The periodic $s\ell(2|1)$ alternating spin chain and Logarithmic Conformal Field Theory at $c = 0$

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with N. Read, H. Saleur, and R. Vasseur

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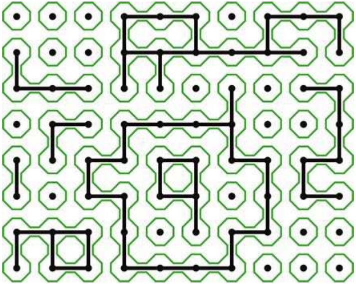
- Non-unitary SUSY spin chains as a local and *managable* model – periodic $s\ell(2|1)$ spin chains for percolation with periodic b.c.
- Logarithmic CFT as the continuum limit and a tool to study these phenomena

2d critical percolation with periodic b.c.

Edges on a lattice are open with probability p and closed with $1 - p$.



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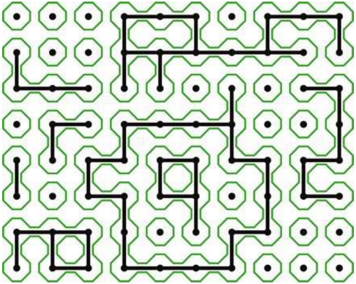
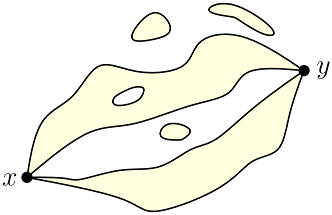


At criticality

occupied and empty edges occur with probability $p = \frac{1}{2}$
$\longrightarrow$ in loop formulation (there is a one to one correspondence between loops and clusters), all loop configurations are equiprobable and that the loops must all be counted with a fugacity equal to one!

Non-local observables

The multi hulls operators $\mathcal{O}_k$ have a natural interpretation in terms of k loops joining two points



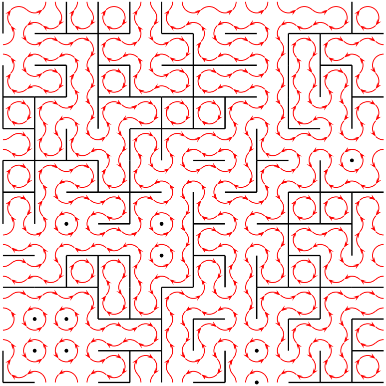
We are looking at quantities like multipoint correlators

$$\left\langle \mathcal{O}_k(\vec{r}_1)\mathcal{O}_k(\vec{r}_2)\right\rangle, \quad \ldots$$

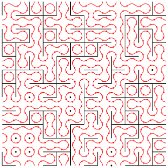
of initially **non-local** observables, for which critical exponents known:

$h_k = \bar{h}_k = \frac{4k^2-1}{24}$ `Duplantier-Saleur, 1987`

Non-local observables but local formulation



The idea is to consider the loops as Feynman diagrams expressing contraction of what is going to be **supergroup variables**.



- Statistics: odd edges carry $\square$ rep of $s\ell(2|1)$, and even ones are $\bar{\square}$ $\longrightarrow$ the fugacity of the loops is then the number of bosonic minus the number of fermionic degrees of freedom.

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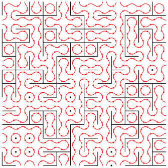

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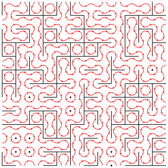

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- The transfer matrix then acts on the graded tensor product

$$\mathcal{H} = \square \otimes \bar{\square} \otimes \square \otimes \bar{\square} \otimes \ldots \otimes \square \otimes \bar{\square}$$

consisting of fundamental and conjugate reps of $s\ell(2|1)$.

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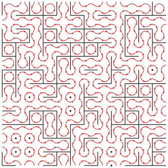

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- *The elementary nearest-neighbor interaction at $(i, i+1)$ corresponds to the action of a Temperley–Lieb algebra generator e_i.*

(periodic) TL algebra

The periodic Temperley–Lieb algebra $\widehat{\mathcal{TL}}_{\mathrm{q},N}$ is an ∞-dim algebra generated by $\mathbf{1}$ and e_i with $1 \leq i \leq N$ (i.e. + one extra generator, e_N)

$$\begin{aligned} [e_i, e_j] &= 0 \qquad (|i-j| \geq 2) \\ e_i^2 &= (\mathrm{q}+\mathrm{q}^{-1})e_i \\ e_i e_{i\pm 1} e_i &= e_i \end{aligned}$$

and indices i interpreted modulo N, and set $\mathrm{q} = e^{i\pi/\beta}$, with $\beta \in \mathbb{R}^*$.

