ± 2^f

[a] GD, Viti, Cardy, '10 ($k \leq 2$)

[b] Domb, Pearce, '76

[c] Aharony, Stauffer, '97

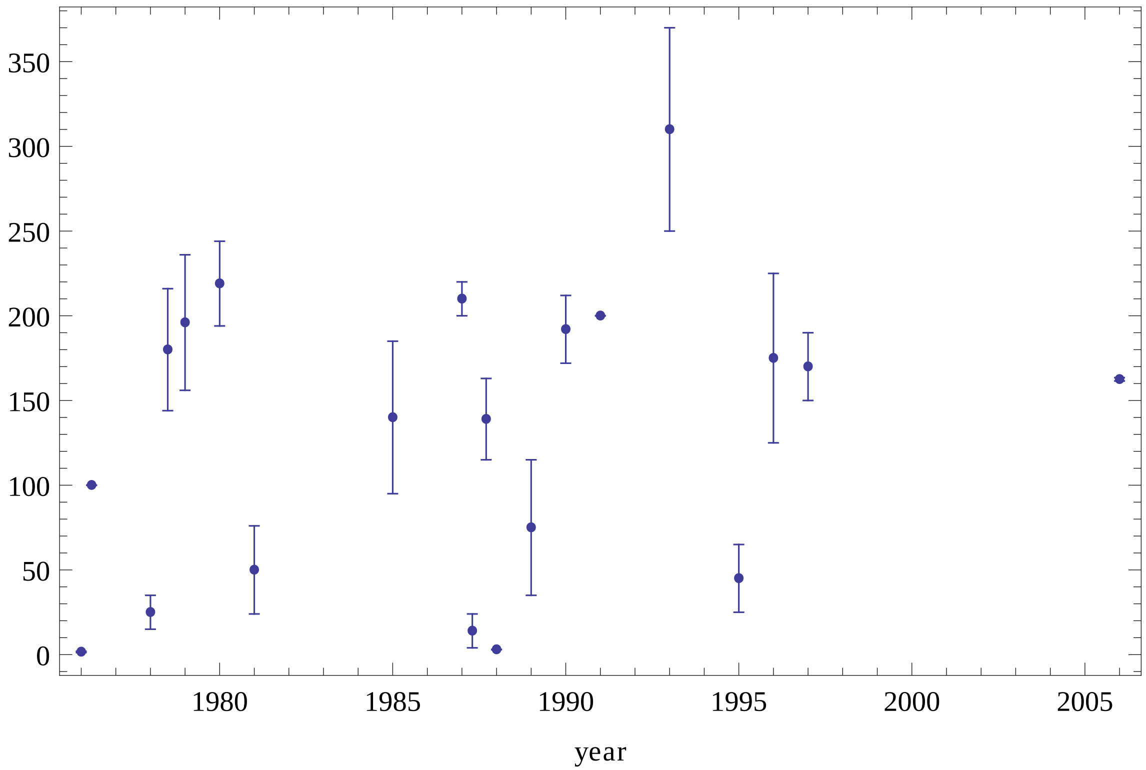
[d] Corsten, Jan, Jerrard, '89

[e] Daboul, Aharony, Stauffer, '00

[f] Jensen, Ziff, '06

30 years of efforts by the lattice community, Γ^-/Γ^+ most elusive

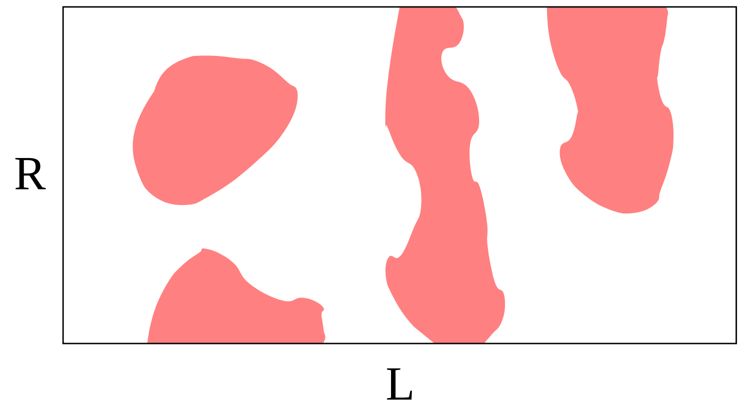
series/MonteCarlo estimates of Γ^-/Γ^+



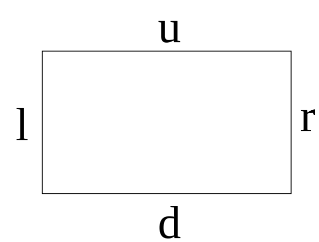
Off-critical crossing clusters [GD, Viti, '12]

$P_v \equiv$ probability there is at least one vertically crossing cluster

