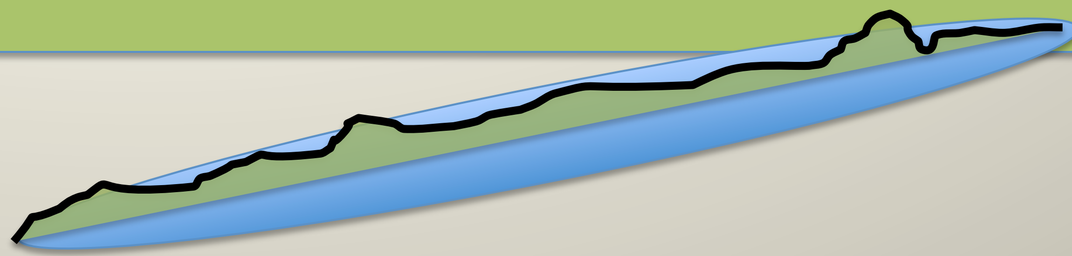
GGE and applications for integrable models

Garry Goldstein

The State University of New Jersey
Rutgers
Department of Physics and Astronomy

With Natan Andrei

What is the Lieb Liniger gas



- 1 dimensional
- Bosons
- Hamiltonian

$$H=\int_{-L/2}^{L/2} dx\left(\partial_x b^+(x)\partial_x b(x)+cb^+(x)b^+(x)b(x)b(x)\right)$$

- "one dimensional bosons with contact interaction"
- Focus on c>0

What do we want to do?

- **Compute** $\langle \Phi | Exp(\alpha Q_{xy}(T)) | \Psi \rangle$

Where $Q_{xy} \equiv \int_x^y b^+(z) b(z) dz$

This leads to correlation functions:

$$\langle b^+(x) b(x) \rangle = -\frac{\partial}{\partial \alpha} \frac{\partial}{\partial x} \langle Exp(\alpha Q_{xy}) \rangle_{\alpha=0}$$

$$\langle b^+(x) b(x) b^+(y) b(y) \rangle = -\frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \frac{\partial^2}{\partial x \partial y} \langle Exp(\alpha Q_{xy}) \rangle_{\alpha=0}$$

How are we going to do it?

$$\left\langle Exp\left(\alpha Q_{xy}\left(T\right)\right)\right\rangle \equiv \left\langle \Phi\right| e^{iTH} Exp\left(\alpha Q_{xy}\right) e^{-iTH} \left|\Psi\right\rangle =$$

$$= \sum_{\vec{q}} \sum_{\vec{k}} \left\langle \Phi | \vec{q} \right\rangle \left\langle \vec{q} \right| Exp\left(\alpha Q_{xy}\right) \left| \vec{k} \right\rangle \left\langle \vec{k} | \Psi \right\rangle e^{i\left(\vec{q}^{\,2} - \vec{k}^{\,2}\right)T}$$

1) Compute the eigenstates – Bethe Ansatz

2) Compute the initial overlaps – Yudson representation

3) Compute matrix elements $\left\langle \vec{q} \right| Exp\left(\alpha Q_{xy}\right) \left| \vec{k} \right\rangle$ - Algebraic Bethe ansatz

4) Sum over states – Turns out to be a determinant

Yudson Representation for finite sized systems

Usual

$$I_N = \sum_{n_1<\dots<n_N} \frac{1}{N\left(k_{n_1},\dots,k_{n_N}\right)} \left|k_{n_1}\dots k_{n_N}\right\rangle\left\langle k_{n_1}\dots k_{n_N}\right|$$

Yudson

$$I_N = \sum_{n_1\dots n_N} \frac{1}{N\left(k_{n_1},\dots,k_{n_N}\right)} \left|k_{n_1}\dots k_{n_N}\right\rangle\left(k_{n_1}\dots k_{n_N} \right\|$$

Here

$$N\left(k_{n_1}\dots k_{n_N}\right) = \left\langle k_{n_1}\dots k_{n_N} \middle| k_{n_1}\dots k_{n_N}\right\rangle$$

$$\left(k_{n_1}\dots k_{n_N}\right| = \prod_i e^{i k_{n_i} x_i}$$

$$\left(\quad \middle\| \quad \right\rangle = \int dx_1 \dots dx_N\, \theta\left(x_1 < x_2 < \dots < x_N\right)$$

$$N\left(k_1\dots k_N\right) = \det\left(M_{jk}\right)$$

$$M_{jk} = \delta_{jk}\left(L + \sum \frac{2c}{c^2+\left(k_j-k_l\right)^2}\right) - \frac{2c}{c^2+\left(k_j-k_k\right)^2}$$

$$Exp\left(ik_iL\right) = \prod_{j\neq i} \frac{k_i - k_j + ic}{k_i - k_j - ic}$$

