Solving the Bethe Ansatz Equations: Quantum Schubert Calculus

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Ginzburg-Kapranov-Vasserot:

Schubert calculus	formal group law	QG / YBE
cohomology	additive	Yangian / rational
K-theory	multiplicative	$U_q(\mathfrak{g})$ / trigonometric
elliptic cohomology	elliptic curves	/ elliptic

Gepner, Vafa, Witten: fusion rings in CFT $\rightsquigarrow$ quantum cohomology

Givental, Kim, Lee: quantum Toda + WDVV $\rightsquigarrow$ quantum cohomology & quantum K-theory

Nekrasov-Shatashvili, Braverman-Maulik-Okounkov: Yangian structure for Nakajima varieties

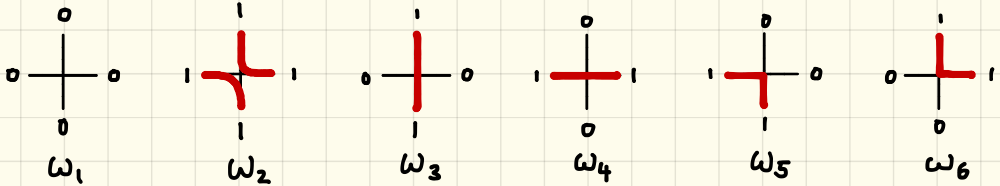
Gorbounov-Rimanyi-Tarasov-Varchenko: cotangent bundle of flag varieties

$\boxed{\text{CK-Stroppel, Gorbounov-CK: quantum Schubert calculus of } Gr_n(\mathbb{C}^N)}$

Outline

1. Vicious & Osculating Walkers
2. DWBC & Factorial Grothendieck Polynomials
3. Algebraic Bethe Ansatz
4. Coordinate ring description
5. GKM Theory: localised Schubert classes

Recall the integrability conditions for the <u>asymmetric six-vertex model</u>:



Two quadrics: $\quad \Delta = \dfrac{\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6}{2\,\omega_1\omega_3}\,, \quad \Gamma = \dfrac{\omega_2\omega_4}{\omega_1\omega_3}$

[Baxter] The equation $R_{12}(\omega)R_{13}(\omega')R_{23}(\omega'') = R_{23}(\omega'')R_{13}(\omega')R_{12}(\omega)$ has a solution if $\omega, \omega', \omega''$ all share the same Δ, Γ.

There are two values of Δ which are special:

$\Delta = 0$ – free fermion point [Lieb-Schultz-Mattis, Fan-Wu, ...]

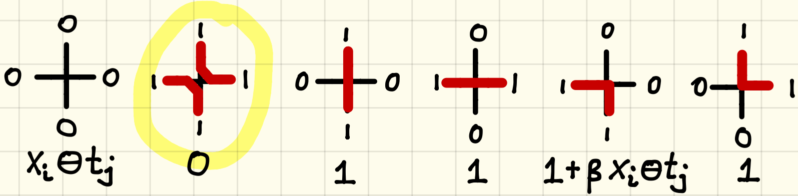
$\Delta = 1/2$ – cubic root of unity [Razumov-Stroganov, ...]

Input from algebraic topology: consider the multiplicative formal group law

$$\boxed{x \oplus y := x + y + \beta x y} \quad \text{and its inverse} \quad \boxed{x \ominus y := \frac{x-y}{1+\beta y}}$$

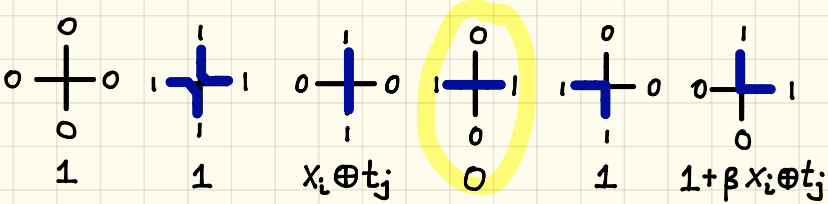
where we want to treat β as a formal variable ("deformation parameter").