- function of R/L at p_c [Cardy, '92]
- function of R/ξ, L/ξ at $p \neq p_c$



$Z_{ud}^{lr} \equiv$ Potts partition function with b.c.

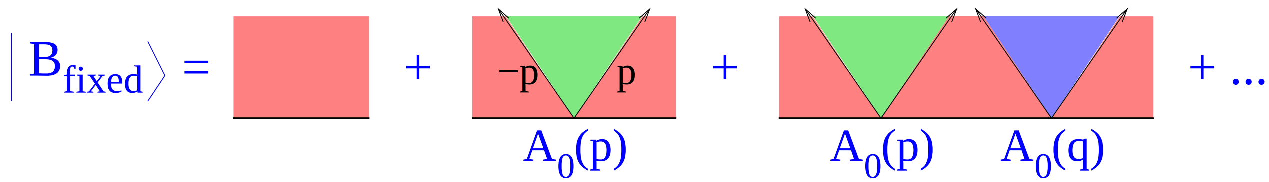


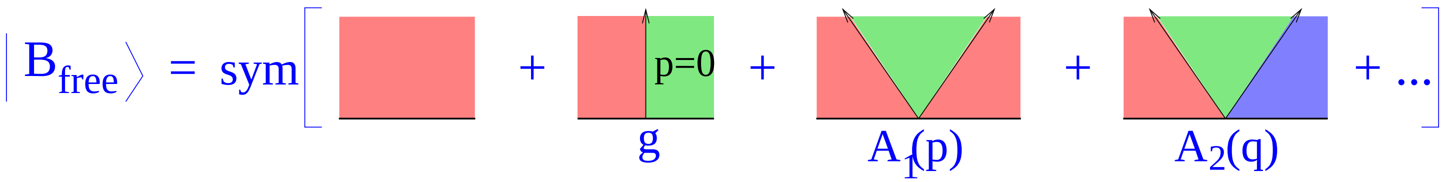
boundary conditions: f=free α=fixed to color α

configurations with vertical crossings do not contribute to $Z_{\alpha\beta}^{ff}$

$$\Longrightarrow \quad P_v = 1 - Z_{\alpha\beta}^{ff}\Big|_{q=1}$$

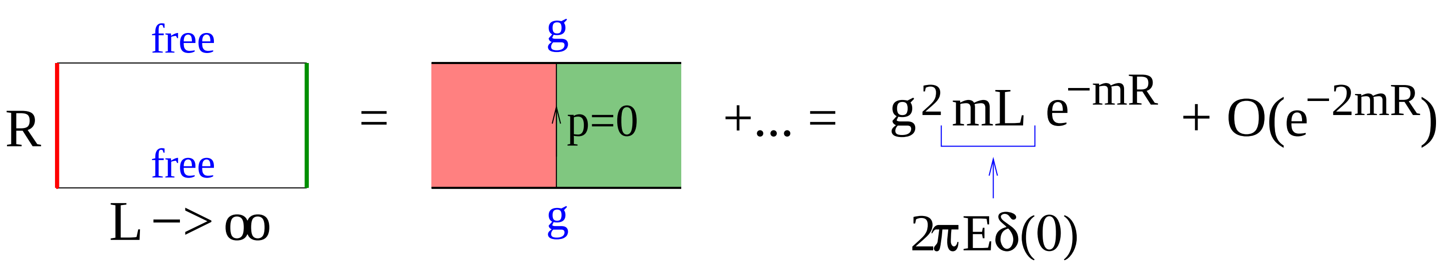
$L \to \infty$, boundary states:





