Form factor approach to correlation functions of the XXZ chain for $\Delta > 1$

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Background

- More than a decade of steady progress in our understanding of correlation functions of integrable models, chief example XXZ chain

$$H = J \sum_{j=-L+1}^{L} \Big(\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta \big(\sigma_{j-1}^z \sigma_j^z - 1 \big) \Big) - \frac{h}{2} \sum_{j=-L+1}^{L} \sigma_j^z$$

$$\Delta = (q + q^{-1})/2 = \mathrm{ch}(\eta)$$

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$\Delta = (q + q^{-1})/2 = \mathrm{ch}(\eta)$

- Long way from multiple integrals to proof of general factorization theorem $\longrightarrow$ understanding of the general structure:

THEOREM (JIMBO, MIWA AND SMIRNOV 2009).
All static correlation functions of the XXZ chain are polynomials in two functions ρ and ω and their derivatives.

THEOREM (BOOS AND FG 2009).
The functions ρ and ω are determined by solutions of linear and non-linear integral equations which are similar to those known from the exact description of the thermodynamics of the model.

Factorization at work

- For $n=4$ (BDGKSW 08)

$$\langle\sigma_1^z\sigma_4^z\rangle_{T,h} = \frac{1}{768q^4\left(-1+q^6\right)\left(1+q^2\right)\eta^2}$$

$$\Big\{384q^4\left(1+q^2\right)^2\left(5-4q^2+5q^4\right)\eta^2\omega-8\left(1+q^4\left(52+64q^2-234q^4+64q^6+52q^8+q^{12}\right)\right)\eta^2\omega_{xy}$$

$$+192q^4\left(-1+q^2\right)^2\left(1+4q^2+q^4\right)\eta^2\omega_{yy}+\left(-1+q^2\right)^4\left(1+q^4\right)\left(1+4q^2+q^4\right)\eta^2\left[-4\omega_{xyyy}+6\omega_{xxyy}\right]$$

$$-768q^4\left(-1-q^2+q^6+q^8\right)\eta\omega'_y+16\left(-1+q^2\right)^3\left(1+6q^2+11q^4+11q^6+6q^8+q^{10}\right)\eta\omega'_{xyy}$$

$$-2\left(-1+q^2\right)^5\left(1+2q^2+2q^4+q^6\right)\eta\omega'_{xxyyy}+8\left(-1+q^2\right)^3\left(1+q^2\right)\left(1+6q^2+34q^4+6q^6+q^8\right)\eta^2\left[\omega_y^2-\omega\omega_{xy}\right]$$

$$+\left(1+4q^2+22q^4+12q^6-12q^{10}-22q^{12}-4q^{14}-q^{16}\right)\eta^2\left[6\omega_{yy}^2-12\omega_{yy}\omega_{xy}-4\omega_y\omega_{yyy}+12\omega_y\omega_{xyy}+4\omega\omega_{xyyy}-6\omega\omega_{xxyy}\right]$$

$$+16\left(-1+q^2\right)^4\left(1+q^2\right)^2\left(1+q^2+q^4\right)\eta\left[\omega_{yy}\omega'_y-\omega_y\omega'_{yy}+\omega\omega'_{xyy}\right]$$

$$+\left(-1+q^4\right)^2\left(1+5q^2+6q^4+5q^6+q^8\right)\eta\Big[4\omega_{xyyy}\omega'_y-6\omega_{xxyy}\omega'_y-2\omega_{yyy}\omega'_{yy}$$

$$+6\omega_{xyy}\omega'_{yy}+2\omega_{yy}\omega'_{yyy}-4\omega_{xy}\omega'_{yyy}-6\omega_{yy}\omega'_{xyy}+4\omega_y\omega'_{xyyy}-2\omega\omega'_{xxyyy}\Big]$$

$$+3\left(-1+q^4\right)^3\left(1+q^2+q^4\right)\left[\omega'_{yyy}\omega'_{xyy}-\omega'_{yy}\omega'_{xyyy}+\omega'_y\omega'_{xxyyy}\right]\Big\}$$