Simplified final answer

$$Exp\left(\alpha Q_{xy}(T)\right)=1+\frac{i\left(e^{\alpha}-1\right)}{2\pi}\left(\int dxdy\frac{Exp\left(-\frac{i}{4T}\left(y^2-x^2\right)\right)}{y-x}\cdot\left(Exp\left(\frac{i(y-x)x}{2T}\right)-Exp\left(\frac{i(y-x)y}{2T}\right)\right)\left\langle b^+(y)Exp\left(i\pi\int_x^y\rho(z)dz\right)b(x)\right\rangle\right)-$$

$$-\frac{\left(e^{\alpha}-1\right)^2}{(2\pi)^2}\left(\int dx_1dy_1\frac{Exp\left(-\frac{i}{4T}\left(y_1{}^2-x_1{}^2\right)\right)}{y_1-x_1}\cdot\left(Exp\left(\frac{i(y_1-x_1)x}{2T}\right)-Exp\left(\frac{i(y_1-x_1)y}{2T}\right)\right)\right)\times$$

$$\times\left(\int dx_2dy_2\frac{Exp\left(-\frac{i}{4T}\left(y_2{}^2-x_2{}^2\right)\right)}{y_2-x_2}\cdot\left(Exp\left(\frac{i(y_2-x_2)x}{2T}\right)-Exp\left(\frac{i(y_2-x_2)y}{2T}\right)\right)\right)\times$$

$$\times\left\langle \operatorname{sgn}(y_2-y_1)\operatorname{sgn}(x_2-x_1)b^+(y_1)b^+(y_2)Exp\left(i\pi\int_{x_1}^{y_1}\rho(z)dz\right)Exp\left(i\pi\int_{x_2}^{y_2}\rho(z)dz\right)b(x_1)b(x_2)\right\rangle-$$

$$-\frac{i2\left(e^{2\alpha}-1\right)}{(2\pi)^2c}\left(\int dx_1dy_1\frac{Exp\left(-\frac{i}{4T}\left(y_1{}^2-x_1{}^2\right)\right)}{y_1-x_1}\cdot Exp\left(\frac{i(y_1-x_1)x}{2T}\right)\int dx_2dy_2\frac{Exp\left(-\frac{i}{4T}\left(y_2{}^2-x_2{}^2\right)\right)}{T}\cdot Exp\left(\frac{i(y_2-x_2)x}{2T}\right)-\right.$$

$$\left.\int dx_1dy_1\frac{Exp\left(-\frac{i}{4T}\left(y_1{}^2-x_1{}^2\right)\right)}{y_1-x_1}\cdot Exp\left(\frac{i(y_1-x_1)y}{2T}\right)\int dx_2dy_2\frac{Exp\left(-\frac{i}{4T}\left(y_2{}^2-x_2{}^2\right)\right)}{T}\cdot Exp\left(\frac{i(y_2-x_2)y}{2T}\right)\right)\times$$

$$\times\left\langle \operatorname{sgn}(y_2-y_1)\operatorname{sgn}(x_2-x_1)b^+(y_1)b^+(y_2)Exp\left(i\pi\int_{x_1}^{y_1}\rho(z)dz\right)Exp\left(i\pi\int_{x_2}^{y_2}\rho(z)dz\right)b(x_1)b(x_2)\right\rangle+\ldots..$$

GGE Hypothesis - Integrable Models

- Integrable models have infinite number of local conserved quantities I_m
- Lieb Liniger model $I_m \left| \vec{k} \right\rangle = \sum k_i^m \left| \vec{k} \right\rangle$
- GGE – Generalized Gibbs Ensemble

$$\left\langle \Theta\left(T \to \infty\right)\right\rangle = Tr\left[\hat{\rho}\Theta\right]$$

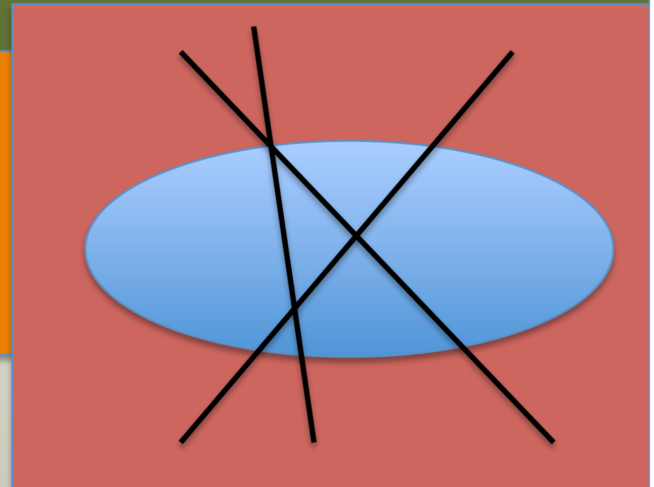
$$\hat{\rho} = Z^{-1} Exp\left(-\sum \alpha_m I_m\right)$$

Initial conditions

$$Tr\left[I_m \hat{\rho}\right] = \left\langle I_m \right\rangle \left(t = 0\right)$$

