Some integrable systems with algebraic separation of variables

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Summary

An integrable systems of mechanical origin with a compact configuration space is called *irreducible* if it does not have any continuous symmetry group and, therefore, cannot be globally reduced to a family of systems with two degrees of freedom. One of the known examples of an irreducible system was found by A.G. Reyman and M.A. Semenov-Tian-Shansky. Today, it is the widest generalization of the classical Kowalevski case and the Kowalevski–Yehia gyrostat.

The integrable systems with two degrees of freedom appear in an irreducible system as invariant almost symplectic submanifolds of dimension 4. They are called critical subsystems. For two of them we present an algebraic separation of variables, i.e., two auxiliary variables satisfy the equations of the Kowalevski type and all initial phase variables are expressed in terms of the separated ones as rational functions of simple radicals with polynomial coefficients. One separation is integrated in elliptic functions, another is hyperelliptic.

Considering the phase space of the separated variables as a 4-dimensional Euclidean space of two coordinates and two momenta, one can derive the polynomial equations for the Hamilton function H. In the elliptic case it is of degree 4 with respect to H.

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General system and its subsystems

Phase space $e(3,2)^*$, equations $\dot{x} = \{x, H\}$, the Hamiltonian and the first integrals [1-3]

$$
\begin{aligned}
H &= \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - \alpha_1 - \beta_2,\\
K &= (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2,\\
G &= (\omega_1\alpha_1 + \omega_2\alpha_2 + \tfrac{1}{2}\omega_3\alpha_3)^2 + (\omega_1\beta_1 + \omega_2\beta_2 + \tfrac{1}{2}\omega_3\beta_3)^2\\
&\quad + \omega_3[\omega_1(\alpha_2\beta_3 - \alpha_3\beta_2) + \omega_2(\alpha_3\beta_1 - \alpha_1\beta_3) + \tfrac{1}{2}\omega_3(\alpha_1\beta_2 - \alpha_2\beta_1)]\\
&\quad - \alpha_1 b^2 - \beta_2 a^2,\\
\alpha^2 &= a^2, \quad \beta^2 = b^2, \quad \alpha\cdot\beta = 0 \quad (a > b > 0).
\end{aligned}
$$

Critical surfaces [4,5]

$$
\begin{aligned}
&\psi_1(h,k,g) := [2g - (a^2+b^2)h]^2 - (a^2-b^2)^2 k = 0,\\
&\psi_2(h,k,g) := \mathrm{Resultant}_s\Big[s^2(k - 3s^2 + 4hs - a^2 - b^2 - h^2) + a^2b^2,\\
&s(g + s^3 - hs^2) - a^2b^2\Big] = 0
\end{aligned}
$$

and critical subsystems

$$
\mathcal{M}_i = \{x : \Theta_i(x) = 0, \mathrm{d}\,\Theta_i(x) = 0\} \quad (\Theta_i(x) = \psi_i(H(x), K(x), G(x)),\ i = 1,2)
$$

First algebraic separation

On $\mathcal{M}_1$, having two independent partial integrals L, M, denote

$$
\Psi = 4ms_1s_2 - 2\ell(s_1+s_2) + \frac{\ell^2-1}{m}, \qquad \Pi(s) = \Psi(s,s),
$$
$$
G_1 = \sqrt{s_1^2 - a^2}, \quad F_1 = \sqrt{-\Pi(s_1)}, \quad G_2 = \sqrt{b^2 - s_2^2}, \quad F_2 = \sqrt{\Pi(s_2)}.
$$

Proposition [6]. In auxiliary variables s_1, s_2, the equations of motion separate

$$
\dot{s}_1 = -\frac{1}{2}G_1F_1, \qquad \dot{s}_2 = \frac{1}{2}G_2F_2
$$

and the phase variables obtain the following expressions in terms of s_1, s_2 and the integral constants

$$
\begin{aligned}
\alpha_1 &= \frac{1}{2(s_1-s_2)^2}[(s_1s_2 - a^2)\Psi - G_1G_2F_1F_2],\\
\alpha_2 &= \frac{1}{2(s_1-s_2)^2}[(a^2 - s_1s_2)F_1F_2 - \Psi G_1G_2],\\
\beta_1 &= \frac{1}{2(s_1-s_2)^2}[\Psi G_1G_2 + (s_1s_2 - b^2)F_1F_2],\\
\beta_2 &= \frac{1}{2(s_1-s_2)^2}[(s_1s_2 - b^2)\Psi - G_1G_2F_1F_2],\\
\omega_1 &= \frac{r(\ell - 2ms_1)F_2}{2(s_1-s_2)}, \quad \omega_2 = \frac{r(2ms_2 - \ell)F_1}{2(s_1-s_2)},\\
\alpha_3 &= \frac{rG_1}{s_1-s_2}, \quad \beta_3 = \frac{rG_2}{s_1-s_2}, \quad \omega_3 = \frac{G_2F_1 - G_1F_2}{s_1-s_2}.
\end{aligned}
$$