Factorization at work

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$$
\begin{aligned}
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& \Big\{ 384 q^4 \left(1+q^2\right)^2 \left(5-4q^2+5q^4\right)\eta^2\omega - 8\left(1+q^4\left(52+64q^2-234q^4+64q^6+52q^8+q^{12}\right)\right)\eta^2\omega_{xy} \\
& + 192 q^4 \left(-1+q^2\right)^2 \left(1+4q^2+q^4\right)\eta^2\omega_{yy} + \left(-1+q^2\right)^4\left(1+q^4\right)\left(1+4q^2+q^4\right)\eta^2\left[-4\omega_{xyyy}+6\omega_{xxyy}\right] \\
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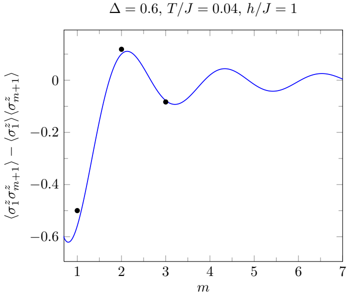
& + 3\left(-1+q^4\right)^3\left(1+q^2+q^4\right)\left[\omega'_{yyy}\omega'_{xyy}-\omega'_{yy}\omega'_{xyyy}+\omega'_{y}\omega'_{xxyyy}\right]\Big\}
\end{aligned}
$$

- Efficient computation of correlation functions at short distances, even for finite temperatures and magnetic fields $\longrightarrow$ applications: e.g. microwave spectroscopy (BGKKW 11, 12), entanglement entropy (SABGKTT 11), thermalization of isolated quantum systems (POZSGAY 13, ESSLER AND FAGOTTI 13)

Form factor approach

- Factorization based techniques cease to be efficient at distances larger than 10 lattice sites. But this is where large-distance asymptotic expansions are expected to start being efficient (example from DGK 14)



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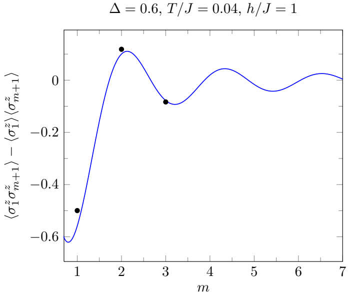
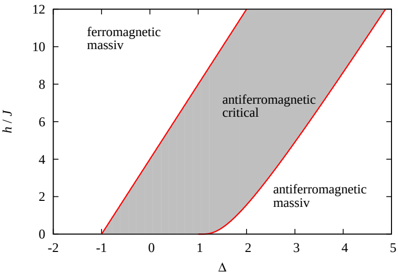


- Large-distance asymptotics at $T = 0$ directly from multiple integral representation (tour de force by KITANINE ET AL. 09)
- Analysis of $T = 0$ large-distance asymptotics in the critical phase from form factor expansion on the lattice (KITANINE ET AL. 11)
- $T > 0$ analysis based on form factors of the quantum transfer matrix. Summation possible for small T based on the above (M DUGAVE, FG, KK KOZLOWSKI 13, 14)

Contents and outlook

- Form factor expansion at finite temperature
- Amplitudes and large-distance asymptotics, $|\Delta| < 1$ and $T \to 0$
- Low-T analysis of Bethe equations in the massive regime $\Delta > 1$, $0 \leq h < h_\ell$
- Summary and outlook



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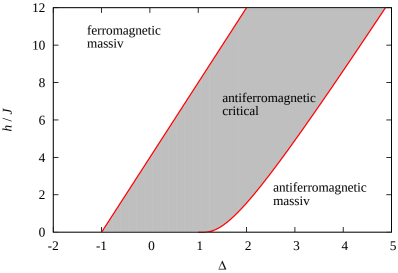

$\Delta > 1$: work in progress with M Dugave, KK Kozlowski and J Suzuki

- Analysis of Bethe equations in massive case for $T = 0$ ✔
- Calculation of form factors of zz-generating function in that case ✔
- Analysis of two-point functions
- Form factors and two-point functions in massless case and at phase boundary ✔
- Analysis of Bethe equations in massive case for $T > 0$ (QTM case) ✘
- $T > 0$ form factors and two-point functions in massive case (✔)

