Quasiclassical expansion of the Slavnov determinant

Ivan Kostov

Institut de Physique Théorique, Saclay

I.K., arXiv:1203.6180, arXiv: 1205.4412 [hep-th];
I.K. and Y. Matsuo, arXiv: 1207.2562 [hep-th]
O.Foda, Y. Jiang, I.K., D. Serban, arXiv:1302.3539 [hep-th]

E. Bettelheim and I.K. , arXiv:1403.0358 [hep-th:1403.0358]

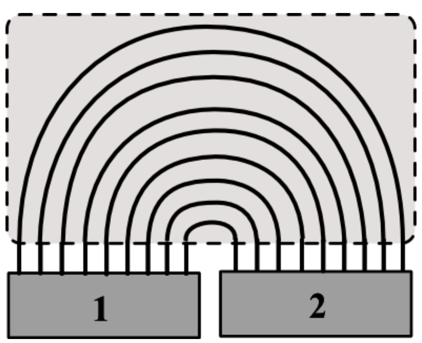
Why to compute scalar products of Bethe states?

- Essential ingredient in computation of correlation functions in 2D integrable models …

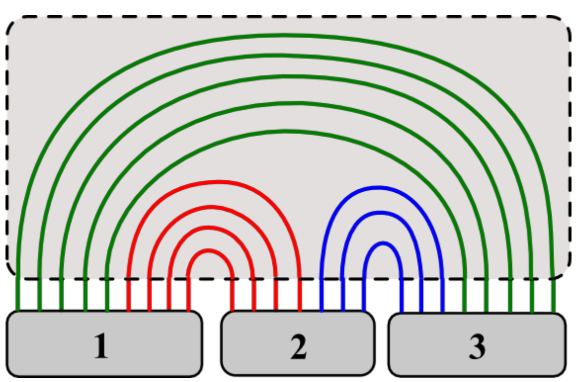
[N. Kitanine, J-M Maillet, V. Terras'07, …, …]
[F. Smirnov, M. Jimbo, T. Miwa'04-11]

- … and, more recently, in N=4 SYM

[J. Escobedo, N. Gromov, A. Sever, P. Vieira'11]
[I.K.'12]
[D. Serban'12]
[O. Foda'12]
[Y. Kazama, S. Komatsu'13]



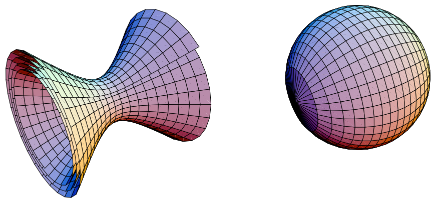
$$\langle \mathbf{u}|\langle \mathbf{u}|V_{12}\rangle = \langle \mathbf{u}|\mathbf{v}\rangle$$



$$|V_{123}\rangle = |V_{12}\rangle|V_{23}\rangle|V_{13}\rangle$$

Why to compute the semi-classical limit?

- Condensation of magnons in bound complexes of large spin above the ferromagnetic vacuum **[Sutherland'95]**
- Condensation of solitons in quantum sine-Gordon to quasi-periodic solutions of KdV **[Babelon, Bernard, Smirnov'96, Smirnov'98**
- Condensation of Cooper pairs in a superconductor **[Bettelheim, Gorohovsky'11]**
- "Heavy" gauge-invariant operators in *N*=4 SYM dual to classical strings in AdS5xS5 **[Beisert, Minahan, Staudacher, Zarembo'03 Kazakov, Marshakov, Minahan, Zarembo'04**



- The standard determinant formulas **[Gaudin, Varchenko, Isergin, Korepin, Slavnov]** become difficult to manage in this limit. New semiclassical methods needed.

New determinant formulas for the scalar product

→

CFT representation in terms of a chiral boson

↓ coarse-graining

effective field theory for the semi-classical limit

←

Semi-classical expansion (leading and subleading terms)

$SU(2)$ spin chain: Algebraic Bethe Ansatz

- Monodromy matrix M(u)

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

- pseudo-vacuum $|\Omega>$:

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle,\ D(u)|\Omega\rangle = d(u)|\Omega\rangle,\ C(u)|\Omega\rangle = 0$$
$$\langle\Omega|A(u) = a(u)\langle\Omega|,\ \langle\Omega|D(u) = d(u)\langle\Omega|,\ \langle\Omega|B(u) = 0$$

- For homogeneous XXX spin ½ chain of length L:

$$\begin{aligned} a(u) &= (u+i/2)^L \\ d(u) &= (u-i/2)^L \end{aligned}$$

- *a(u)* and *d(u)* will be considered as unrestricted functional variables (generalised SU(2) model)