Consider the following 5-vertex degenerations of the asymmetric 6-vertex model:



Vicious Walkers

$\Delta = -\beta/2 \quad \Gamma = 0$



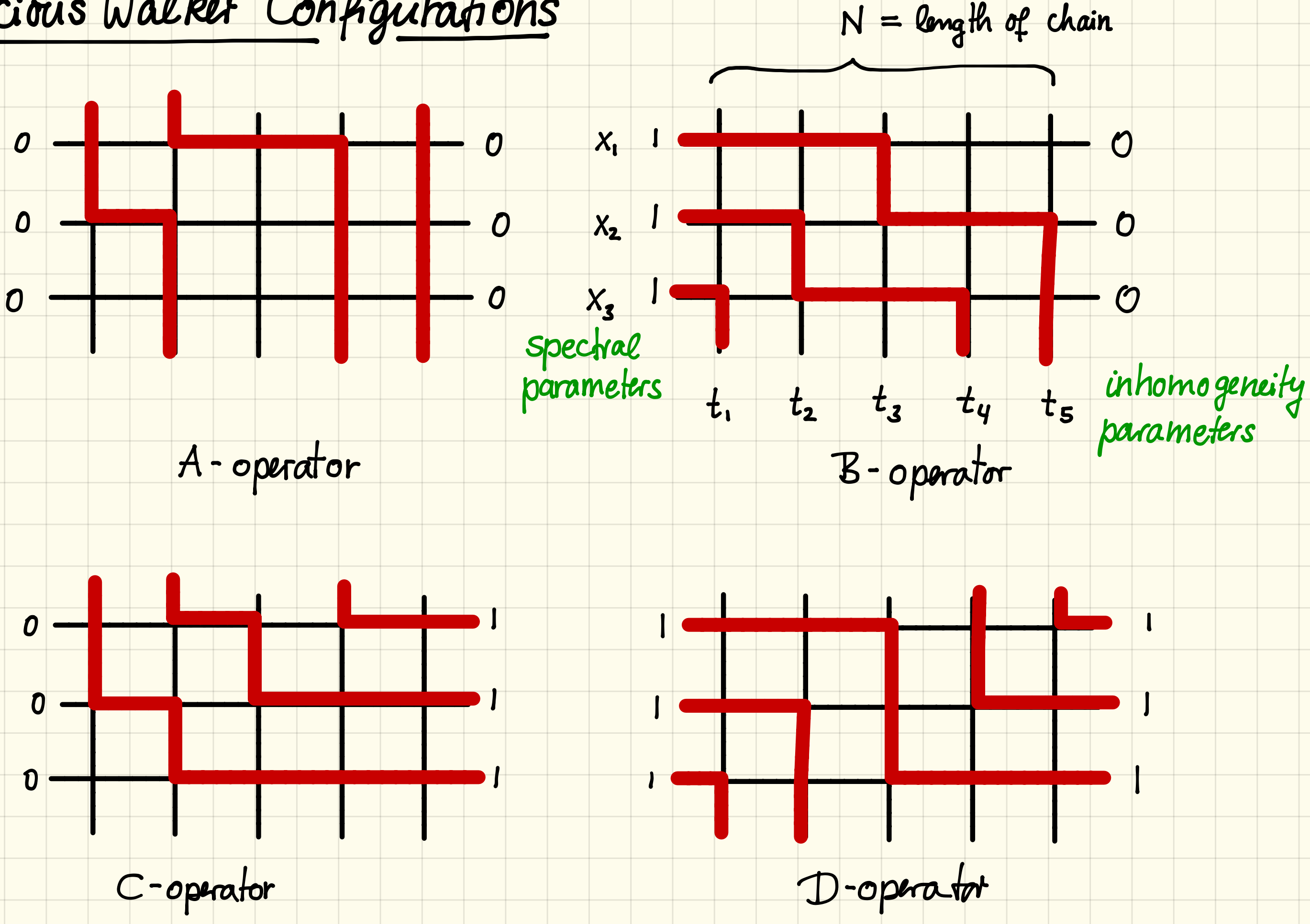
Osculating Walkers

$\Delta = -\beta/2 \quad \Gamma = 0$

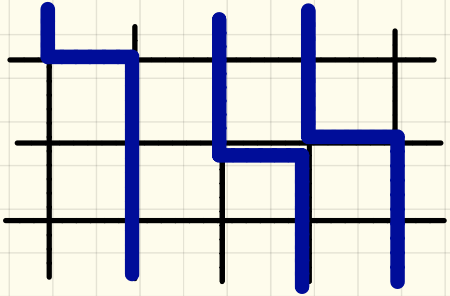
Special value $\beta = 0$: quantum equivariant cohomology $QH^T(Gr(n, \mathbb{C}^N))$