R matrix and statistical operator

- R matrix (solution of YANG-BAXTER equation)

$$R(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{aligned} b(\lambda,\mu) &= \frac{\mathrm{sh}(\lambda-\mu)}{\mathrm{sh}(\lambda-\mu+\eta)} \\ c(\lambda,\mu) &= \frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda-\mu+\eta)} \end{aligned}$$

- For the exact calculation of thermal averages we need to express the statistical operator $\mathrm{e}^{-H/T}$ in terms of $R(\lambda,\mu)$

- For this purpose we introduce an auxiliary vertex model with monodromy matrix

$$T_j(\lambda) = q^{\kappa\sigma_j^z} R_{j\overline{N}}(\lambda, \tfrac{\beta}{N}) R^{t_1}_{\overline{N-1}j}(-\tfrac{\beta}{N},\lambda) \dots R_{j\overline{2}}(\lambda, \tfrac{\beta}{N}) R^{t_1}_{\overline{1}j}(-\tfrac{\beta}{N},\lambda)$$

$j = -L+1,\dots,L$ and $N \in 2\mathbb{N}$. Here

$$\beta = \frac{2J\,\mathrm{sh}(\eta)}{T}, \qquad \kappa = \frac{h}{2\eta T}$$

Correlation functions

- Then

$$e^{-H/T} = \lim_{N\to\infty} \underbrace{\mathrm{tr}_{\bar{1}\dots\bar{N}}\{T_{-L+1}(0)\dots T_L(0)\}}_{=\rho_{N,L}}$$

- And expectation values are approximated as

$$\langle \mathcal{O}_1^{(1)}\dots\mathcal{O}_m^{(m)}\rangle_N = \lim_{L\to\infty} \frac{\mathrm{Tr}_{-L+1\dots L}\{\rho_{N,L}\mathcal{O}_1^{(1)}\dots\mathcal{O}_m^{(m)}\}}{\mathrm{Tr}_{-L+1\dots L}\{\rho_{N,L}\}}$$

$$= \lim_{L\to\infty} \frac{\mathrm{Tr}_{\bar{1}\dots\bar{N}}\{\mathrm{Tr}^L\{T(0)\}\,\mathrm{Tr}\{\mathcal{O}^{(1)}T(0)\}\dots\mathrm{Tr}\{\mathcal{O}^{(m)}T(0)\}\,\mathrm{Tr}^{L-m}\{T(0)\}\}}{\mathrm{Tr}_{\bar{1}\dots\bar{N}}\{\mathrm{tr}^{2L}\{T(0)\}\}}$$

$$= \frac{\langle\Psi_0|\,\mathrm{Tr}\{\mathcal{O}^{(1)}T(0)\}\dots\mathrm{Tr}\{\mathcal{O}^{(m)}T(0)\}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\Lambda_0^m(0)}$$

where $\Lambda_0(0)$ is the unique eigenvalue of largest modulus of the quantum transfer matrix $t(\lambda) = \mathrm{Tr}\,T(\lambda)$ and $|\Psi_0\rangle$ is the corresponding eigenvector

Form factor series for transversal two-point functions

- Considering $\mathcal{O}^{(1)} = \sigma_1^-$ and $\mathcal{O}^{(m)} = \sigma_m^+$ and inserting a complete set of states we obtain a finite temperature asymptotic expansion for the transversal correlation functions

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle_N = \sum_{n=1}^{N_M} A_n^{-+} \rho_n^m ,$$

where

$$\rho_n = \frac{\Lambda_n(0)}{\Lambda_0(0)} , \quad A_n^{-+} = \frac{\langle \Psi_0 | B(0) | \Psi_n \rangle}{\Lambda_n(0) \langle \Psi_0 | \Psi_0 \rangle} \frac{\langle \Psi_n | C(0) | \Psi_0 \rangle}{\Lambda_0(0) \langle \Psi_n | \Psi_n \rangle}$$

- Bottom line: Thermal correlation functions can be expanded into series of form factors of the quantum transfer matrix. The form factor series is an asymptotic series, since $|\rho_n| < 1$ for $n \in \mathbb{Z}_+$. Instead of form factors of local operators, form factors of ABA operators appear.