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$\widehat{\mathcal{TL}}_{q,N}$ action on $\mathcal{H} = \square \otimes \bar{\square} \otimes \square \otimes \ldots \otimes \square \otimes \bar{\square}$

by projections onto the $s\ell(2|1)$ singlet $\mathbb{C}$ in the tensor product

$$\square \otimes \bar{\square} = \mathrm{Ad} \oplus \mathbb{C}$$

– the Heisenberg like interaction on $\mathcal{H}$.

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- The Hamiltonian $H = -\sum_i e_i$ of our model is just particular element of $\widehat{\mathcal{TL}}_{\mathrm{q},N}$ with $\mathrm{q} = e^{i\pi/3}$ (fugacity one)
- But the algebra has more operators – excitation operators. We call this type of algebra ***the dynamical symmetry of the model***

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∞-dim Lie algebra Vir_β generated by L_n's with $n \in \mathbb{Z}$ and satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

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- The energy-momentum tensor of CFT has L_n's as its Fourier modes!
- Quantum fields are grouped into appropriate chiral-antichiral Virasoro representations.

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TL vs Vir (important part)

Lattice dynamical symmetries (e.g. Temperley–Lieb algebras with the fugacity $n = 2\cos\beta$) go in the continuum limit $N \to \infty$ to CFT's dynamical symmetries, i.e., to the Virasoro algebra!

Lattice vs Continuum

- TL algebra gives a regularization of the energy-momentum tensor

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$$P(n) = \sum_{j=1}^{N-1} \exp\left(i\pi\frac{nj}{N}\right) [e_j, e_{j+1}] \quad \xrightarrow{N\to\infty} \quad L_n - \bar{L}_{-n}$$

In particular, the Hamiltonian $H \longrightarrow L_0 + \bar{L}_0$ and the momentum $P \longrightarrow L_0 - \bar{L}_0$, but we can also see the excitations operators L_n and $\bar{L}_n$'s!

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In the limit $N \to \infty$ the commutators of $H(n)$ and $P(n)$'s give the commutation relations of the Virasoro algebra. Koo-Saleur, 1994, G-Read-Saleur, 2011-12

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The $s\ell(2|1)$ spin chain $\mathcal{H}_N$ decomposition

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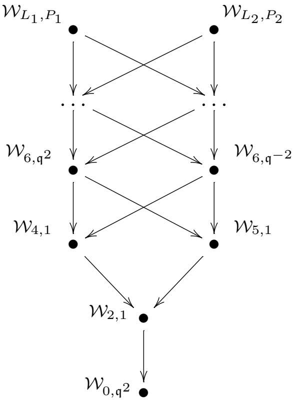
- The multiplicities $n_{j,P}$ correspond to combinations of many representations of $s\ell(2|1)$. *Note: the full symmetry or the centralizer is much bigger than $s\ell(2|1)$.*

The "ladder" structure of the tilting vacuum module

$\mathcal{T}_0$ is a "glueing" of many standard modules for periodic TL

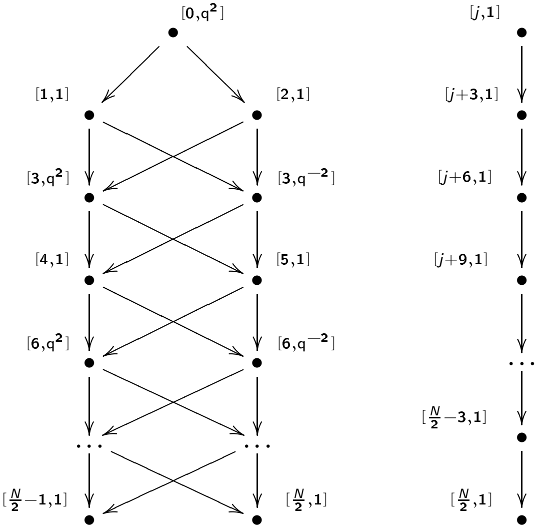
$\mathcal{T}_0 \quad =$



the periodic TL algebra acts following the arrows

The "ladder" structure of the standard modules

$\mathcal{W}_{j,P}$ at $q = e^{\frac{i\pi}{3}}$ and $[j, P]$ are irreducible subquotients