- **Bethe states:** $|\mathbf{u}\rangle = B(u_1)\ldots B(u_N)|\Omega\rangle$ $\qquad \mathbf{u} = \{u_1,\ldots,u_M\}$

- A Bethe state is characterised completely by its **pseudo-momentum** $p(u)$ defined by

$$2ip(u) = \log\frac{Q_{\mathbf{u}}(u+i\varepsilon)}{Q_{\mathbf{u}}(u-i\varepsilon)} - \log\frac{a(u_j)}{d(u_j)} + \log\kappa$$

- Eigenstates of the transfer matrix $T(u)$ = Tr $M(u)$ must satisfy the **on-shell condition**

$$\frac{a(u_j)}{d(u_j)} + \kappa\,\frac{Q_{\mathbf{u}}(u_j+i\varepsilon)}{Q_{\mathbf{u}}(u_j-i\varepsilon)} = 0 \qquad (j=1,\ldots,M) \qquad \text{or} \qquad e^{2ip(u_j)}+1=0 \quad (j=1,\ldots,M).$$

Inner product in the SU(2) model

$$\langle \mathbf{v}, \mathbf{u} \rangle = \langle \Omega | \prod_{j=1}^{M} C(v_j) \prod_{j=1}^{M} B(u_j) | \Omega \rangle$$

Two facts:

- When the u-rapidities are on shell, the scalar product is given by *M*x*M* **determinant**

[N.Slavnov'89]

- Off-shell/on-shell inner product is completely **symmetric** function of the total set of rapidities

$$\mathbf{w} \equiv \{w_1, \ldots, w_{2M}\} = \{u_1, \ldots, u_M, v_1, \ldots, v_M\}$$

Jimbo, Miwa, Smirnov, arXiv:0811.0439 [math-ph]
[I,K. , Y. Matsuo, 2012]

Proof:

$$|\mathbf{u}\rangle \equiv \prod_{j=1}^{M} B(u_j) |\Omega\rangle \sim \prod_{j=1}^{M} C(u_j) \, (S_+)^{2M} |\Omega\rangle$$

A consequence: the scalar product depends only on the sum of the two pseudo-momenta:

$$p_{\mathbf{w}} = p_{\mathbf{u}} + p_{\mathbf{v}}$$

Can we make this property explicit? Yes!

Symmetric determinant formulas for the inner product

$$\langle \mathbf{v} | \mathbf{u} \rangle = \prod_{j=1}^{M} a(v_j) d(u_j) \, \mathscr{A}_{\mathbf{w}}, \qquad \mathbf{w} = \mathbf{u} \cup \mathbf{v}$$

1) Vandermonde-like *N*x*N*, *N*=2*M*:

[I.K. 2012]

$$\mathscr{A}_{\mathbf{w}} = \det_{jk} \left(w_j^{k-1} - \kappa \frac{d(w_j)}{a(w_j)} (w_j + i)^{k-1} \right) / \det_{jk} \left(w_j^{k-1} \right)$$

2) Fredholm-like *N*x*N*, *N*=2*M*:

[E. Bettelheim, I.K. 2014]

$$\mathscr{A}_{\mathbf{w}} = \det \left(1 - K \right),$$

$$K_{jk} = \frac{Q_j}{w_j - w_k + i} \qquad (j, k = 1, \ldots, N)$$

$$Q_j \equiv \operatorname*{Res}_{z \to w_j} \mathcal{Q}(z), \qquad \mathcal{Q}(z) \equiv \frac{d(z)}{a(z)} \prod_{j=1}^{N} \frac{z - w_j + i}{z - w_j} \qquad (N = 2M)$$

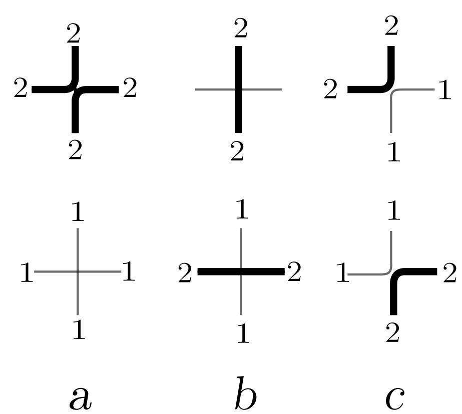
The case of a XXX ½ spin chain with inhomogeneities

$$\mathbf{z} = \{z_1, \ldots, z_L\}.$$

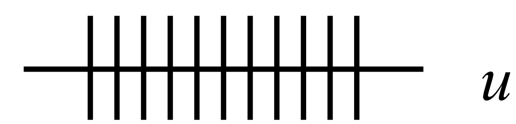
$$a(z) = \prod_{l=1}^{L} (u - z_l + i), \quad d(z) = \prod_{l=1}^{L} (u - z_l + i)$$