Algebraic Bethe ansatz solution

- Eigenvalue ratios ρ_n and amplitudes A_n^{-+} are parameterized by sets of Bethe roots λ_j, $j = 1, \dots, M$, satisfying the Bethe ansatz equations

$$\frac{a(\lambda_j)}{d(\lambda_j)} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{\operatorname{sh}(\lambda_j - \lambda_k + \eta)}{\operatorname{sh}(\lambda_j - \lambda_k - \eta)}$$

where

$$a(\lambda) = q^{\kappa} \left(\frac{\operatorname{sh}(\lambda + \beta/N)}{\operatorname{sh}(\lambda + \beta/N - \eta)} \right)^{N/2}, \quad d(\lambda) = q^{-\kappa} \left(\frac{\operatorname{sh}(\lambda - \beta/N)}{\operatorname{sh}(\lambda - \beta/N + \eta)} \right)^{N/2}$$

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- Task is to calculate ρ_n and A_n^{-+} in the Trotter limit $N \to \infty$

Auxiliary functions

- For every solution of the Bethe equations we define an auxiliary function

$$\mathfrak{a}_n(\lambda|\{\lambda_j\}) = \mathfrak{a}_n(\lambda) = \frac{d(\lambda)}{a(\lambda)} \prod_{j=1}^{M} \frac{\mathrm{sh}(\lambda - \lambda_j + \eta)}{\mathrm{sh}(\lambda - \lambda_j - \eta)}$$

  This function has a well-defined Trotter limit and determines the Bethe via $\mathfrak{a}_n(\lambda|\{\lambda_j\}) = -1$. For technical reasons we introduce a twist $\kappa \to \kappa + \alpha$ in the complete set of states used in the form factor expansion and write $\mathfrak{a}_n(\lambda|\alpha)$, $\Lambda_n(\lambda|\alpha)$ for the twisted quantities

- In the Trotter limit the functions $\mathfrak{a}_n(\lambda)$ are uniquely determined by non-linear integral equations (A KLÜMPER 93)

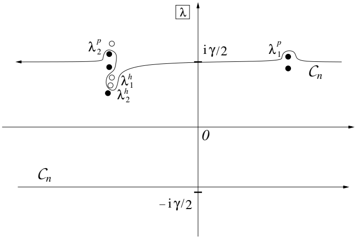
Contour and functions ρ_n

- The functions $\rho_n(\lambda|\alpha) = \Lambda_n(\lambda|\alpha)/\Lambda_0(\lambda)$ can be represented as contour integrals over the auxiliary functions

$$\rho_n(\lambda|\alpha) = q^{\alpha+\frac{N}{2}-M} \exp\left\{ \int_{\mathcal{C}_n} \frac{d\mu}{2\pi i}\, e(\mu-\lambda) \ln\left(\frac{1+\mathfrak{a}_n(\mu|\alpha)}{1+\mathfrak{a}_0(\mu)} \right) \right\}$$

λ is located inside the contour $\mathcal{C}_n$. The number $N/2-M$ is the eigenvalue of the conserved z-component of the pseudo spin $\eta^z = \frac{1}{2}\sum_{j=1}^N (-1)^j \sigma_j^z$

- The Contour $\mathcal{C}_n$ (here $\eta = -i\gamma$, $\gamma \in (-\pi/2, \pi/2)$, critical case)



The bare energy

$$e(\lambda) = \mathrm{cth}(\lambda) - \mathrm{cth}(\lambda+\eta)$$

Form factors for transversal correlation functions

- We consider the twisted, inhomogeneous amplitudes

$$A_n^{-+}(\xi) = \frac{\langle \Psi_0 | B(\xi) | \Psi_n^\alpha \rangle}{\Lambda_n(\xi|\alpha) \langle \Psi_0 | \Psi_0 \rangle} \frac{\langle \Psi_n^\alpha | C(\xi) | \Psi_0 \rangle}{\Lambda_0(\xi) \langle \Psi_n^\alpha | \Psi_n^\alpha \rangle}$$

