Thermal form factors of the anisotropic Heisenberg chain

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Abstract

We derive expressions for the form factors of the quantum transfer matrix of the spin-$\frac{1}{2}$ XXZ chain which allow us to take the infinite Trotter number limit. This solves the longstanding problem of describing analytically the amplitudes in the leading asymptotics of the finite temperature correlation functions of the model.
In the zero-temperature limit, we recover the predictions of conformal field theory (CFT) and Luttinger liquid (LL) approach concerning the large-distance behaviour of the correlators in the massless phase. As a first result for the massive regime, we obtain Baxter's spontaneous magnetization.

Introduction

The Hamiltonian of the XXZ chain in a longitudinal magnetic field h is given by

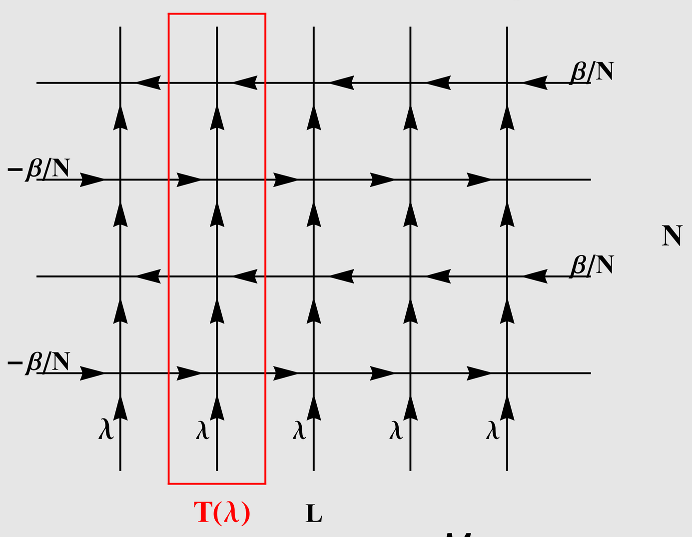
$$H = J\sum_{j=1}^{L}(\sigma_{j-1}^x\sigma_j^x + \sigma_{j-1}^y\sigma_j^y + \Delta(\sigma_{j-1}^z\sigma_j^z - 1)) - \frac{h}{2}\sum_{j=1}^{L}\sigma_j^z \,,$$

where periodic boundary conditions are implied. $J > 0$ fixes the energy scale and $\Delta = \mathrm{ch}(\eta) \in \mathbb{R}$ is the anisotropy parameter. We investigate the large-distance asymptotics of longitudinal and transversal correlation functions in the thermodynamic limit $L \to \infty$ at finite temperature T,

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle\,,\ \langle\sigma_1^-\sigma_{m+1}^+\rangle \underset{m\to\infty}{\sim} ? \quad \text{with} \quad \langle\mathcal{O}\rangle = \mathrm{Tr}(\mathrm{e}^{-\beta H}\mathcal{O})/\mathrm{Tr}(\mathrm{e}^{-\beta H}) \,.$$

Quantum transfer matrix and Bethe ansatz

We study finite temperature properties of the system by means of the quantum transfer matrix (QTM) $t(\lambda)$ which is defined as the column-to-column transfer matrix of a certain inhomogeneous six-vertex model.



The monodromy matrix $T(\lambda)$ is a 2×2 matrix with matrix elements $A(\lambda); B(\lambda); C(\lambda); D(\lambda) \in \mathrm{End}(\mathbb{C}_2^{\otimes N})$
The QTM $t(\lambda) = \mathrm{Tr}\, T(\lambda) = A(\lambda) + D(\lambda)$ can be diagonalized using the algebraic Bethe ansatz (ABA). Eigenvectors have the form $|\Psi_n\rangle = B(\lambda_M)\dots B(\lambda_1)|0\rangle$ where the Bethe roots $\{\lambda_j\}_{j=1}^M$ have to satisfy the Bethe ansatz equations (BAE).
The QTM $t(0)$ has a unique eigenvalue $\Lambda_0(0)$ with largest modulus which, together with the corresponding eigenvector $|\Psi_0\rangle$ determines the state of thermal equilibrium completely for $L\to\infty$. Correlation functions can be calculated via

$$\langle\mathcal{O}_1^{(1)}\dots\mathcal{O}_m^{(m)}\rangle = \lim_{N\to\infty}\frac{\langle\Psi_0|\,\mathrm{Tr}\{\mathcal{O}^{(1)}T(0)\}\dots\mathrm{Tr}\{\mathcal{O}^{(m)}T(0)\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\Lambda_0^m(0)} \,.$$