Relation to six-vertex partition functions …

six-vertex representation of the R-matrix:

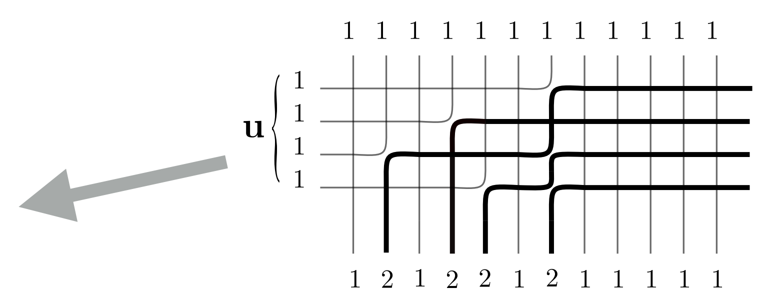


Monodromy matrix:

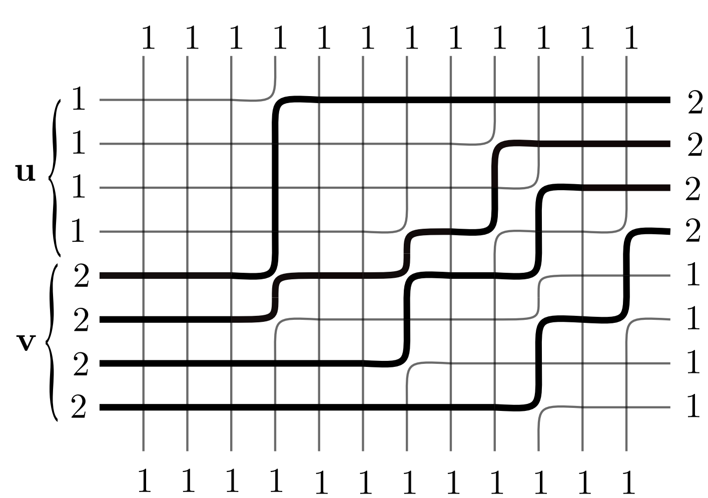


Bethe states:

$$|\mathbf{u}\rangle = \sum_{s_1,\ldots,s_L=1,2} \psi_{s_1,\ldots,s_L}(\mathbf{u}) |s_1, \ldots, s_L\rangle$$



Scalar product $\langle \mathbf{v}, \mathbf{u} \rangle$:

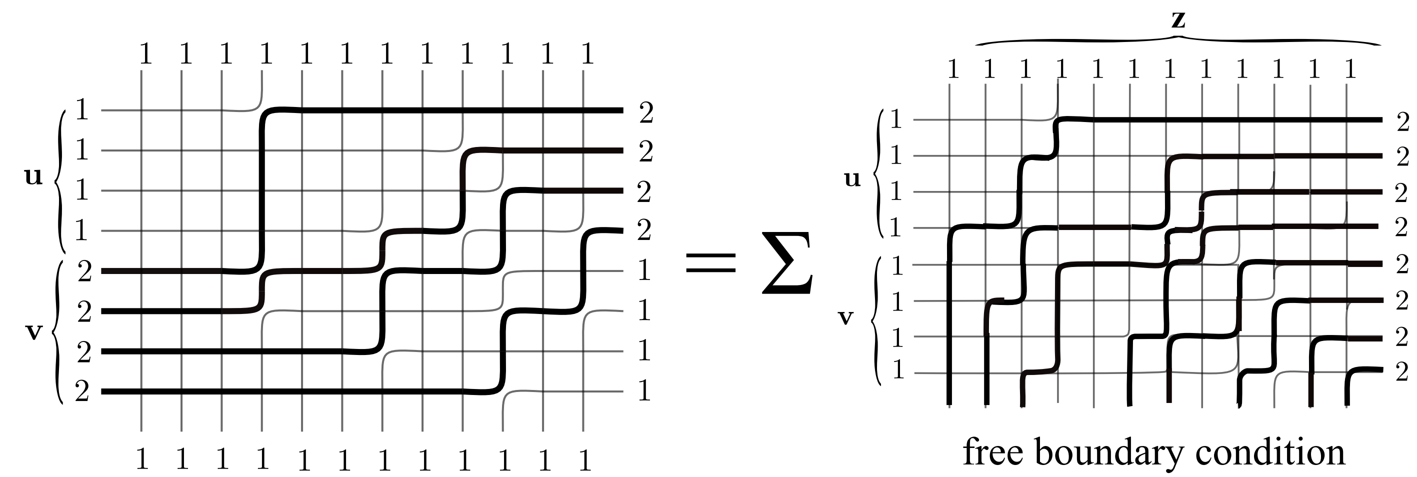


… and w-z symmetry:

- The scalar product is given by Partial Domain Wall Partition Functions (**PDWPF)** studied by **Foda and Wheeler'2012**

$$|\mathbf{u}\rangle \equiv \prod_{j=1}^{M} B(u_j)|\Omega\rangle \sim \prod_{j=1}^{M} C(u_j)\,(S_+)^{2M}|\Omega\rangle \quad => $$

[I.K, Y. Matsuo'2012]



- ***N*=*L*** (=2*M*): Izergin-Korepin determinant (=**DWPF**) is a particular case of the scalar product

$$\mathscr{Z}_{\mathbf{w},\mathbf{z}} = \frac{\det_{jk} t(w_j - z_k)}{\det_{jk} \frac{1}{w_j - z_k + i\varepsilon}}, \quad t(u) = \frac{1}{u} - \frac{1}{u + i\varepsilon}$$ — symmetric under exchanging **u** and **z**

- ***N*<*L*** The **u**—**z** symmetry still holds:

=> *N*x*N* versus *L*x*L* determinant representations of the scalar product

CFT representation of the Fredholm determinant

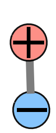
$$\det(1-K) = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \prod_{j=1}^{n} \oint_{\mathcal{C}_{\mathbf{w}}} \frac{dz_j}{2\pi i} \; \frac{\mathcal{Q}(z_j)}{i} \; \prod_{j<k}^{n} \frac{(z_j - z_k)^2}{(z_j-z_k)^2+1}$$

- **This is an expectation value for a chiral bosonic field**

$$\langle 0|\phi(x)\phi(y)|0\rangle = \log(x-y)$$

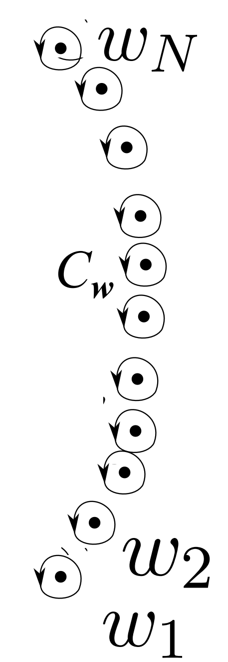
$$e^{-\phi(x)}\, e^{\phi(u)} \sim \frac{1}{x-u} e^{\phi(u)-\phi(x)}$$

$$\det(1-K) = \langle 0| \exp\left(-\frac{1}{i}\oint_{\mathcal{C}_{\mathbf{w}}} \frac{dz}{2\pi i}\, \mathcal{Q}(z)\, \mathcal{V}(z)\right)|0\rangle$$



$$\mathcal{V}(z) \equiv e^{\phi(z+i)-\phi(z)}$$

encircles the roots w_j and leaves outside all other singularities of the integrand