$\beta = -1$: quantum (?) equivariant K-theory $QK_T(Gr(n, \mathbb{C}^N))$

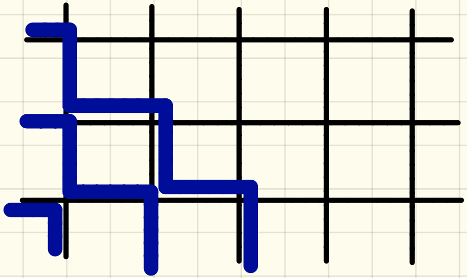
Vicious Walker Configurations



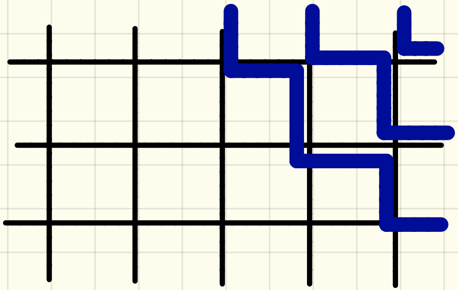
Osculating Walker Configurations



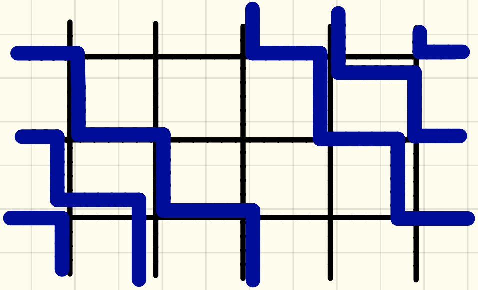
A'-operator



B'-operator



C'-operator



D'-operator

Factorial Grothendieck polynomials ($\rightarrow$ Bethe wave function)

Let λ be a partition with at most n parts and $\mathbb{P}_n$ the power set of $[n] := \{1, \dots, n\}$.

DEF [Buch] A set-valued tableau of shape λ is a map $T : \lambda \longrightarrow \mathbb{P}_n$ such that

$$\max T(i,j) \leq \min T(i,j+1) \quad \text{and} \quad \max T(i,j) < \min T(i+1,j)$$

(weakly increasing in rows) (strictly increasing in columns)

EXAMPLE

1,2	2,5
3	

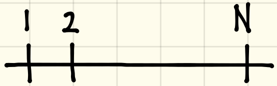
$\lambda = (2,1,0,0,0)$
$n = 5$

$$\beta^{2+2+1-3}(x_1 \oplus t_1)(x_2 \oplus t_2)(x_2 \oplus t_3)(x_5 \oplus t_6)(x_3 \oplus t_2)$$

DEF [Buch][McNamara] The factorial Grothendieck polynomial is the weighted sum

$$\boxed{\mathcal{G}_\lambda(x|t) = \sum_T \beta^{|T|-|\lambda|} \prod_{\substack{(i,j)\in\lambda \\ r\in T(i,j)}} x_r \oplus t_{r+j-i}}$$

N.B. there exist also determinant formulae [Ikeda, Naruse], $t_j = 0$ [Motegi, Sakai]

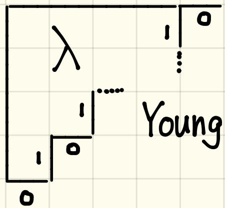
Define the momodromy matrix $M = L_N \cdots L_1 =$ 

$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

and (row) <u>Yang-Baxter algebra</u> for both models in the usual manner.

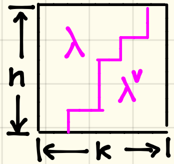
Let $\mathcal{R}$ be the ring of rational functions in β with integer coefficients and regular at $\beta=0,-1$. Set $\boxed{\mathcal{V} = \mathcal{R}(t_1,\dots,t_N)\otimes V^{\otimes N}}$ with $V=\mathbb{Z}v_0\oplus\mathbb{Z}v_1$ and define the standard basis

$$v_\lambda = v_0\otimes v_1\otimes\cdots\otimes v_1\otimes v_0$$



Young diagram

<u>THEOREM</u> (i) $C(x_1)\cdots C(x_n)\,v_\lambda = \mathcal{G}_\lambda(x|\ominus t)\, v_0\otimes v_0\otimes\cdots\otimes v_0$

$B(x_1)\cdots B(x_k)\,v_\lambda = \mathcal{G}_{\lambda'}(x|t')\, v_1\otimes v_1\otimes\cdots\otimes v_1$ for each 

$\lambda^* = (\lambda^\vee)'$

(ii) $B(x_1)\cdots B(x_n)|0\rangle = \sum_{\lambda\subset(k^n)} \mathcal{G}_{\lambda^\vee}(x|\ominus t')\frac{\Pi(x)}{\Pi(t_\lambda)}\, v_\lambda$

$C(x_1)\cdots C(x_k)|N\rangle = \sum_{\lambda\in(k^n)} \mathcal{G}_{\lambda^*}(x|t)\,\Pi(x)\,\Pi(t_{\lambda^*})\, v_\lambda$

$\Pi(x) = \prod_{i=1}^{n}(1+\beta x_i)$ etc.