To every solution of the BAE, one associates an auxiliary function $\mathfrak{a}_n(\lambda)$. In the Trotter limit $N\to\infty$, it is determined by the non-linear integral equation

$$\ln\mathfrak{a}_n(\lambda) = -(\kappa + \tfrac{N}{2} - M)2\eta - \beta\,\mathrm{e}(\lambda) - \int_{\mathcal{C}_n}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}K(\lambda-\mu)\ln(1+\mathfrak{a}_n(\mu)) \,,$$

where $\beta = 2J\,\mathrm{sh}(\eta)/T$ and $\kappa = h/(2\eta T)$. The contour $\mathcal{C}_n$ encircles all Bethe roots. The kernel $K(\lambda)$ and the bare energy $\mathrm{e}(\lambda)$ are defined by

$$K(\lambda) = K_0(\lambda)\,,\ K_\alpha(\lambda) = q^{-\alpha}\,\mathrm{cth}(\lambda-\eta) - q^\alpha\,\mathrm{cth}(\lambda+\eta)\,,\ \mathrm{e}(\lambda) = \mathrm{cth}(\lambda) - \mathrm{cth}(\lambda+\eta)\,,$$

with $q = \mathrm{e}^\eta$. The eigenvalue can be expressed as

$$\ln\Lambda_n(\lambda) = (\kappa + \tfrac{N}{2} - M)\eta - \int_{\mathcal{C}_n}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\mathrm{e}(\mu-\lambda)\ln(1+\mathfrak{a}_n(\mu)) \,.$$

Form factor expansion

For the longitudinal two-point functions we start with a generating function, which is closely related to the twisted QTM $t(\lambda|\alpha) = q^\alpha A(\lambda) + q^{-\alpha}D(\lambda)$. Setting $S(m) = \frac{1}{2}\sum_{j=1}^m \sigma_j^z$, the generating function is defined as $\langle q^{2\alpha S(m)}\rangle$, where $\alpha\in\mathbb{C}$. This correlation function generates the longitudinal two-point functions via

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle = \tfrac{1}{2}D_m^2\partial_{\eta\alpha}^2\langle q^{2\alpha S(m+1)}\rangle|_{\alpha=0} \,,$$

where D_m is the difference operator defined by $D_m f(m) = f(m) - f(m-1)$. It follows from the general formula that

$$\langle q^{2\alpha S(m)}\rangle_N = \frac{\langle\Psi_0|t^m(0|\alpha)|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\Lambda_0^m(0)}$$

is the finite Trotter number approximant of the generating function. Inserting a complete set of eigenstates $\{|\Psi_n^\alpha\rangle\}$ of the twisted QTM (with eigenvalues $\Lambda_n(\lambda|\alpha)$ and auxiliary functions $\mathfrak{a}_n(\lambda|\alpha)$), we obtain the form factor expansion

$$\langle q^{2\alpha S(m)}\rangle_N = \sum_n A_n(\alpha)\,\mathrm{e}^{-m/\xi_n} \,,$$

$$A_n(\alpha) = \frac{\langle\Psi_0|\Psi_n^\alpha\rangle\langle\Psi_n^\alpha|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\langle\Psi_n^\alpha|\Psi_n^\alpha\rangle}\ ,\quad -1/\xi_n = \ln\rho_n(0|\alpha)\ ,\quad \rho_n(\lambda|\alpha) = \frac{\Lambda_n(\lambda|\alpha)}{\Lambda_0(\lambda)} \,.$$

Similarly, the form factor expansion for the transversal correlators reads

$$\langle\sigma_1^-\sigma_{m+1}^+\rangle_N = \sum_n A_n^{-+}(0|0)\,\mathrm{e}^{-m/\xi_n}\,,\ A_n^{-+}(\xi|\alpha) = \frac{\langle\Psi_0|B(\xi)|\Psi_n^\alpha\rangle}{\Lambda_n(\xi|\alpha)\langle\Psi_0|\Psi_0\rangle}\frac{\langle\Psi_n^\alpha|C(\xi)|\Psi_0\rangle}{\Lambda_0(\xi)\langle\Psi_n^\alpha|\Psi_n^\alpha\rangle} \,.$$