- We obtain the following formula in the Trotter limit

$$A_n^{-+}(\xi) = \frac{\overline{G}_+^-(\xi) \overline{G}_-^+(\xi)}{(q^{1+\alpha} - q^{-1-\alpha})(q^\alpha - q^{-\alpha})}$$
$$\times \exp\left\{ - \int_{\mathcal{C}_n} \frac{d\lambda}{2\pi i} \ln\big(\rho_n(\lambda|\alpha)\big) \partial_\lambda \ln\left(\frac{1 + \mathfrak{a}_n(\lambda|\alpha)}{1 + \mathfrak{a}_0(\lambda)} \right) \right\}$$
$$\times \frac{\det_{dm_+^\alpha, \mathcal{C}_n}\left\{1 - \widehat{K}_{1-\alpha}\right\} \det_{dm_-^\alpha, \mathcal{C}_n}\left\{1 - \widehat{K}_{1+\alpha}\right\}}{\det_{dm_0^\alpha, \mathcal{C}_n}\left\{1 - \widehat{K}\right\} \det_{dm, \mathcal{C}_n}\left\{1 - \widehat{K}\right\}}$$

- Conjecture: Form is universal

Measures, kernels, G functions

- We introduced the measures

$$\mathrm{d}m_-^\alpha(\lambda) = \frac{\mathrm{d}\lambda\, \rho_n^{-1}(\lambda|\alpha)}{2\pi\mathrm{i}(1 + \mathfrak{a}_0(\lambda))}\,, \quad \mathrm{d}m_+^\alpha(\lambda) = \frac{\mathrm{d}\lambda\, \rho_n(\lambda|\alpha)}{2\pi\mathrm{i}(1 + \mathfrak{a}_n(\lambda|\alpha))}$$

$$\mathrm{d}m(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}(1 + \mathfrak{a}_0(\lambda))}\,, \quad \mathrm{d}m_0^\alpha(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}(1 + \mathfrak{a}_n(\lambda|\alpha))}$$

and the kernels

$$K(\lambda) = K_0(\lambda)\,, \quad K_\alpha(\lambda) = q^{-\alpha}\,\mathrm{cth}(\lambda - \eta) - q^\alpha\,\mathrm{cth}(\lambda + \eta)$$

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$$K(\lambda) = K_0(\lambda)\,, \quad K_\alpha(\lambda) = q^{-\alpha}\,\mathrm{cth}(\lambda-\eta) - q^\alpha\,\mathrm{cth}(\lambda+\eta)$$

- For $s = \pm$,

$$\overline{G}_s^\pm(\xi) = \lim_{\mathrm{Re}\,\lambda\to\pm\infty} \overline{G}_s(\lambda,\xi)$$

and $\overline{G}_s(\lambda,\xi)$ is the solution of the linear integral equation

$$\overline{G}_s(\lambda,\xi) = -\mathrm{cth}(\lambda-\xi) + q^{\alpha-s}\rho_n^s(\xi|\alpha)\,\mathrm{cth}(\lambda-\xi-\eta) + \int_{\mathcal{C}_n} \mathrm{d}m_s^\alpha(\mu)\,\overline{G}_s(\mu,\xi)\,K_{\alpha-s}(\mu-\lambda)$$

Low temperature analysis for $|\Delta| < 1$

- High temperatures: Only a few terms of the form factor series are needed to obtain an accurate description of the two-point functions
- Low temperatures: The patterns of Bethe roots for the QTM become regular, and an identification and summation of the (infinitely many) relevant excitations becomes possible

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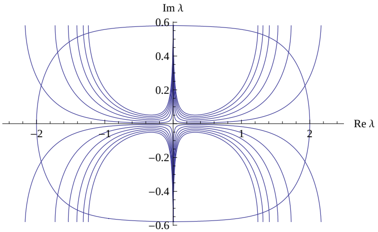
- High temperatures: Only a few terms of the form factor series are needed to obtain an accurate description of the two-point functions
- Low temperatures: The patterns of Bethe roots for the QTM become regular, and an identification and summation of the (infinitely many) relevant excitations becomes possible
- $\mathfrak{a}_0(\lambda) \sim e^{-\varepsilon(\lambda - i\gamma/2)/T}$ for small T (ε dressed energy)