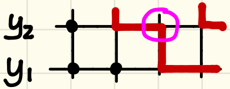
$t'_i = t_{N+1-i}$

EXAMPLES

(i)

$N=4$, $n=2$

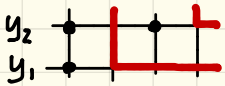
$\lambda = 2,1$



$(y_2 \ominus t_1)(y_1 \ominus t_1)(y_1 \ominus t_2)(1+\beta y_2 \ominus t_3)$

1	1,2
2	

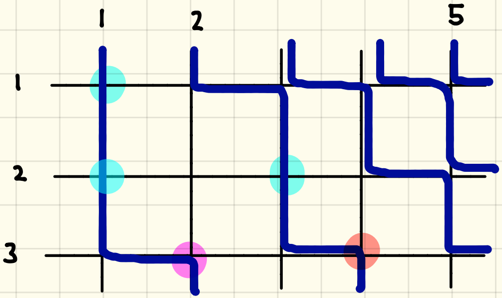
$\lambda = 2,1$



$(y_2 \ominus t_1)(y_2 \ominus t_3)(y_1 \ominus t_1)$

1	2
2	

(ii) $N=5$, $k=3$



1	2,3
2,3	

$\lambda^* = 2,1$

$(x_1 \oplus t_1)(x_2 \oplus t_1)(x_2 \oplus t_3)(1+\beta x_3 \oplus t_2)(1+\beta x_3 \oplus t_4)$

The Bethe Ansatz

Define $H = A + q D$ and $E = A' + q D'$

THEOREM (i) Let $|y\rangle = B(y_1) \cdots B(y_n)|0\rangle$ then

(osculating walkers)

$$H(x)|y\rangle = \frac{\prod_{j=1}^{N}(x \ominus t_j) + (-1)^n q \prod_{i=1}^{n} \frac{1+\beta x}{1+\beta y_i}}{(x\ominus y_1)(x\ominus y_2)\cdots(x\ominus y_n)} |y\rangle$$

and

$$E(x)|y\rangle = \prod_{i=1}^{n}(x \oplus y_i)\,|y\rangle$$

provided that

$$\boxed{\prod_{j=1}^{n}(y_i \ominus t_j) + q(-1)^n \prod_{j=1}^{n} \frac{1+\beta y_i}{1+\beta y_j} = 0 \quad \forall\, i = 1, 2, \dots, n}$$

(ii) The Bethe ansatz is complete and the solutions have the expansion

$$\boxed{y_i = t_i + q\,(-1)^n \frac{(1+\beta t_i)^{n+1}}{\prod(t_\lambda)} \prod_{j\neq i} \frac{1}{t_i \ominus t_j} + \mathcal{O}(q^2)}$$

In particular, $y_i \in \mathbb{Z}[\![q]\!] \otimes \mathbb{R}(t_1, \dots, t_N)$.

Functional relation

"Hamiltonian"

PROP $$\boxed{H(x)E(\ominus x) = \Big(\prod_{i=1}^{n} t_i \ominus x\Big)\Big(\prod_{i=n+1}^{N} x\ominus t_i\Big)(1+\beta H_1) + q\cdot 1}$$

$$(1+\beta H_1)\, v_\lambda = \prod_{i=1}^{n} \frac{1+\beta t_{n+\lambda_i - i}}{1+\beta t_i} \sum_{\lambda \rightrightarrows \mu} \beta^{|\mu-\lambda|} q^d\, v_\mu$$

not allowed

$0\cdots 010\cdots 0100\cdots$

λ

add at most one box
in each row & column

Define "higher Hamiltonians" by setting (in each fixed spin sector)

$$\boxed{H(x) = \sum_{r=0}^{k} (H_r + \beta H_{r+1}) \prod_{i=1}^{k-r} x\ominus t_{N+1-i}, \qquad E(x) = \sum_{r=0}^{n} (E_r + \beta E_{r+1}) \prod_{i=1}^{n} x\oplus t_i}$$

where $E_0 = H_0 = 1$ and $H_r = 0$ for $r > k$, $E_r = 0$ for $r > n$.

OBSERVATION The functional relation implies polynomial eqns in $\{E_r\},\{H_r\}$!

Solving the Bethe ansatz equations

Consider the (abstract) polynomial algebra generated by $\{e_r\}_{r=1}^{n} \cup \{h_r\}_{r=1}^{k}$ over $\mathcal{R}(t,q)$ modulo the relations implied by the functional relation

$$qh_n^* = \mathcal{R}(t,q)[e_1,\dots,e_n,h_1,\dots,h_k]/\sim \text{functional relation}$$

THM The map defined by $e_r \mapsto E_r$ and $h_r \mapsto H_r$ is a ring isomorphism $qh_n^* \cong \mathcal{A}_n \subset \mathrm{End}(\mathcal{V}_n)$ where $\mathcal{A}_n$ is the subalgebra generated by $\{E_r\}, \{H_r\}$ ("Bethe algebra")

What have we gained? How does this constitute a "solution"?