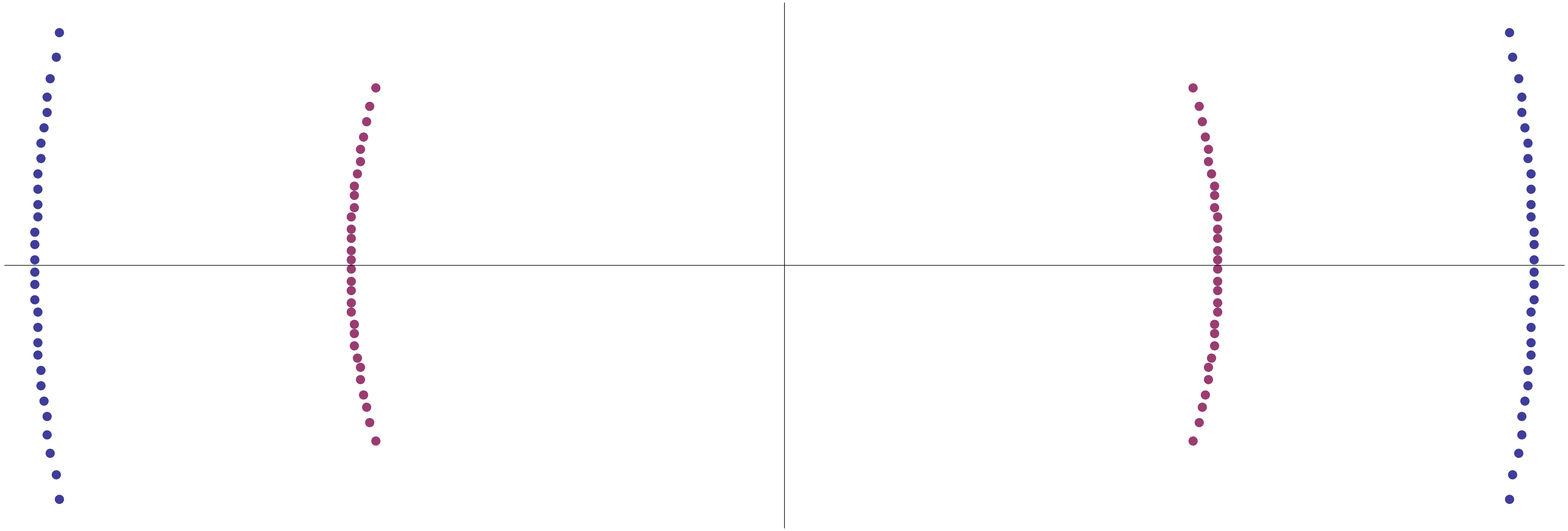


The semiclassical limit

1) *L, M* large,

2) M/*L* ~ 1,

3) Bethe roots form a small number of bound complexes of macroscopic size



Macroscopic Bethe strings ==> cuts of the finite-zone solution

Effective IR field theory in the semi-classical limit

Semi-classical expansion: $\log\det(1-K) = F_0 + F_1 + F_2 + \cdots + \mathcal{O}(e^{-\Lambda})$

$$F_n \sim L^{1-n}$$

- We will solve exactly the UV limit and find an effective IR theory

$$\phi(x) \;=\; \phi_{\text{slow}} + \phi_{\text{fast}}$$

1) **Deform the contour** away from the roots w_j, to have slowly varying weight function $Q(x)$
2) Introduce intermediate scale $1 \ll \Lambda \ll N$ and split the field into a **slow** and **fast** components:
3) Integrate out the fast component and obtain an **effective action** for the slow component. Effective action as the sum of all connected correlators (**cumulants**)

The *n*-th cumulant:

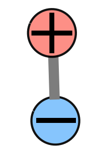
$$\Xi_n(z) = -\frac{1}{i}\,\frac{\mathcal{Q}(z)\mathcal{Q}(z+i)\ldots\mathcal{Q}(z+in)}{n^2}\;e^{\phi(z+in)-\phi(z)}$$

Does not depend on the cutoff!

Intuitive picture: non-ideal gas of dipoles

$$\mathcal{V}_\varepsilon(z) \equiv e^{\phi(z+i\varepsilon)-\phi(z)}$$

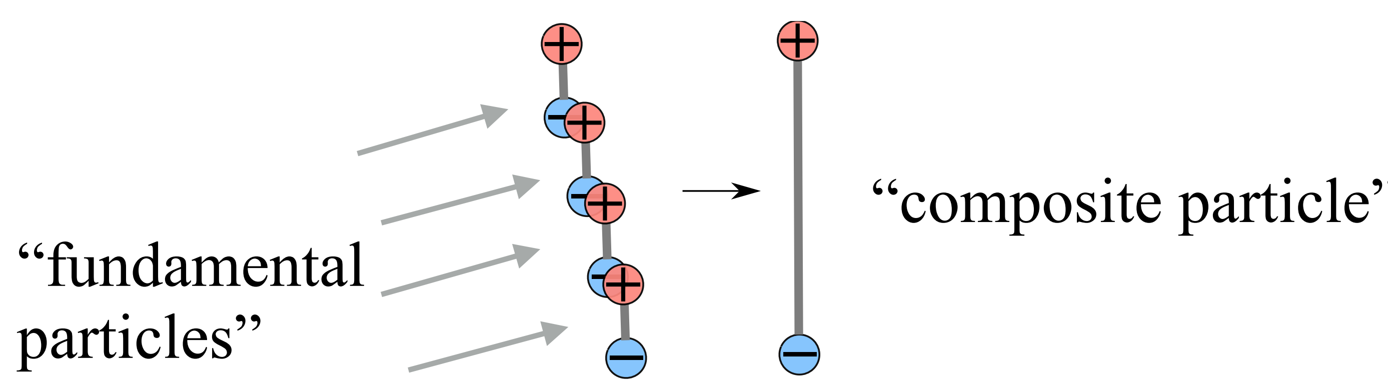
a pair of oppositely oriented Coulomb charges (a dipole)