Solutions of the equation

$$\varepsilon(\lambda) = -(2n-1)\pi i T$$

in the complex plane depicted by the intersections of the curve $\mathrm{Re}\,\varepsilon(\lambda) = 0$ (closed curve encircling the origin) with the curves $\mathrm{Im}\,\varepsilon(\lambda) = -(2n-1)\pi T$ (open curves). $J = 1$, $T = 0.01$, $\Delta = 0.4$, $h = 0.051$ and $n = -5, -4, \dots, 6$ in this example

After summation

- The leading low-temperature large-distance asymptotics of the transversal correlation functions

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle \sim \frac{e^{\frac{1}{2}\theta(2Q)+E(-2Q)+C_-[w]+C_+[w]}}{4\gamma \operatorname{sh}(\eta)} \mathcal{D}(0)\, \overline{G}_+^-(0)\, \partial_\alpha \overline{G}_-^+(0)\big|_{\alpha=0}$$

$$\times G^2\Big(1+\frac{1}{2\mathcal{Z}}\Big) G^2\Big(1-\frac{1}{2\mathcal{Z}}\Big) \left(\frac{e^{\frac{1}{2}\theta(2Q)-E(2Q)}}{2\pi\rho(Q)\operatorname{sh}(\eta)}\right)^{\frac{1}{2\mathcal{Z}^2}} (-1)^m \left(\frac{\pi T/v_0}{\operatorname{sh}(\pi m T/v_0)}\right)^{\frac{1}{2\mathcal{Z}^2}}$$

This is the main result of DGK 14. This formula is numerically efficient and matches well with known results

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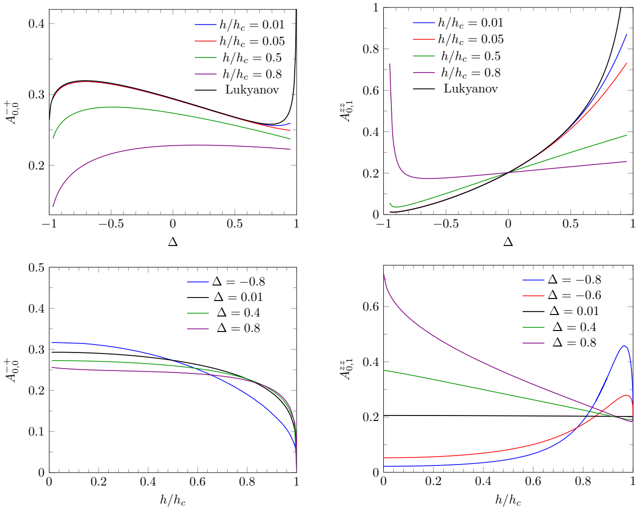
- Note that

$$\mathcal{D}(0) = \frac{\det_{\mathrm{d}\Delta_+^0,[-Q,Q]}\big\{1-\widehat{K}_1\big\}\det_{\mathrm{d}\Delta_-^0,[-Q,Q]}\big\{1-\widehat{K}_1\big\}}{\det^2_{\mathrm{d}\lambda/2\pi\mathrm{i},[-Q,Q]}\big\{1-\widehat{K}\big\}}$$