1. Note that e_r, h_r act by multiplication in qh_n^* ("eigenvalues" of E_r, H_r)
2. Need to find a basis $\{g_\lambda\}_{\lambda \subset k^n}$ in qh_n^* and its structure constants

$$g_\lambda \cdot g_\mu = \sum_\nu C_{\lambda\mu}^{\nu}(t,q)\, g_\nu$$

$\rightsquigarrow$ K-theoretic Gromov-Witten invariants

Reformulation of the Bethe ansatz equations (to find a basis for qh_n^*)

PROP [Ikeda-Naruse]

$$G_\lambda(x|t) = \frac{\det\left[(x_j|t)^{n+\lambda_i-i}(1+\beta x)^{i-1}\right]_{1\le i,j\le n}}{\det\left(x_j^{n-i}\right)_{1\le i,j\le n}}$$

with $(x|t)^m = \prod_{i=1}^{m} x \oplus t_i$

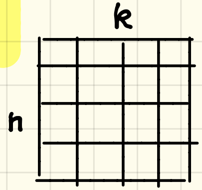
COR (straightening rule)

$$G_{\dots,\lambda_i,\lambda_{i+1},\dots} = -\beta\, G_{\dots,\lambda_i+1,\lambda_{i+1},\dots} - \frac{1+\beta t_{n+\lambda_i-i+1}}{1+\beta t_{n+\lambda_{i+1}-i}}\left(G_{\dots,\lambda_{i+1}-1,\lambda_i+1,\dots} + \beta\, G_{\dots,\lambda_{i+1},\lambda_i+1,\dots}\right)$$

PROP Let $\lambda = (\lambda_1,\dots,\lambda_n)$, $\lambda_1 > k$ and $y = (y_1,\dots,y_n)$ be a soln of the BAE.

$$G_\lambda(y|\ominus t) = q\sum_{r=0}^{\lambda_1-1-k} h_{\lambda_1-k-1-r}(t_1,\dots,t_{r+1})\, G_{\lambda_2-1,\dots,\lambda_n-1,r}(y|\ominus t)$$

The Bethe ansatz equations imply that we can restrict to $\lambda \subset$



PROP Set $e(x) = \sum_{r=0}^{n} (e_r + \beta e_{r+1})(x|\ominus t)^{n-r} = \prod_{i=1}^{n} x \oplus y_i$

then $\{g_\lambda = G_\lambda(y|\ominus t)\}$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda_1 \leq k$

form a basis of qh_n^* and $e_r = g_{1^r}$, $h_r = g_r$.

Generalised rim-hook algorithm

In the homogeneous limit $t_j \to 0$ the (K-theoretic) Littlewood-Richardson rule

$$G_\lambda(x) G_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu G_\nu(x)$$

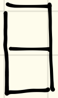
is known. [Buch]

not restricted to bounding box

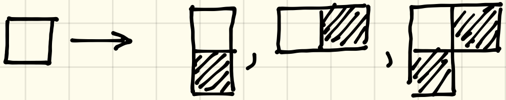
① Compute (non-restricted) product: $G_\lambda \cdot G_\mu = \sum_{\nu \subset k^n} c_{\lambda\mu}^\nu G_\nu + \sum_{\nu \not\subset k^n} c_{\lambda\mu}^\nu G_\nu$

② Reduce terms in the second sum using BAE + straightening rule to find

$$g_\lambda g_\mu = \sum_{\nu \subset k^n} C_{\lambda\mu}^\nu(q) g_\nu \quad \text{with } g_\lambda = G_\lambda \text{ for } \underline{\lambda \subset k^n}$$

Example $N=3$, $n=2$, $k=1$  $\supset \lambda = \emptyset, (1,0), (1,1)$

$$G_1 \cdot G_1 = G_{1,1} + G_{2,0} + \beta G_{2,1}$$



$$G_{2,0} = q\, G_{-1,0} = -q\beta\, G_{0,0} = -q\beta$$

$$G_{2,1} = q\, G_{0,0}$$

$$y_1^3 = -q\,\frac{1+\beta y_1}{1+\beta y_2}$$

$$y_2^3 = -q\,\frac{1+\beta y_2}{1+\beta y_1}$$

$$G_{2,1} = \frac{\begin{vmatrix} y_1^3 & y_2^3 \\ y_1(1+\beta y_1) & y_2(1+\beta y_2) \end{vmatrix}}{y_1 - y_2} = q\,\frac{\begin{vmatrix} y_1(1+\beta y_1) & y_2(1+\beta y_2) \\ \frac{1+\beta y_1}{1+\beta y_2} & \frac{1+\beta y_2}{1+\beta y_1} \end{vmatrix}}{y_1 - y_2}$$

$$= q\,\frac{y_1(1+\beta y_2) - y_2(1+\beta y_1)}{y_1 - y_2}$$

$$= q \cdot 1 = q \cdot G_{0,0}$$

$$\boxed{\Rightarrow \quad G_1 \cdot G_1 \overset{\text{BAE}}{=} G_{1,1} + q(\beta - \beta)\, G_{0,0} = G_{1,1}}$$

"Fusion matrices"

Q: How do we compute matrix elements of operators using qh_n^* ?