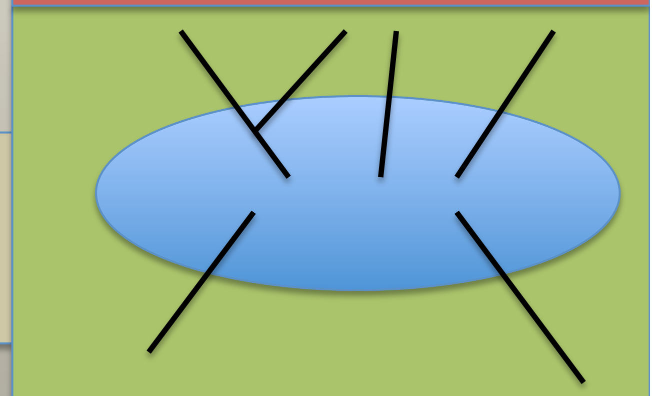
GGGE Hypothesis – Integrable models

$$\rho_{GGGE} = \tilde{Z}^{-1} Exp\left[-\sum_{m_1, m_2 \ldots.} \alpha_{m_1, m_2, \ldots} I_{m_1} I_{m_2} \ldots.. \right]$$



GGGE equivalent to diagonal ensemble hypothesis and assumption:

$$\langle \vec{k} | \Theta | \vec{k} \rangle = c_0 + c_1 \sum_i k_i + c_{1,1} \sum_{ij} k_i k_j + c_2 \sum_i k_i^2 + \ldots..$$



It will turn out

$$All \left\langle I_{m_1} I_{m_2} \ldots. \right\rangle \; are \; relevant$$

Initial conditions

$$Tr\left[I_{m_1} I_{m_2} \ldots..\hat{\rho} \right] = \left\langle I_{m_1} I_{m_2} \ldots.. \right\rangle (t=0)$$

Proof GGGE equivalent to diagonal

$$\rho_D = \sum_i p_{|\{\lambda_i\}\rangle} \left|\{\lambda_i\}\right\rangle\left\langle\{\lambda_i\}\right|$$

$$\langle I_1 \rangle = \sum_{\{\lambda_i\}} p_{|\{\lambda_i\}\rangle} \sum_i \lambda_i, \langle I_1^2 \rangle = \sum_{\{\lambda_i\}} p_{|\{\lambda_i\}\rangle} \sum_{i,j} \lambda_i \lambda_j, \langle I_2 \rangle = \sum_{\{\lambda_i\}} p_{|\{\lambda_i\}\rangle} \sum_i \lambda_i^2 \ldots\ldots$$

$$\langle \Theta \rangle = Tr\left[\rho_D \Theta\right] = c_0 + c_1 \sum P_{|\{\lambda_i\}\rangle} \sum \lambda_i + c_{1,1} \sum P_{|\{\lambda_i\}\rangle} \left(\sum \lambda_i\right)^2 + c_2 \sum P_{|\{\lambda_i\}\rangle} \sum \lambda_i^2 + \ldots. =$$

$$= c_0 + c_1 \langle I_1 \rangle + c_{1,1} \langle I_1^2 \rangle + c_2 \langle I_2 \rangle + \ldots. = Tr\left[\rho_{GGGE} \Theta\right]$$

GGE vs GGGE

$$\langle I_1^2\rangle = \langle I_1\rangle^2$$
$$\langle I_1 I_2\rangle = \langle I_1\rangle\langle I_2\rangle$$
$$\langle I_1^3\rangle = \langle I_1\rangle^3$$

Then GGGE → GGE

Which is true if:

$$\langle ib^+(x)\partial_x b(x) ib^+(y)b(y)\rangle \to \langle ib^+(x)\partial_x b(x)\rangle\langle ib^+(y)b(y)\rangle$$

$$|x-y| \to \infty$$

But its easy to come up with a counter example

$$\rho = \frac{1}{T}\int_0^T \frac{e^{-\frac{H}{t}}}{Z_t}dt$$

$$\langle I_2^2\rangle = \frac{1}{T}\int_0^T Tr\left[H^2\frac{e^{-\frac{H}{t}}}{Z_t}\right]dt, \langle I_2\rangle^2 = \left(\frac{1}{T}\int_0^T Tr\left[H\frac{e^{-\frac{H}{t}}}{Z_t}\right]\right)^2$$

$$I_1 = \int dx ib^+(x)\partial_x b(x)$$

$$I_2 = \int dx\left(\partial_x b^+\partial_x b + cb^+(x)b^+(x)b(x)b(x)\right)$$

$$I_3 = \int dx\left(b^+(x)\partial_{x^3}^3 b(x) - \frac{3c}{2}\left(b^+(x)\right)^2\partial_x\left(b(x)\right)^2\right)$$