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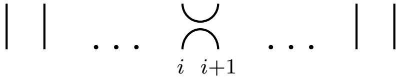
$e_i =$ 

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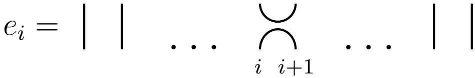

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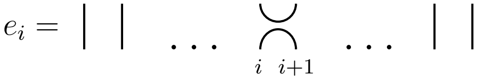

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Link or standard representations $\mathcal{W}_{j,P}$

are paramterized by the number of through-lines $2j$
and twist P (a point on a unit circle):

- through-lines connect the bottom boundary of the cylinder with $2j$ sites and the top with N sites
- the rest is connected by arcs without crossing

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This action gives rise to a generically irreducible module, which we denote by $\mathcal{W}_{j,P}$ – *the standard modules.*

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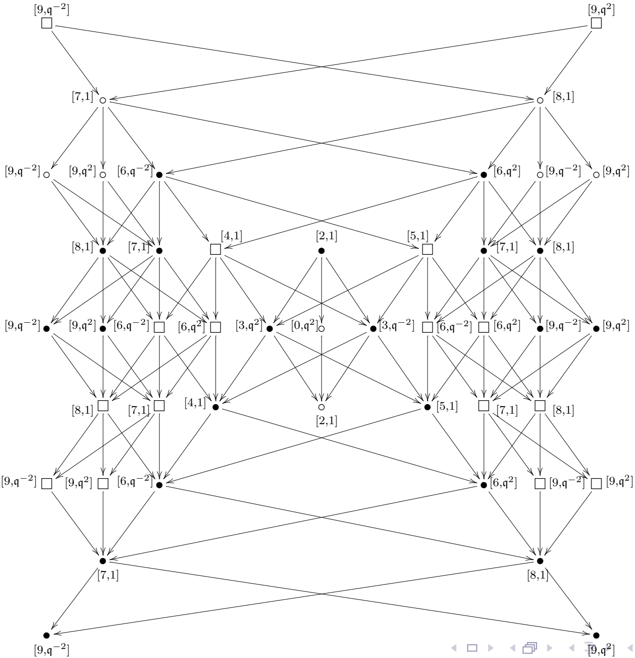
high-rank Jordan blocks

For the vacuum module $\mathcal{T}_0$ and states in its irreducible subquotients $[j, P]$:

$$\text{rank of } H \;\geq \begin{cases} 2\left\lceil\frac{j-2}{3}\right\rceil - 1, & j \bmod 3 = 0, \quad P = \mathrm{q}^{\pm 2}, \\ 2\left\lceil\frac{j-2}{3}\right\rceil, & j \bmod 3 = 1 \text{ or } 2, \quad P = 1. \end{cases}$$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[9,q2]
[7,1]
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[7,1]
[4,1]
[2,1]
[5,1]
[7,1]
[8,1]
[9,q−2]
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Connection between $s\ell(2|1)$ spin chains and twisted XXZ

Many properties of the $s\ell(2|1)$ spin chain can be obtained exactly, by combining

- the algebraic analysis (representation theory of affine TL)

$$\mathcal{H}_N = \bigoplus_{(j,P)} n_{j,P} \times \mathcal{T}_{j,P}$$

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  and each $\mathcal{T}_{j,P}$ is composed of $\mathcal{W}_{j,P}$
- with the Bethe ansatz of an integrable spin-chain (twisted XXZ) where each $\mathcal{W}_{j,P}$ also appears but as a direct summand.

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but the generators of the periodic TL are now in the XXZ representation

$$e_{i<N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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As a representation of periodic TL, the sector with the total spin $S_z = j$ is isomorphic to the standard module $\mathcal{W}_{|j|,e^{\pm i\phi}}$, where '$+$' is for positive j and '$-$' is for negative value of j.