Define an operator $\mathbb{G}_\lambda : \mathcal{V}_n^q \to \mathcal{V}_n^q$ via $\mathbb{G}_\lambda = \sum_\alpha \beta^{|\alpha|} \phi_\alpha(\lambda) \det(H_{\lambda_i - i + j})$

① The $\mathbb{G}_\lambda$ is diagonal in the Bethe vectors, $\mathbb{G}_\lambda Y_\mu = g_\lambda(y_\mu) Y_\mu$

$$\boxed{\mathbb{G}_1 = H_1 \overset{t_j=0}{=} \sum_j \sigma_j^- \sigma_{j+1}^+ + \beta \sum_{j_1<j_2} \sigma_{j_1}^- \sigma_{j_1+1}^+ \sigma_{j_2}^- \sigma_{j_2+1}^+ + \beta^2 \sum_{j_1<j_2<j_3} \sigma_{j_1}^- \sigma_{j_1+1}^+ \sigma_{j_2}^- \sigma_{j_2+1}^+ \sigma_{j_3}^- \sigma_{j_3+1}^+ + \cdots}$$

② The $\mathbb{G}_\lambda$'s give a faithful representation of qh_n^*, i.e.

$$\boxed{\mathbb{G}_\lambda \cdot \mathbb{G}_\mu = \sum_{\nu \subset k^n} C_{\lambda\mu}^\nu(t,q)\, \mathbb{G}_\nu} \qquad \mathbb{G}_r = H_r, \quad \mathbb{G}_{1^r} = E_r$$

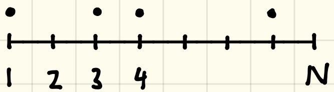
with $C_{\lambda\mu}^\nu(t,q) = \langle \nu | \mathbb{G}_\lambda | \mu \rangle = \sum_\alpha \dfrac{g_\lambda(y_\alpha) g_\mu(y_\alpha) g_{\nu^*}(\ominus y_{\alpha^*})}{\langle Y_\alpha, Y_\alpha \rangle}$

Geometric Interpretation : Goresky-Kottwitz-MacPherson Theory

Let $Gr(n,\mathbb{C}^N)$ be the variety of all n-dimensional subspaces in $\mathbb{C}^N$.

There is a natural action of $GL_N(\mathbb{C})$ on $Gr(n,\mathbb{C}^N)$. Fix a torus $T \subset GL_N(\mathbb{C})$.

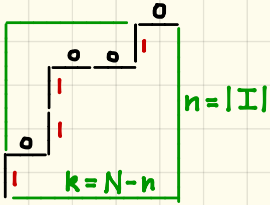
The fixed pts under the torus action are described in terms of binary strings and there is a natural Weyl group action ($W = S_N$) on them.



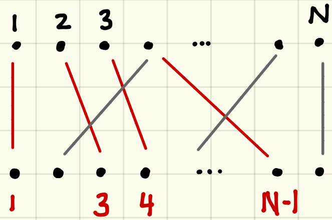
set of integers $I \subset [N]$

$$\underbrace{10110010}_{N}$$

binary string $b = b_1 \cdots b_N$



partition λ



minimal length rep w

cosets $[w] \in S_N / S_n \times S_k \rightsquigarrow$ natural S_N-action: $(s_j, [w]) \mapsto [s_j w]$

A localised Schubert class $\mathcal{O}_\lambda$ is a sequence $\mathcal{O}_\lambda = (\mathcal{O}_\lambda|_{w_1}, \ldots, \mathcal{O}_\lambda|_{w_r})$ with the w_i's being the fixed pts, $\mathcal{O}_\lambda \in \mathcal{R}(t_1, \ldots, t_N)$ and $*$ = pointwise multiplication

Identification: localised Schubert class $\mathcal{O}_\lambda|_\mu \longleftrightarrow$ eigenvalue $g_\lambda(y_\mu)$

fixed point under torus action $\longleftrightarrow$ Bethe roots y_μ

THEOREM [Gorbounov, C.K.]

i) Set $\beta=0$ (free fermion pt) then $qh_n^* \cong QH^T(Gr_{n,N})$ equivariant quantum cohomology

ii) Set $t_j=0$, $\beta=-1$, then $qh_n^* \cong QK(Gr_{n,N})$ quantum K-theory

iii) Set $q=0$, $\beta=-1$, then $qh_n^* \cong K_T(Gr_{n,N})$ equivariant K-theory

CONJECTURE qh_n^* for $\beta=-1$, $q, t_j \neq 0$ describes <u>quantum equivariant K-theory</u>.