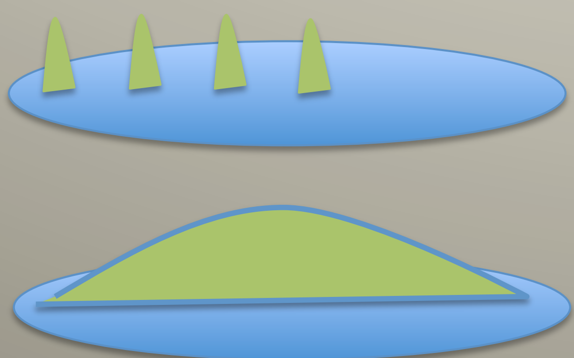
GGGE for Lieb Liniger

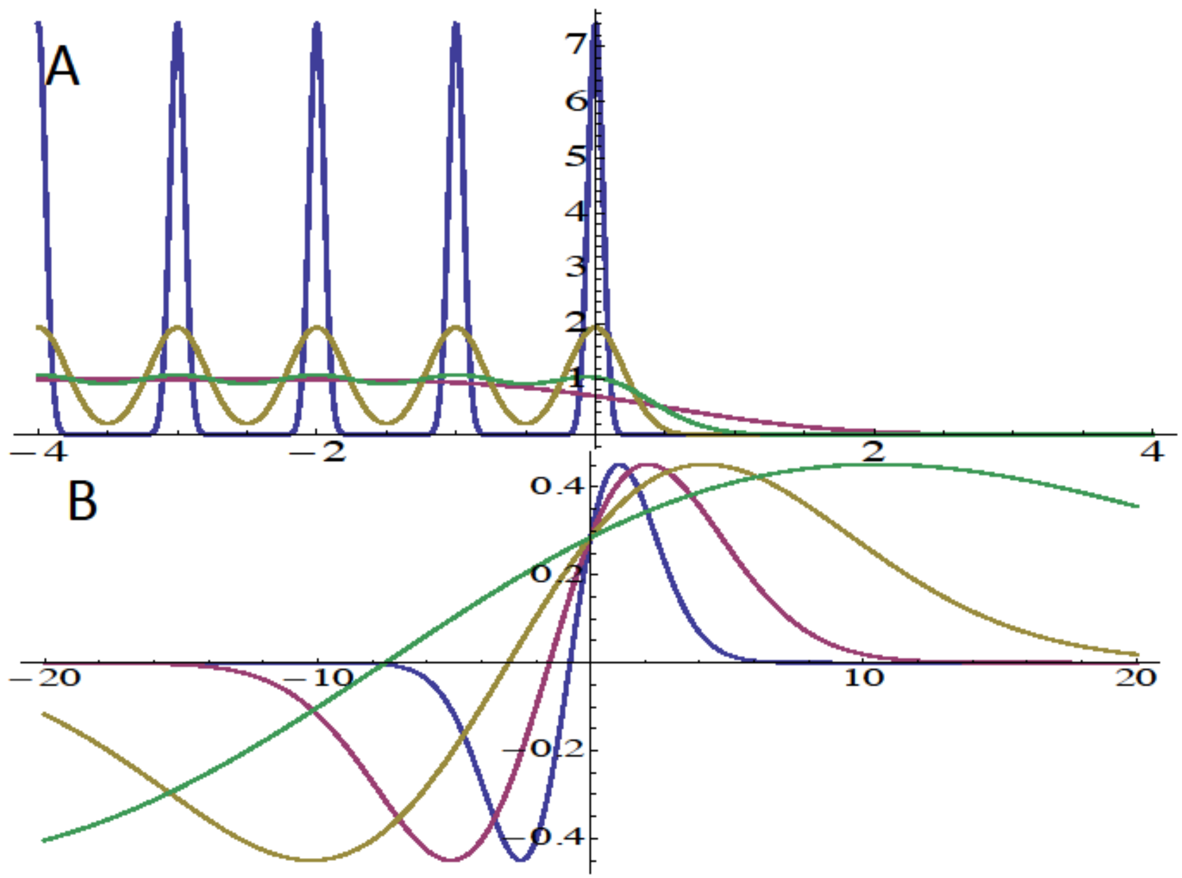
- Need only prove diagonal ensemble – every term $\propto Exp\left(i\left(\sum q_i^2 - \sum k_i^2\right)T\right)$
- Taylor expansion of observables see Algebraic Bethe ansatz.