One-dimensional Coulomb gas of dipoles in a common external potential $\Phi(u)$

$$\mathcal{Q}(z) = e^{\Phi(z+i)-\Phi(z)}, \qquad \Phi(z) = \sum_{j=1}^{N} \log(z - w_j) - \log \frac{a(z)}{d(z)}$$

- **Clustering:** The "fundamental" dipoles form bound complexes of length n =1,2,3,…



The weight for composite particles: $\mathcal{Q}(z)\mathcal{Q}(z+i)\ldots\mathcal{Q}(z+in) = e^{\Phi(z+in)-\Phi(z)}$

Claim: In the semi-classical limit of large L with N/L~1, the perturbative expansion in ε ~ $1/L$ is given by the expectation value for the slow component

$$\det(1-K) \approx \left\langle \exp\left(\sum_{n=1}^{\Lambda} \frac{1}{n^2} \oint_{\mathcal{C}} \frac{dz}{2\pi}\, e^{\Phi(z+in)-\Phi(z)}\, e^{\phi(z+in)-\phi(z)} \right) \right\rangle$$

(≈ means "equal up to non-perturbative terms)

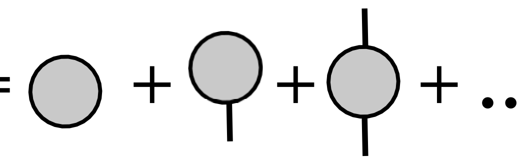
Operator form: $\mathbb{D} \equiv e^{i\partial}$ $\mathrm{Li}_2(\mathbb{D}) = \sum_{n=1}^{\infty} \frac{\mathbb{D}^n}{n^2}.$

$$\det(1-K) = \left\langle \exp \oint_{\mathcal{C}} \frac{dz}{2\pi}\, \mathcal{W}[\phi] \right\rangle$$

$$\mathcal{W}[\phi] = :e^{-\Phi(z)-\phi(z)}\, \mathrm{Li}_2(\mathbb{D})\, e^{\Phi(z)+\phi(z)}:$$

Semiclassical expansion

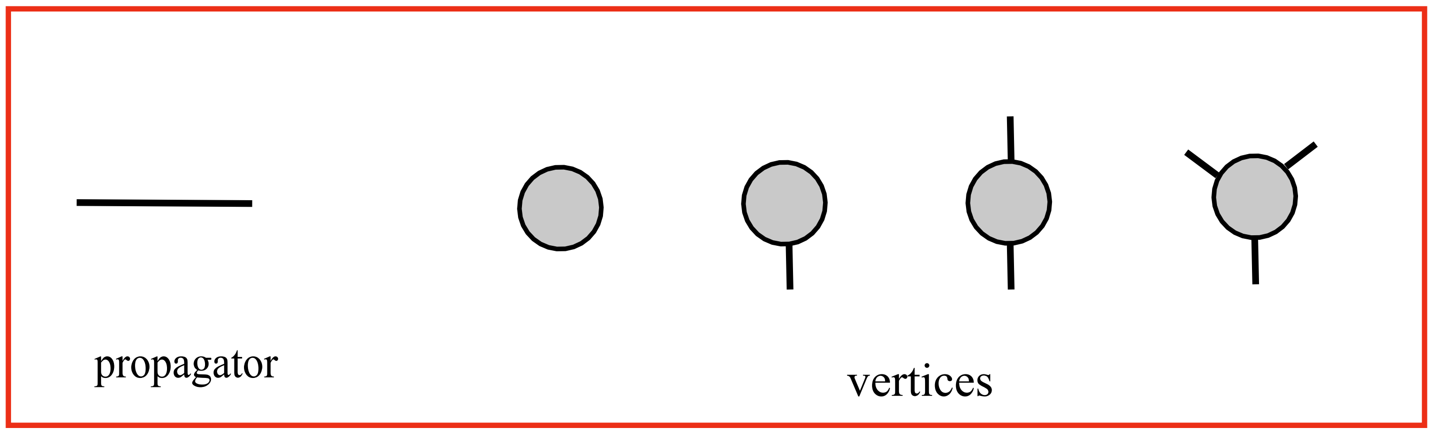
- **effective action**

$$W[\phi] = \mathrm{Li}_2(\mathcal{Q}) + i\log(1-\mathcal{Q})\,\partial\phi - \frac{1}{1-\mathcal{Q}}((\partial\phi)^2 + \partial^2\phi) + \dots =$$ 

external potential

tadpole

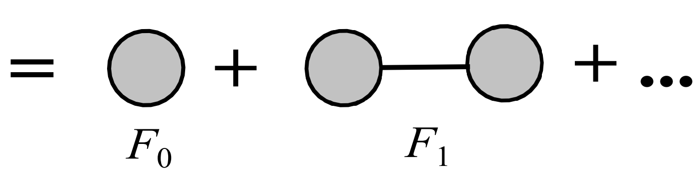
$$\mathcal{Q} \approx e^{i\partial\Phi}$$