Exactly solvable lattice model $\rightsquigarrow$ algebraic geometry / topology

Theorem The partition function $Z^\lambda_\mu = \langle v_\lambda, H(x_1)\cdots H(x_n) v_\mu \rangle$ is the weighted sum over all cylindric (set-valued) tableaux and

i) $\beta=0$: $$Z^\lambda_\mu(\underline{x}) = \sum_\nu q^d \, C^\lambda_{\mu\nu}(\underline{t}) \, s_{\nu}(\underline{x}|-\underline{t}')$$

Gromov-Witten invariants

factorial Schur function

ii) $\beta=-1$ $(q=0)$: $$Z^\lambda_\mu(\underline{x}) = \sum_\nu c^\lambda_{\mu\nu}(\underline{t}) \, \mathcal{G}_{\nu'}(\underline{x}|\ominus t')$$

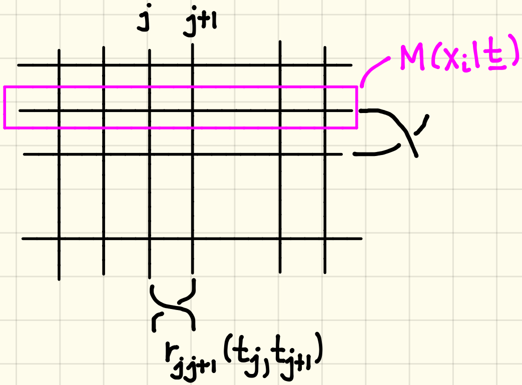
K-theoretic Littlewood-Richardson coefficients

⇝ Richardson variety: $[X^\lambda_\mu] = \sum_\nu c^\lambda_{\mu\nu}(\underline{t})[X^\nu]$

⇝ determinant formula for $C^\nu_{\lambda\mu}$ in terms of Z^λ_μ!

Description via Iwahori-Hecke algebras

Recall the definition of the monodromy matrix $M(x|\underline{t}) = L_N(x,t_N)\cdots L_1(x,t_1)$



column braiding: $\hat{r}_{jj+1} M(x|\underline{t}) = M(x|s_j\underline{t})\,\hat{r}_{jj+1}$

Let $\underline{s}_j = (s_j \otimes 1)\,\hat{r}_{jj+1}$ with s_j acting on $\mathcal{V} = \mathcal{R}(t_1,\dots,t_N)\otimes V^N$ by swapping $t_j \leftrightarrow t_{j+1}$.

Prop (Schur-Weyl duality) The $\underline{s}_j$'s define an S_N-action on each $\mathcal{V}_n$ and

1) this action commutes with the action of the Yang-Baxter algebra

$$A,D : \mathcal{V}_n \longrightarrow \mathcal{V}_n \quad \text{and} \quad B,C : \mathcal{V}_n \longrightarrow \mathcal{V}_{n\pm 1}$$

2) it is the natural action on the idempotents (Bethe vectors)

$$\underline{s}_j Y_\lambda = Y_{s_j\lambda}$$

Define the following generalised divided difference operator on each $\mathcal{V}_n$

$$\delta_j = (1+\beta t_j)\frac{1-\hat{r}_j}{t_j - t_{j+1}} \qquad \partial_j = (1+\beta t_j)\frac{1-s_j}{t_j - t_{j+1}} \circlearrowright \mathcal{R}(t_1,\dots,t_N)$$

[Lascoux, Schützenberger]

PROP The δ_j's define a representation of the Iwahori-Hecke algebra $\mathcal{H}_N(\beta)$, i.e.

$$\delta_j^2 = \beta\,\delta_j \qquad \text{and} \qquad \delta_j\,\delta_{j+1}\,\delta_j = \delta_{j+1}\,\delta_j\,\delta_{j+1}$$

N.B. $\mathcal{H}_N(0)$ is the nil-Coxeter and $\mathcal{H}_N(-1)$ the nil-Hecke algebra.