- Thermal correlation functions can be expanded into series of form factors of the QTM. Instead of form factors of local operators, those of ABA operators appear.
- For $T>0$ only a few terms contribute to the form factor expansion and correlators decay exponentially
- Correlation lengths ξ_n have been studied extensively [Klümper et al. '01], so far little was known about the amplitudes A_n

Amplitudes for longitudinal correlators

Our main result is the following analytic expression for the amplitudes in the Trotter limit:

$$A_n(\alpha) = \overline{\sigma}_+^-\overline{\sigma}_-^-\,\exp\left[\int_{\mathcal{C}_n}\frac{\mathrm{d}\lambda\,\rho_n'(\lambda|\alpha)}{2\pi\mathrm{i}\,\rho_n(\lambda|\alpha)}\ln\left(\frac{1+\mathfrak{a}_n(\lambda|\alpha)}{1+\mathfrak{a}_0(\lambda)}\right)\right]$$
$$\times\frac{\det_{\mathrm{d}m_+^\alpha,\mathcal{C}_n}\left[1-\widehat{K}_{-\alpha}\right]\det_{\mathrm{d}m_-^\alpha,\mathcal{C}_n}\left[1-\widehat{K}_\alpha\right]}{\det_{\mathrm{d}m_0^\alpha,\mathcal{C}_n}\left[1-\widehat{K}\right]\det_{\mathrm{d}m,\mathcal{C}_n}\left[1-\widehat{K}\right]} \,.$$

The determinants have to be understood as Fredholm determinants, e.g.

$$\det_{\mathrm{d}m_+^\alpha,\mathcal{C}_n}\left[1-\widehat{K}_{-\alpha}\right] = 1 + \sum_{k=1}^\infty\frac{(-1)^k}{k!}\left[\prod_{j=1}^k\int_{\mathcal{C}_n}\mathrm{d}m_+^\alpha(\omega_j)\right]\det_k[K_{-\alpha}(\omega_l-\omega_m)] \,,$$

and similarly for the other determinants. The measures are given by

$$\mathrm{d}m(\omega) = \frac{\mathrm{d}\omega}{2\pi\mathrm{i}(1+\mathfrak{a}_0(\omega))}\,,\ \mathrm{d}m_0^\alpha(\omega) = \frac{\mathrm{d}\omega}{2\pi\mathrm{i}(1+\mathfrak{a}_n(\omega|\alpha))}\,,$$
$$\mathrm{d}m_-^\alpha(\omega) = \frac{\mathrm{d}\omega\,\rho_n^{-1}(\omega|\alpha)}{2\pi\mathrm{i}(1+\mathfrak{a}_0(\omega))}\,,\ \mathrm{d}m_+^\alpha(\omega) = \frac{\mathrm{d}\omega\,\rho_n(\omega|\alpha)}{2\pi\mathrm{i}(1+\mathfrak{a}_n(\omega|\alpha))} \,.$$

Furthermore, our result involves solutions of linear integral equations,

$$\overline{\sigma}_\pm^- = \lim_{\mathrm{Re}\,\lambda\to-\infty}\overline{\sigma}_\pm(\lambda)\,,\quad \overline{\sigma}_\pm(\lambda) = 1 + \int_{\mathcal{C}_n}\mathrm{d}m_\pm^\alpha(\mu)K_{\pm\alpha}(\lambda-\mu)\overline{\sigma}_\pm(\mu) \,.$$

Amplitudes for transversal correlators

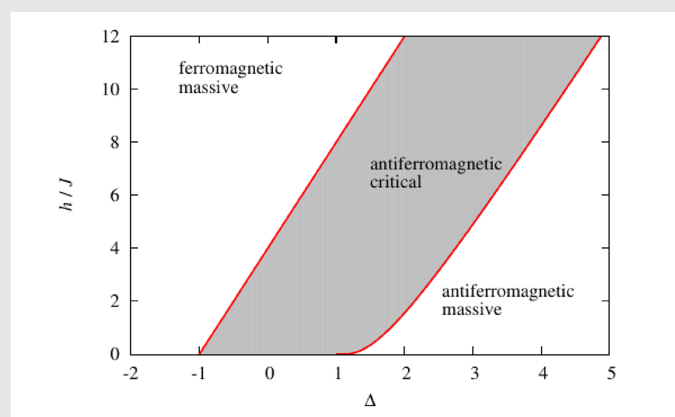
The expressions for the amplitudes pertaining to the transversal correlation functions have a remarkably similar structure,