  where

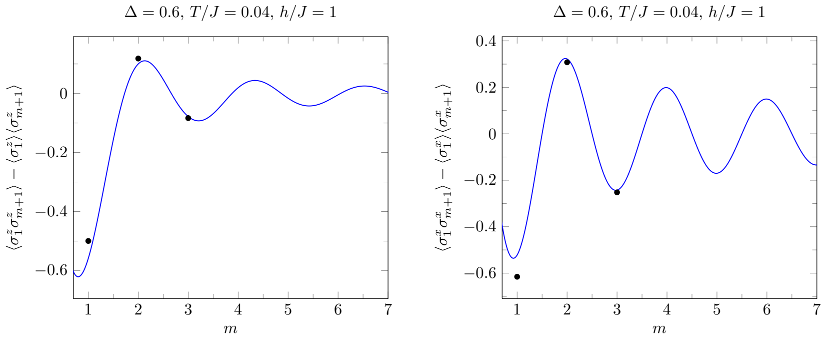
$$\mathrm{d}\Delta_\pm^0(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} e^{\pm\int_{-Q}^{Q}\mathrm{d}\mu\, e(\mu-\lambda)(w(\mu)-w(\lambda))}\, e^{\pm\{(w(\lambda)+1/2)E(Q-\lambda)-(w(\lambda)-1/2)E(-Q-\lambda)\}}$$

Leading amplitudes for transverse and logitudinal two-point functions



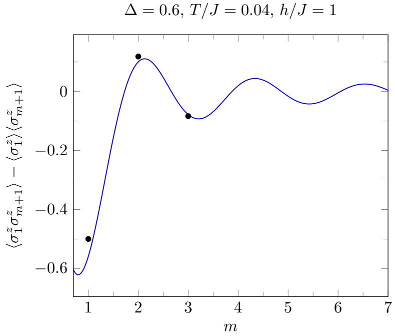
Amplitudes in the leading asymptotic terms as functions of the anisotropy parameter (upper panels) and as functions of the magnetic field (lower panels)

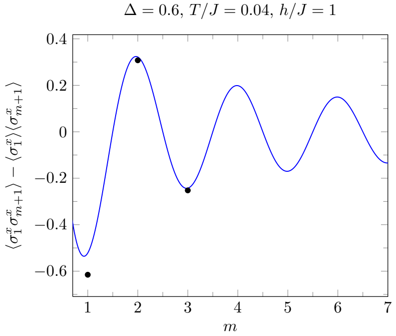
Asymptotics of two-point functions



Correlation functions according to our asymptotic formulae. Black dots depict the exact values at short distances obtained by BDGKSW 08

Asymptotics of two-point functions





Correlation functions according to our asymptotic formulae. Black dots depict the exact values at short distances obtained by BDGKSW 08

- We have carried out a similar analysis in the critical phase at $\Delta > 1$, $h_\ell < h < h_u$ (DGKS in preparation)
- On the critical line $h = h_\ell$ all quantities become explicit (in terms of q-functions) and we obtain a beautiful scaling form of the asymptotic formula (DGKS in preparation, see poster DUGAVE)

Low-T analysis of Bethe equations for $\Delta > 1$

- We have reconsidered the analysis of the Bethe ansatz equations of the usual transfer matrix at $\Delta > 1$, $h_\ell < h < h_u$ (DGKS in preparation) originally carried out by O BABELON, HJ DE VEGA, CM VIALLET 82 and by A VIROSZTEK AND F WOYNAROVICH 84. We confirm the picture of two-strings, quartets and wide pairs appearing in the large-L limit
- By way of contrast, for any finite $h \in (0, h_\ell)$ the spectrum of the QTM for $T \to 0$ can be interpreted in terms of particle-hole excitations (cf JUNJI's talk)

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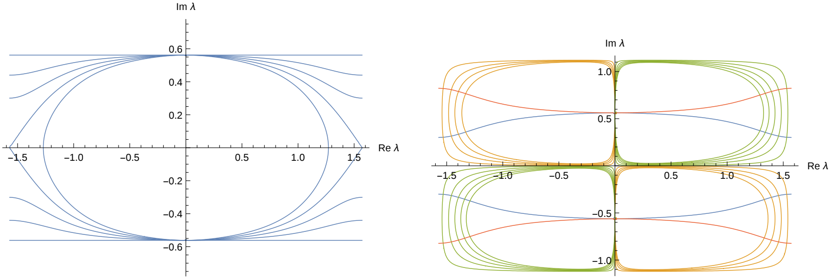

Locus of Bethe roots in the dominant state for $J = 1, \Delta = 1.7$. Left panel $h/h_\ell = 4/3, 1, 2/3, 1/3, 0$. Right panel $h/h_\ell = 2/3$, $T = 0.01$, $n = -3, \ldots, 4$.