Feynman rules for the effective field theory

- **semi-classical expansion of the "vacuum energy":**

$$\log\det(1-K) = F_0 + F_1 + \cdots + \mathcal{O}(e^{-\Lambda})$$



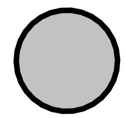
$$F_n \sim L^{1-n}$$

= sum of the connected diagrams with
propagators - #vertices = n-1

The first two terms of the semiclassical expansion

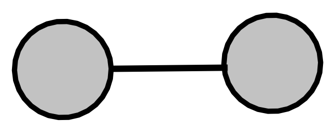
Leading term (known)

[Gromov, Sever, Vieira 2011, IK'2012]



$$F_0 = \oint_{\mathcal{C}} \frac{dx}{2\pi} \, \mathrm{Li}_2[\mathcal{Q}(x)]$$

Subleading term (new)



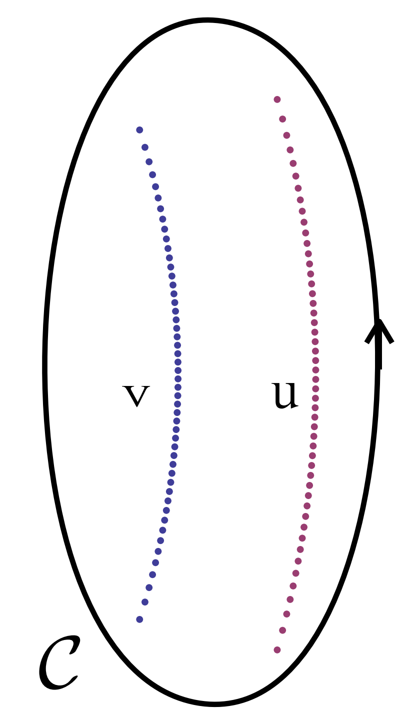
$$F_1 = -\frac{1}{2} \oint_{\mathcal{C}\times\mathcal{C}} \frac{dx\, du}{(2\pi)^2} \, \frac{\log\left[1-\mathcal{Q}(x)\right]\, \log\left[1-\mathcal{Q}(u)\right]}{(x-u)^2}$$

- **for the scalar product < u|v > :** $\mathcal{Q} = e^{ip_{\mathbf{u}} + ip_{\mathbf{v}}}$

on-shell off-shell

$$(\mathbf{u},\mathbf{v}) = \exp\left[\oint_{\mathcal{C}} \frac{dx}{2\pi} \, \mathrm{Li}_2\left(e^{ip_{\mathbf{u}}(x) + ip_{\mathbf{v}}(x)}\right)\right]$$

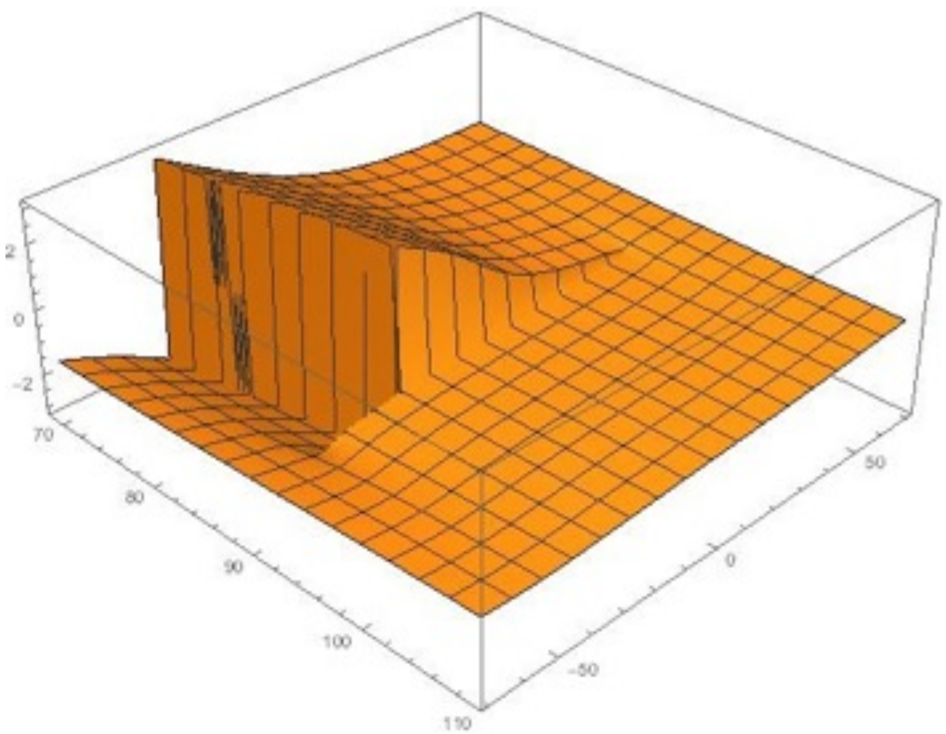
$$(\mathbf{u},\mathbf{u}) = \exp\left[\oint_{\mathcal{C}} \frac{dx}{2\pi} \, \mathrm{Li}_2\left(e^{2ip_{\mathbf{u}}(x)}\right)\right]$$