$$A_n^{-+}(\xi|\alpha) = \frac{q^{\alpha+\kappa-1}+q^{-\alpha-\kappa+1}}{(q^{1+\alpha}-q^{-1-\alpha})(q^\alpha-q^{-\alpha})(q^\kappa+q^{-\kappa})}\overline{G}_+^-(\xi)\overline{G}_-^+(\xi)$$
$$\times\exp\left[\int_{\mathcal{C}_n}\frac{\mathrm{d}\lambda\,\rho_n'(\lambda|\alpha)}{2\pi\mathrm{i}\,\rho_n(\lambda|\alpha)}\ln\left(\frac{1+\mathfrak{a}_n(\lambda|\alpha)}{1+\mathfrak{a}_0(\lambda)}\right)\right]$$
$$\times\frac{\det_{\mathrm{d}m_+^\alpha,\mathcal{C}_n}\left[1-\widehat{K}_{1-\alpha}\right]\det_{\mathrm{d}m_-^\alpha,\mathcal{C}_n}\left[1-\widehat{K}_{1+\alpha}\right]}{\det_{\mathrm{d}m_0^\alpha,\mathcal{C}_n}\left[1-\widehat{K}\right]\det_{\mathrm{d}m,\mathcal{C}_n}\left[1-\widehat{K}\right]} \,,$$

where $\overline{G}_s^\pm(\xi)$ (for $s=\pm$) is defined by linear integral equations of the same type as in the longitudinal case.
Conjecture: Structure is universal and persists for more general form factors.

Zero-temperature limit for $\Delta > 1$

Massless regime ($\mathbf{h_{c_1} < h < h_{c_2}}$)
In this regime, infinitely many terms contribute to the form factor series, which implies that the individual amplitudes must vanish as $T\to 0$. Therefore, the low-T analysis requires the following steps:



- Calculation of amplitudes and correlation lengths for small but finite T
- Summation of the form factor series with the formula of [Kitanine et al. '11]

Result:

$$\langle\sigma_1^-\sigma_{m+1}^+\rangle \sim (-1)^m A_{0,0}^{-+}\left[\frac{\pi T/v_0}{\mathrm{sh}(m\pi T/v_0)}\right]^{\frac{1}{2\mathcal{Z}^2}}$$

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle \sim \langle\sigma_1^z\rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2}\left[\frac{\pi T/v_0}{\mathrm{sh}(m\pi T/v_0)}\right]^2 + A_{0,1}^{zz}\cos(2mk_F)\left[\frac{\pi T/v_0}{\mathrm{sh}(m\pi T/v_0)}\right]^{2\mathcal{Z}^2}$$

with k_F and v_0 being the Fermi momentum and sound velocity. The critical exponent $\mathcal{Z}$ is defined by linear integral equations.

- We have confirmed the predictions of CFT and LL theory concerning the large-distance behaviour of the correlation functions by an exact calculation
- The non-universal amplitudes $A_{0,0}^{-+}$ and $A_{0,1}^{zz}$ are known explicitly and can be easily computed numerically (cf. figures below)

In the vicinity of the lower phase boundary ($\mathbf{h} \searrow \mathbf{h_{c_1}}$) we obtain the explicit result

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle \sim \frac{\sqrt{e}\,2^{1/6}}{A^6}\left(\frac{2k}{1-k^2}\right)^{1/4}\left(\prod_{n=1}^\infty\mathrm{th}(\eta n)^4\right)\left(\frac{h}{h_{c_1}}-1\right)^{-1/4}\frac{(-1)^m}{\sqrt{m}} \,,$$

where A is Glaisher's constant and k is given by $k = \vartheta_2^2(0,q))/\vartheta_3^2(0,q)$.
This divergence for $h\searrow h_{c_1}$ indicates the transition to longer-ranged order.

Massive regime ($\mathbf{0 < h < h_{c_1}}$)
As a first result, we obtained the spontaneous magnetization

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle \sim (-1)^m\prod_{n=1}^\infty\mathrm{th}(\eta n)^4 \,,$$

in agreement with Baxter's formula for the spontaneous staggered polarization.
Open problem: Obtain the next-leading term of the large-distance asymptotics!