Low-T analysis of Bethe equations for $\Delta > 1$

$$\mathfrak{a}(x) = \begin{cases} \mathrm{e}^{-\frac{h}{T}} \left[\prod_{j=1}^{n_h} \frac{\sin(x-x_j^h)}{\sin(x-x_j^h-\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_c} \frac{\sin(x-x_j^c-\mathrm{i}\gamma)}{\sin(x-x_j^c+\mathrm{i}\gamma)}\right] \\ \qquad \left[\prod_{j=1}^{n_+} \frac{\sin(x-x_j^+-\mathrm{i}\gamma)}{\sin(x-x_j^++\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_-} \frac{\sin(x-x_j^--2\mathrm{i}\gamma)}{\sin(x-x_j^-)}\right] & \text{for } x \in D_+ \\ (-1)^k \mathrm{e}^{-\frac{\varepsilon(x)}{T} - \sum_{j=1}^{n_h} \phi(x,x_j^h)} \left[\prod_{j=1}^{n_c} \frac{\sin(x-x_j^c)}{\sin(x-x_j^c+\mathrm{i}\gamma)}\right] \\ \qquad \left[\prod_{j=1}^{n_+} \frac{\sin(x-x_j^+)}{\sin(x-x_j^++\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_-} \frac{\sin(x-x_j^--\mathrm{i}\gamma)}{\sin(x-x_j^-)}\right] & \text{for } x \in D_0 \\ \mathrm{e}^{-\frac{h}{T}} \left[\prod_{j=1}^{n_h} \frac{\sin(x-x_j^h+\mathrm{i}\gamma)}{\sin(x-x_j^h)}\right] \left[\prod_{j=1}^{n_c} \frac{\sin(x-x_j^c)}{\sin(x-x_j^c+2\mathrm{i}\gamma)}\right] \\ \qquad \left[\prod_{j=1}^{n_+} \frac{\sin(x-x_j^+)}{\sin(x-x_j^++2\mathrm{i}\gamma)}\right] \left[\prod_{j=1}^{n_-} \frac{\sin(x-x_j^--\mathrm{i}\gamma)}{\sin(x-x_j^-+\mathrm{i}\gamma)}\right] & \text{for } x \in D_- \end{cases}$$

Higher level Bethe equations with $n_h - 2n_c - 2n_f = 2s$ and

$$\phi(x,z) = \mathrm{i}\left(\frac{\pi}{2} + x - z\right) + \ln\left(\frac{\Gamma_{q^4}\left(\frac{1}{2} - \frac{\mathrm{i}(x-z)}{2\eta}\right)\Gamma_{q^4}\left(1 + \frac{\mathrm{i}(x-z)}{2\eta}\right)}{\Gamma_{q^4}\left(\frac{1}{2} + \frac{\mathrm{i}(x-z)}{2\eta}\right)\Gamma_{q^4}\left(1 - \frac{\mathrm{i}(x-z)}{2\eta}\right)}\right)$$

Summary and further outlook

- We have derived expressions for thermal form factors of the XXZ chain in the Trotter limit
- These determine the amplitude in the asymptotic expansion of the correlation functions at finite temperature
- The formulae for the amplitudes seem to be of universal form
- Asymptotic series based on form factors of the QTM are efficient for calculating the large-distance asymptotics of two-point functions
- We can extract the critical behaviour at $T \to 0$ from our formulae. Upon summation we obtain the Tomonaga-Luttinger liquid correlation functions and efficient expressions for the non-universal amplitudes at finite magnetic field
- A low-T analysis at finite magnetic field h seems to be efficient in the massive phase as well, where strings seem to be avoidable

Based on published work (JSTAT 2013 P07010, 2014 P04012) with MAXIME DUGAVE (Wuppertal) and KAROL K. KOZLOWSKI (Dijon) and on work in progress jointly with the above authors and JUNJI SUZUKI (Shizuoka)