Other Applications

Mott insulator on the half line

A) Density Tonks Gas
B) 1/c corrections to density



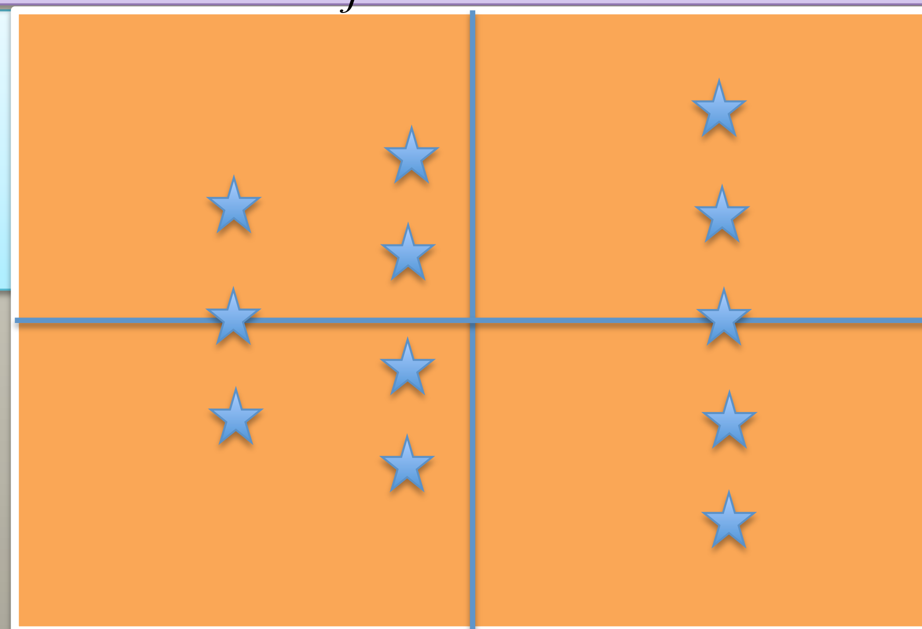


Does the GGE work for other models? …No

The long time dynamics of models with bound states are not controlled by the GGE

$$\left|\Psi(k_1,\ldots k_N)\right\rangle = \int dx_1\ldots dx_N \times \prod_{i<j} Z_{ij}^{x}(k_i - k_j) \prod_j e^{ik_j x_j} b^{+}(x_j)\left|0\right\rangle$$

Here k's are complex numbers arranged in strings in the complex plane. Each string is a bound state



The strings are chosen such that

$$S(k_1 - k_2) = 0,$$

As such the wavefunction is normalisable

$$S(k) = \frac{k - ic}{k + ic}$$

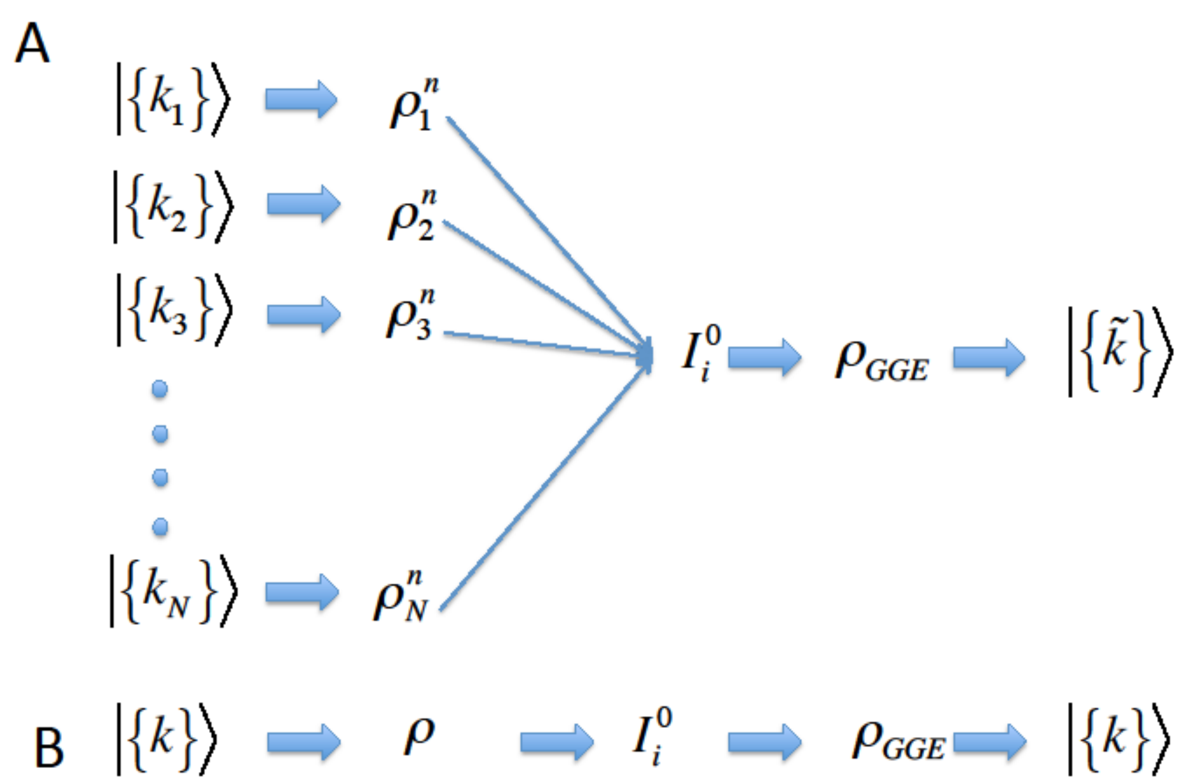
Multi to one mapping

We can introduce densities of string excitations

$$L\rho_p^n(k)dk$$

Number of strings in the interval [k,k+dk]

Expectation values of all local observables are determined by the string densities.