g, A_0, A_1, A_2 known [Ghoshal, A.Zamolodchikov, '94; Chim, '95]

$$\lim_{L\to\infty} \left[Z^{lr}_{ud}\right]^{+} = \langle B_u|e^{-R\,H}|B_d\rangle_{l,r}$$



duality: vertical crossing below p_c $\Rightarrow$ no horizontal crossing above p_c

$$P_v^- = 1 - P_h^+ = \left[Z_{ff}^{\alpha\beta}\right]_{q=1}^+ \sim A\, m L\, e^{-mR} \qquad A \equiv g^2|_{q=1} = \frac{3-\sqrt{3}}{2}$$

$$P_v^+ = 1 - P_h^- = 1 - P_v^-|_{R\leftrightarrow L} \sim 1 - A\, m R\, e^{-mL}$$

$$m = m_0\, |p - p_c|^{4/3} = \begin{cases} 1/\xi\,, & p < p_c \\ 1/2\xi\,, & p > p_c \end{cases} \qquad P_2(r) \propto r^{-a}\, e^{-r/\xi}\,, \quad r \to \infty$$

mean number of crossing clusters :

$$\bar{N}_v^- \sim A\, m L\, e^{-mR}$$

$$\bar{N}_v^+ \sim 1 + A\, m L\, e^{-mR}$$

- next to leading term also computed

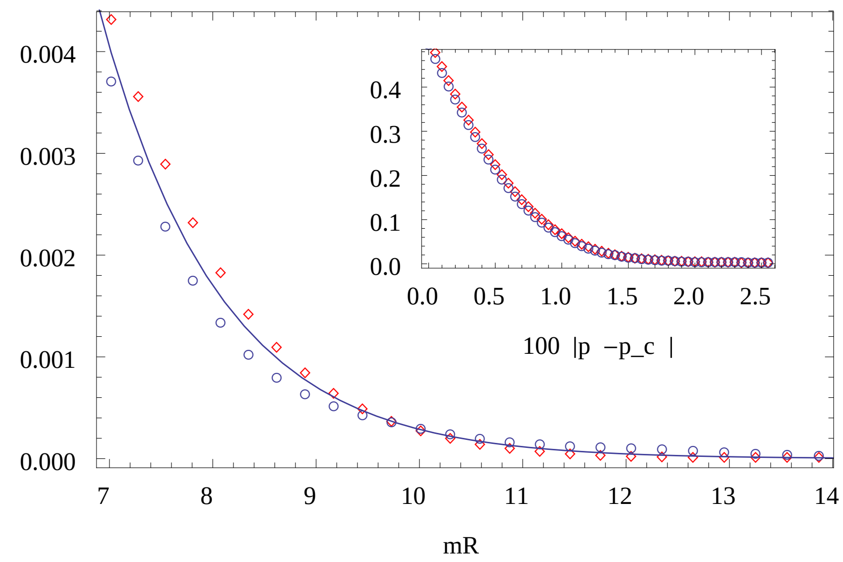
comparison with Monte Carlo data of Watanabe et al, '04

256 x 256 square lattice

- ◇ P_v^-
- ○ $1 - P_v^+$
- — $\frac{3-\sqrt{3}}{2} mR\, e^{-mR}$

$m_0 = 5.5 \pm ?$ (numerics + theory)

$m_0^{fit} = 5.8 \pm ?$



data agree with theory within available numerical accuracy

Conclusion

• percolation tests the limits of what we know about 2D QFT

• despite subtleties conformal and integrable field theory can be used to obtain universal results for clusters connectivities

some natural questions

• $p = p_c$: four-point bulk connectivities

• $p \neq p_c$: beyond large distance