Second algebraic separation

On $\mathcal{M}_2$, having two independent partial integrals S, T, denote

$$
\begin{aligned}
&\sigma = \tau^2 - 2p^2\tau + r^4, \quad \chi = \sqrt{\frac{4s^2\tau + \sigma}{4s^2}}, \quad \varkappa = \sqrt{\sigma},\\
&K_1 = \sqrt{t_1 + \varkappa}, \quad K_2 = \sqrt{t_2 + \varkappa},\\
&L_1 = \sqrt{t_1 - \varkappa}, \quad L_2 = \sqrt{t_2 - \varkappa},\\
&M_1 = \sqrt{t_1 + \tau + r^2}, \quad M_2 = \sqrt{t_2 + \tau + r^2},\\
&N_1 = \sqrt{t_1 + \tau - r^2}, \quad N_2 = \sqrt{t_2 + \tau - r^2},\\
&V_1 = \sqrt{\frac{t_1^2 - 4s^2\chi^2}{s\tau}}, \quad V_2 = \sqrt{\frac{t_2^2 - 4s^2\chi^2}{s\tau}}\\
&U_1 = K_1L_1, \quad U_2 = K_2L_2, \quad R = K_1K_2 + L_1L_2,\\
&\mathcal{A} = [(t_1 + \tau + r^2)(t_2 + \tau + r^2) - 2(p^2 + r^2)r^2]\tau,\\
&\mathcal{B} = [(t_1 + \tau - r^2)(t_2 + \tau - r^2) + 2(p^2 - r^2)r^2]\tau.
\end{aligned}
$$

Proposition [7]. In auxiliary variables t_1, t_2, the equations of motion separate

$$
(t_1 - t_2)\dot{t}_1 = \sqrt{\frac{1}{2s\tau}(t_1^2 - 4s^2\chi^2)(t_1^2 - \sigma)[(t_1+\tau)^2 - r^4]},
$$
$$
(t_1 - t_2)\dot{t}_2 = \sqrt{\frac{1}{2s\tau}(t_2^2 - 4s^2\chi^2)(t_2^2 - \sigma)[(t_2+\tau)^2 - r^4]}.
$$

and the phase variables obtain the following expressions in terms of t_1, t_2 and the integral constants

$$
\begin{aligned}
\alpha_1 &= (U_1 - U_2)^2\frac{(\mathcal{A} - r^2U_1U_2)(4s^2\tau + U_1U_2) - (\tau + r^2)s\tau M_1N_1V_1M_2N_2V_2}{4r^2s\tau(t_1^2 - t_2^2)^2},\\
\alpha_2 &= \mathrm{i}(U_1 - U_2)^2\frac{(\mathcal{A} - r^2U_1U_2)s\tau V_1V_2 - (4s^2\tau + U_1U_2)(\tau + r^2)M_1N_1M_2N_2}{4r^2s\tau(t_1^2 - t_2^2)^2},\\
\alpha_3 &= \frac{R}{2r}\frac{M_1M_2}{t_1 + t_2},\\
\beta_1 &= \mathrm{i}(U_1 - U_2)^2\frac{(\mathcal{B} + r^2U_1U_2)s\tau V_1V_2 - (4s^2\tau + U_1U_2)(\tau - r^2)M_1N_1M_2N_2}{4r^2s\tau(t_1^2 - t_2^2)^2},\\
\beta_2 &= -(U_1 - U_2)^2\frac{(\mathcal{B} + r^2U_1U_2)(4s^2\tau + U_1U_2) - (\tau - r^2)s\tau M_1N_1V_1M_2N_2V_2}{4r^2s\tau(t_1^2 - t_2^2)^2},\\
\beta_3 &= -\mathrm{i}\frac{R}{2r}\frac{N_1N_2}{t_1 + t_2},\\
\omega_1 &= \mathrm{i}\frac{R}{4rs\sqrt{2}}\frac{U_1N_1M_2V_2 - M_1V_1U_2N_2}{t_1^2 - t_2^2},\\
\omega_2 &= \frac{R}{4rs\sqrt{2}}\frac{U_1M_1N_2V_2 - N_1V_1U_2M_2}{t_1^2 - t_2^2},\\
\omega_3 &= -\mathrm{i}\frac{U_1 - U_2}{\sqrt{2}}\frac{V_1M_2N_2 + M_1N_1V_2}{t_1^2 - t_2^2}.
\end{aligned}
$$

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