Attractive Lieb-Liniger conservation laws

J_l^n moments of ρ_p^n

$$\sum_{n=0}^{\infty}\sum_{l=0}^{i} J_l^n \left(\frac{ic}{2}\right)^{i-l} \sum_{j=0}^{n}(n-2j)^{i-l} = I_i^0$$

How to detect a GGE? Universal correlations

$$c = \infty$$

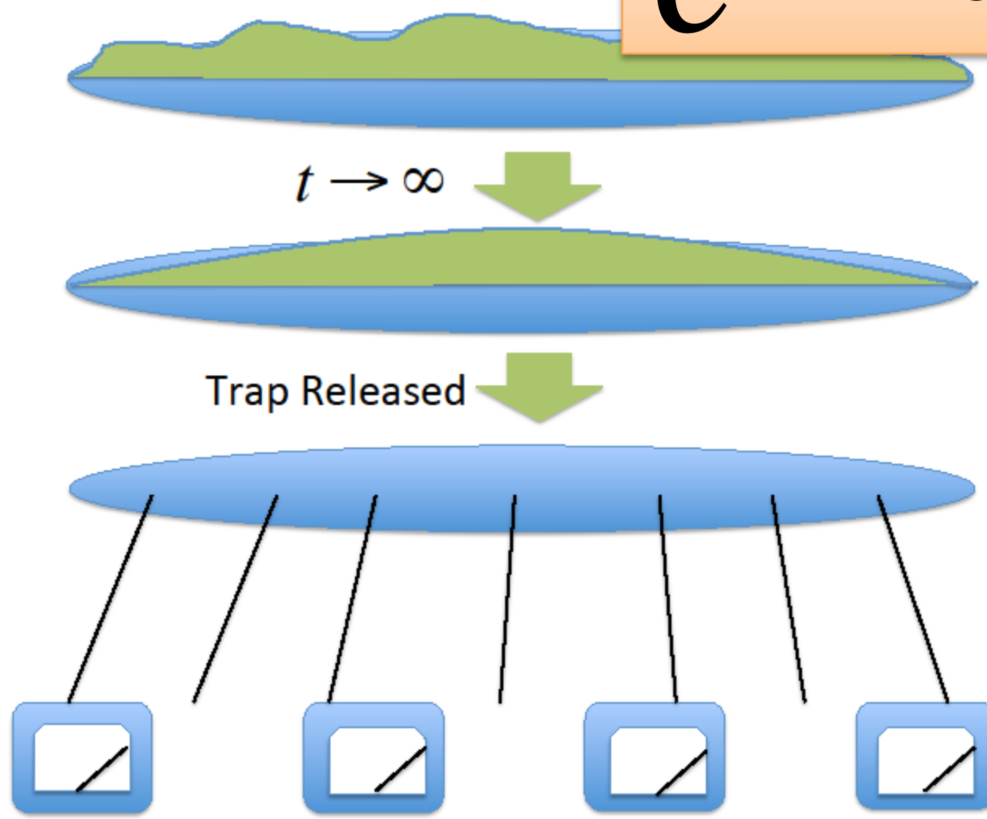
Initialize in some non-equilibrium state