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spectrum equivalence

The idea is that there is a spectrum equivalence (for the set of eigenvalues of H, up to multiplicities) between these two models

Aufgebauer-Brockmann-Nuding-Klümper, 2010

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$$H = \sum_{i=1}^{N} e_i$$

but

in other aspects (the action of the whole dynamical symmetry algebra and thus the structure of modules) the two models are **quite different!**

For the $N \to \infty$ limit, we are interested in the generating function $F_{j,e^{2iK}}$ of energy levels for each standard periodic TL module $\mathcal{W}_{j,e^{2iK}}$

Scaling limit of $s\ell(2|1)$ spin chains

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The generating function of the energy and momentum spectra is related to conformal spectra (for the critical Hamiltonian at $|q| = 1$) as

$$\operatorname{Tr} e^{-\beta_R(H-Ne_0)} e^{-i\beta_I P} \xrightarrow{N\to\infty} \operatorname{Tr} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}$$

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The scaling limit of each S_z sector and twist e^{2iK}

$$F_{j,e^{2iK}} = \frac{q^{-c/24} \bar{q}^{-c/24}}{P(q)P(\bar{q})} \sum_{n\in\mathbb{Z}} q^{h_{n+K/\pi,-j}} \bar{q}^{h_{n+K/\pi,j}}$$

and

$$P(q) = \prod_{n=1}^{\infty} (1 - q^n)$$

Pasquier-Saleur, 1990

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Putting all the ingredients together

Using this information about scaling properties of $\mathcal{W}_{j,P}$ and having the explicit structure of the tilting modules for periodic TL, we obtain explicit structure of $\mathfrak{vir} \oplus \overline{\mathfrak{vir}}$ modules in the scaling limit – bulk LCFT at $c = 0$

Results in the scaling limit

- Complete description of the operator content (including the multi hulls operators) of our theory. Several important conclusions follow, among which:

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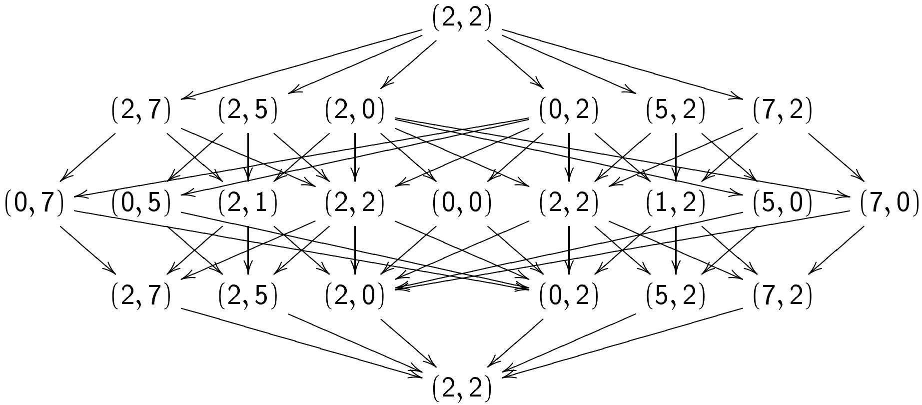
- Jordan cells of arbitrarily high rank occur in the scaling limit for large enough conformal weights. These ranks were calculated in the paper.

Vacuum module structure for percolation model in bulk (periodic b.c)

The scaling limit of the vacuum tilting module gives the structure of the vacuum module over the chiral-antichiral Virasoro at $c = 0$:



G-Read-Saleur-Vasseur 2014

Vacuum module structure for percolation model in bulk (periodic b.c)

The structure of Hamiltonian's Jordan cells!

$$H = \begin{pmatrix} E_j & 1 \\ 0 & E_j \end{pmatrix} \qquad L_0 = \begin{pmatrix} h_{1,j} & 1 \\ 0 & h_{1,j} \end{pmatrix}$$

and the Jordan cell in the vacuum module

$$L_0 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

gives log 's in two point functions

$$\langle T(z)T(0)\rangle = 0, \qquad \langle T(z)t(0,0)\rangle = -\frac{5}{z^4}, \qquad \langle t(z,\bar{z})t(0)\rangle = \frac{10\log|z|^2}{z^4}$$

Vasseur-G-Jacobsen-Saleur 2012

So...

Using lattice regularizations of LCFTs – quantum spin-chains with non-diagonalizable Hamiltonian – the representation theory of lattice algebras and connection with integrable twisted XXZ models,
we explored many problems of the following types:

- Structure of indecomposable modules over $\mathfrak{vir} \oplus \overline{\mathfrak{vir}}$ in bulk (or full) LCFTs where chiral and antichiral sectors are mixed in a highly non-trivial way.

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- Logarithmic couplings – numbers that describe correlations functions.

Thank you!