COROLLARY The factorial Grothendieck polynomials evaluated at the Bethe roots, $g_\lambda(y_\mu)$, are localised Schubert classes. They obey the GKM condition

$$\boxed{s_j\, g_\lambda - g_\lambda = (t_j \ominus t_{j+1})\,\delta_j\, g_\lambda}$$

Remark The last result implies that the localised Schubert classes or eigenvalues $g_\lambda(y_\mu)$ can be succesively generated from the "top class"

$$g_{k^n}(y_\mu) = \prod_{j=1}^{k} \prod_{i\in I_\mu} y_i \ominus t_j = \prod_{j=1}^{k} e(t_j)$$

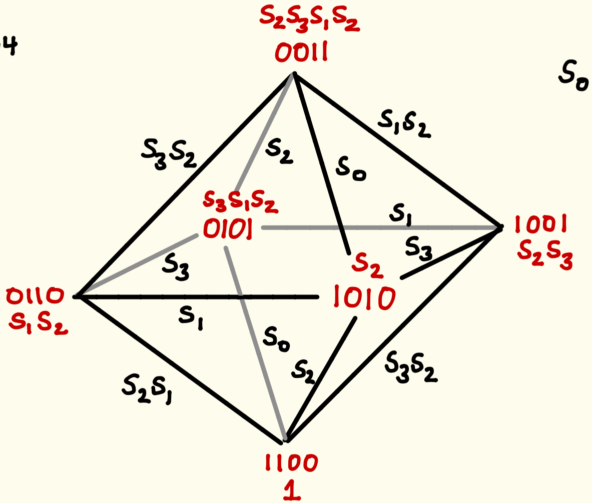
transfer matrix eigenvalues at $x = t_j$

Each λ is a 01-string which can be "generated" from $\underbrace{0\cdots 0}_{k}\underbrace{1\cdots 1}_{n} = k^n$ by moving 1-letters to the left using the GKM condition.

Let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be such that $\lambda = w\, k^n$ then

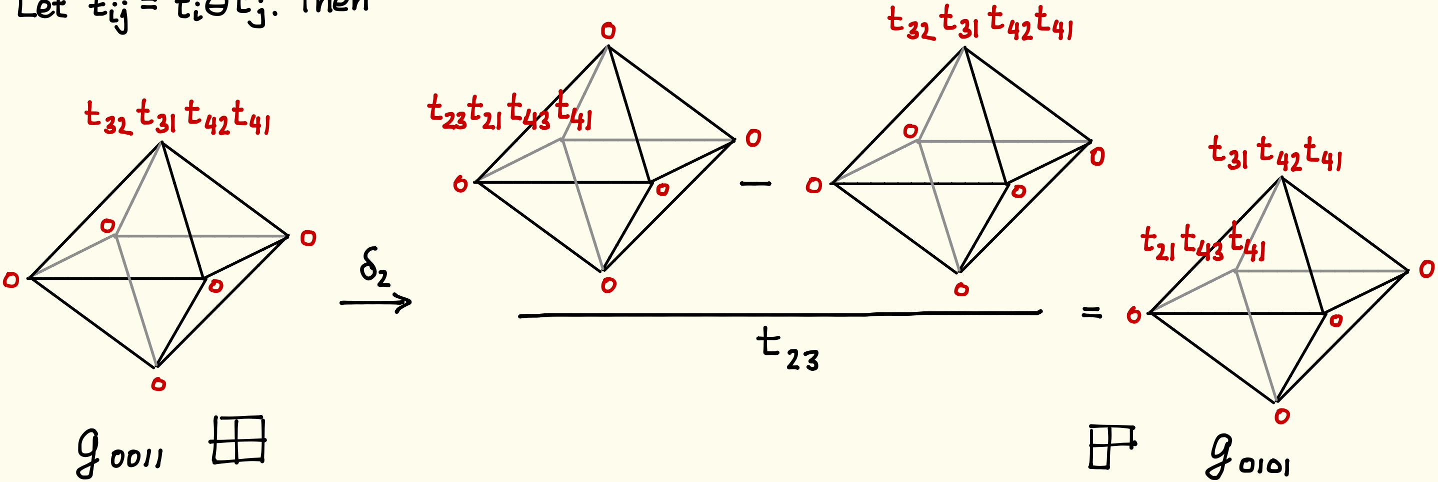
$$\delta_w\, g_{k^n} = g_\lambda + \cdots \qquad \text{where} \qquad \delta_w = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_r}$$

Example $Gr_{2,4}$



$s_0 \equiv s_3 s_1 s_2$

Let $t_{ij} = t_i \ominus t_j$. Then



Outlook

Why is this formulation important?

MATHS:

⇝ geometric action of the YB algebra

geometry of flag varieties $\xleftrightarrow[\text{integrability}]{}$ Lie theory

PHYSICS: exact computation of form factors using Schubert calculus?

THANK YOU FOR YOUR ATTENTION!