$$|\Phi\rangle \to e^{-itH}|\Phi\rangle \equiv |\Psi\rangle$$

Establish a GGE after long time evolution. Measure correlation functions

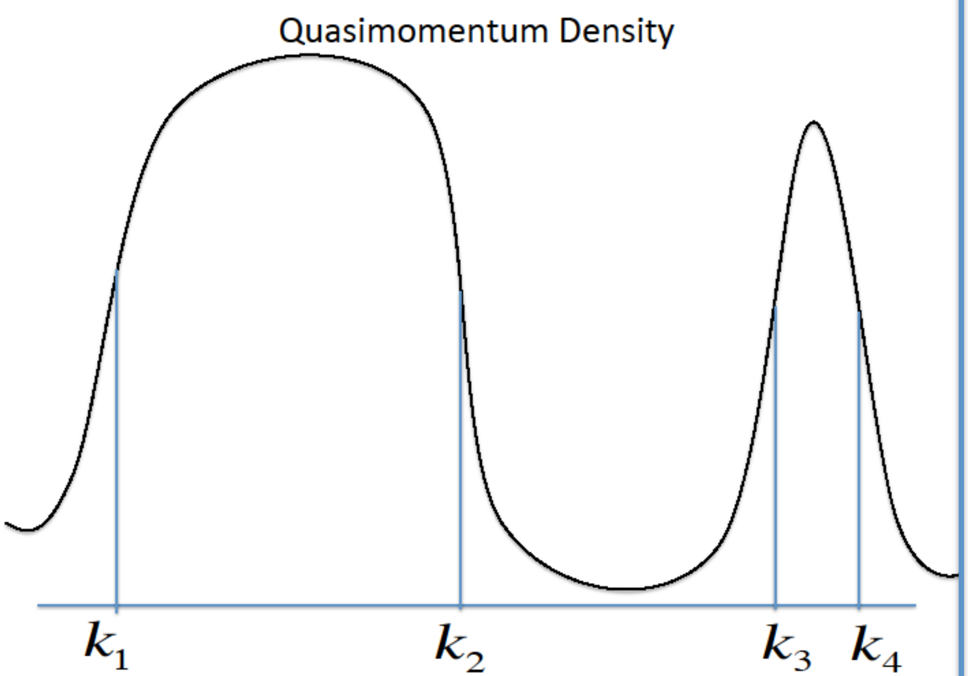
$$\rho(x)\rho(0)$$

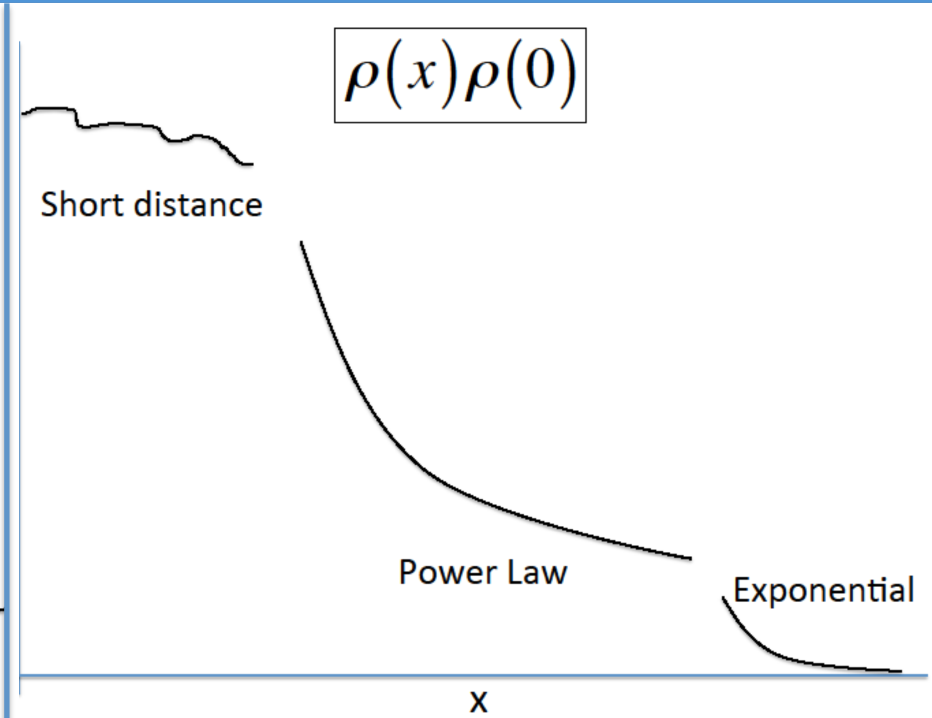
When Yang Yang entropy is low get universal results



$$\rho(x)\rho(0) \sim \frac{1}{x^2}$$

Results





when Yang Yang entropy is low density density correlations have 3 regions. Complex short distance behavior , power law $1/x^2$ intermediate distance behavior and exponential long distance behavior.

The power law region is controlled by the momenta when the quasiparticle density has rapid changes.

Derivation

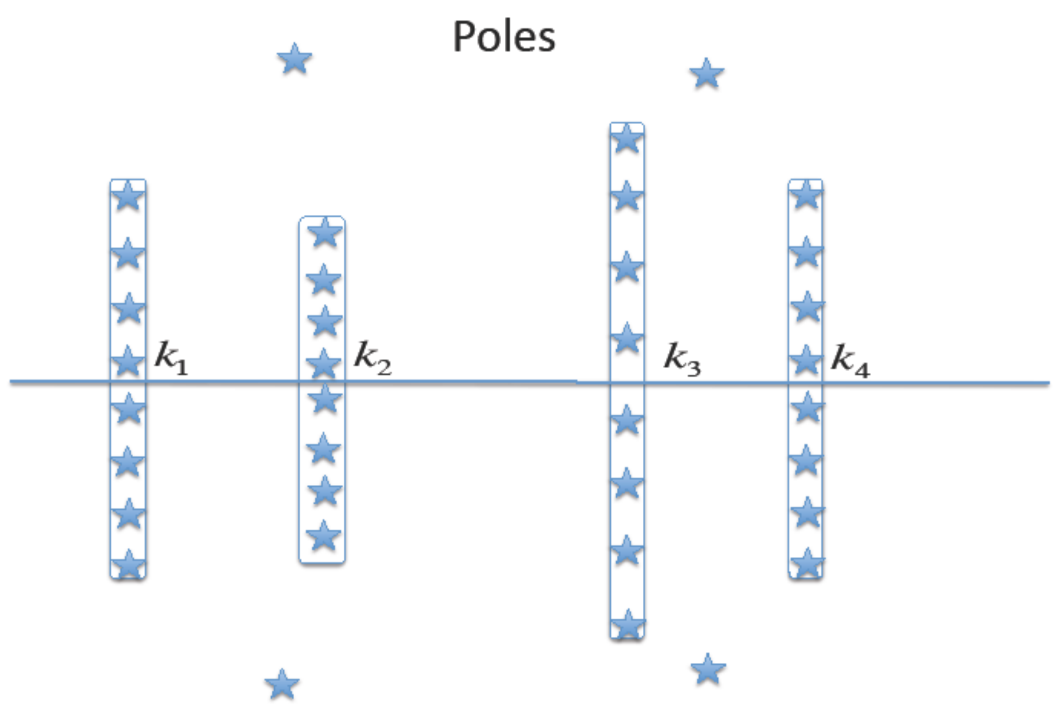
The GGE corresponds to an eigenstate $\left|\vec{k}_0\right\rangle$