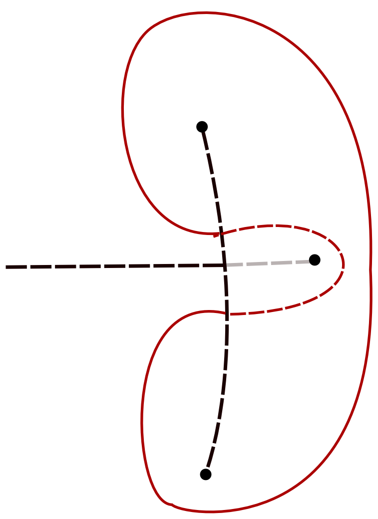


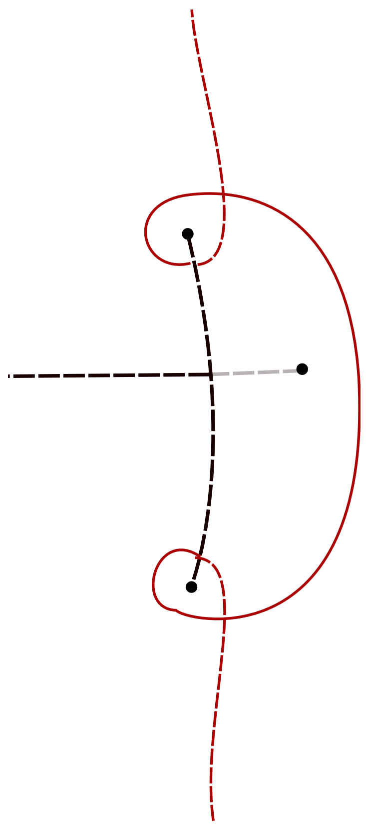
Remark 1: Subtleties concerning the integration contour

Infinitely many “resonance points” where argument of Li(Q) =1 and the integrand has **logarithmic branch points.** Is it possible to place the contour far from all these points?

- **Prescription for the contour:**







With this prescription the contour integral for the leading term matches the numerical fit (with M=1, …, 60, M/L= 16) with accuracy 12 digits **[N. Gromov, unpublished]**

Conclusion

- The scalar product $<\mathbf{u}|\mathbf{v}>$ depends in an universal way on the sum of the quasi-momenta of the two states $p_{\mathbf{w}} = p_{\mathbf{u}} + p_{\mathbf{v}}$
- Unexpectedly simple expression for the first two terms of the semi-classical expansion (checked numerically with high precision)

Unclear points:

- XXZ??
- Why there is no contribution from the "resonance points" $u = \gamma_n$?

$$p_{\mathbf{u}}(\gamma_n) + p_{\mathbf{v}}(\gamma_n) = 2\pi n, \quad n \in \mathbb{Z}$$

(cf the semiclassical form factors in the quasi-periodic sine-Gordon **[F. Smirnov'98]**)

- The semi-classical expression does not seem to be symmetric under exchanging rapidities and inhomogeneities. Why?
- How to obtain the same result from the SoV representation of the scalar product

$$\langle\, \mathbf{u}|\mathbf{v}\,\rangle \sim \oint_{\mathcal{C}} \prod_{l=1}^{L-1} \frac{dy_l}{2\pi i} \prod_{k<l}^{L-1} (y_k - y_l)\ \sinh \pi(y_k - y_l)\ \frac{\prod_{j=1}^{L}\prod_{k=1}^{M}(y_j - u_k)(y_j - v_k)}{\prod_{j,l=1}^{L}(y_j - z_l)(y_j - z_l + i)}$$

[Kazama, Komatsu, Nishimura'2013, ???]