$$\frac{\rho_t(k)-\rho_p(k)}{\rho_p(k)} = e^{\sum \alpha_i k^i} \equiv e^{\varepsilon(k)}$$

We can calculate

$$\langle \rho(x)\rho(0)\rangle = \rho^2 + \left|\sum \frac{e^{ix\varepsilon(s_j)}}{\varepsilon'(s_j)}\right|^2$$

Where $\varepsilon(s_j) = (2n+1)\pi i, \mathrm{Im}(s_j) > 0$



When the Yang Yang entropy is low there are lots of solutions to $\varepsilon(s_j) = (2n+1)\pi i$ Near $\varepsilon(k_n) = 0$ then

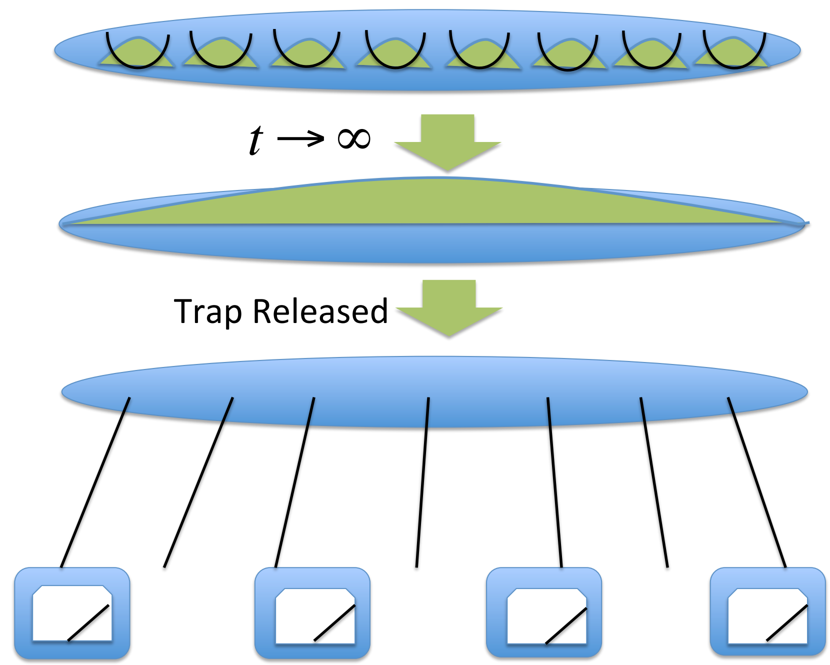
$$\langle \rho(x)\rho(0)\rangle = \rho^2 + \frac{1}{4\pi^2 x^2}\left|\sum \mathrm{sgn}(\varepsilon'(k_n))e^{ik_n x}\right|^2$$

Trap Release Mott Insulator

$$\left|\Phi_0\right\rangle = \prod_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x+j\cdot l)\,b^{+}(x)\left|0\right\rangle$$

$$\varphi(x) = \frac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$$

$$L\int dk\,\rho_p(k) = I_n(t=0)$$



$$I_n(t=0) = \frac{L}{l}\left\langle 0\right|\int dx\varphi^*(x)b(x)I_n\int dy\varphi(y)b^{+}(y)\left|0\right\rangle$$

$$\rho_p(k) = \frac{\sigma^{1/2}}{\pi^{1/2}l}Exp\left(-\frac{k^2\sigma}{2}\right)$$

$$\left\langle \rho(x)\rho(0)\right\rangle = \rho^2 + \frac{1}{4\pi^2 l^2}Exp\left(-\frac{x^2}{\sigma}\right)$$

Conclusions

1) Showed finite sized Yudson formula
2) Presented formulas for correlators for Lieb Liniger and Tonks Girardeau Gas
3) Used them to prove GGGE and diagonal ensemble
4) Showed failure of GGE for systems with bound states
5) Universality in Tonks quench

Future Trends

1) Field field correlators $\langle b^{+}(x)b(y)\rangle(T)$
2) Two time correlation functions $\langle\rho(x,T)\rho(y,T')\rangle$
3) Imaginary time formalism $T \to i\tau$
4) Kubo Formula
5) Other intial states
6) What operators work for systems with bound states?