Spin Chains and Three-Point Functions in $\mathcal{N} = 4$ Super Yang–Mills Theory

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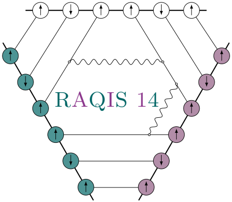

work with

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Ivan Kostov,

Didina Serban

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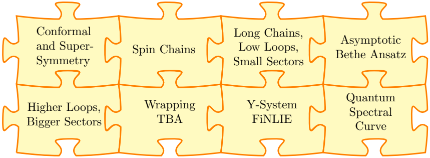
Goal: Solve Planar $\mathcal{N} = 4$ Super Yang–Mills Theory

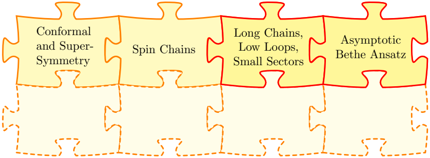
Conformal Field Theory $\rightarrow$ We want Spectrum and Three-Point Functions!

Use Integrability [see e.g. Review 2012 Beisert et al.]

1. **Spectrum** (almost solved):



2. **Three-Point Functions:**



Here:

- Simplest subsector: Complex scalar fields X, Z in $\mathfrak{su}(2)$ sectors.
- Study one-loop correction (previously obtained in [Gromov Vieira, 12]).
- Compare to AdS/CFT-dual string theory result.

Spectrum: $\mathfrak{su}(2)$-Sector

One Loop: Dilatation Operator = Heisenberg Hamiltonian $\mathcal{Q}_2 = H_2$

Spin Chains (cyclic): $\leftrightarrow$ Gauge invariant states:

$$|{\downarrow\uparrow\uparrow}\ldots{\downarrow}\rangle(x) \quad \leftrightarrow \quad \mathcal{O}(x) = \mathrm{Tr}(XZZ\ldots X)(x).$$

Excitations: Characterized by sets of rapidities $\mathbf{u} = \{u_1, u_2, \ldots, u_M\}$:

$$\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} = e^{ip_j} \quad \rightarrow \quad |\mathbf{u}\rangle \sim |\ldots\uparrow\uparrow\uparrow \overset{e^{ip_j}}{\downarrow} \uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow \overset{e^{ip_k}}{\downarrow} \uparrow\uparrow\uparrow\ldots\ldots\rangle$$

Integrability : Tower of commuting charges: $\mathcal{Q}_r$ with $\mathcal{Q}_2 = \mathcal{D}$

$\Rightarrow$ Dilatation Operator diagonalized by Bethe Ansatz [Minahan Zarembo, 02]

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}$$

Asymptotic Spectrum: $\mathfrak{su}(2)$-Sector

Higher Loops: Dilatation Operator $\mathcal{Q}_2(g^2) = H_2 + g^2 \ldots$ (long-ranged)

↖ 't Hooft coupling

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$$\frac{x(u_j + \frac{i}{2})}{x(u_j - \frac{i}{2})} = e^{ip_j} \quad \rightarrow \quad |\mathbf{u}\rangle \sim |\ldots\uparrow\uparrow\uparrow \overset{e^{ip_j}}{\downarrow} \uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow \overset{e^{ip_k}}{\downarrow} \uparrow\uparrow\uparrow\ldots\ldots\rangle$$

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BDS: [Beisert, Dippel Staudacher, 04]

Dressing: [Arutyunov, Beisert Eden, Frolov, Hernández López, Staudacher, ...]

- Valid up to wrapping order!
- Dressing Phase starts at four loops

Asymptotic Spectrum: $\mathfrak{su}(2)$-Sector

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$$\frac{x(u_j + \frac{i}{2})}{x(u_j - \frac{i}{2})} = e^{ip_j} \quad \to \quad |\mathbf{u}\rangle \sim |\ldots{\uparrow}{\uparrow}{\uparrow} \overset{e^{ip_j}}{\downarrow} {\uparrow}{\uparrow}{\uparrow}\ldots{\uparrow}{\uparrow}{\uparrow} \overset{e^{ip_k}}{\downarrow} {\uparrow}{\uparrow}{\uparrow}\ldots\ldots\rangle$$

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BDS: [Beisert, Dippel Staudacher, 04]

- Valid up to wrapping order!
- Dressing Phase starts at four loops $\to$ switch off for the moment

Two Perturbative Definitions of Higher-Loop Spin Chains I

I. Deformations using Boost Operators

Start with Heisenberg (XXX) spin chain with local charges H_r:

$$\text{One loop: } \mathcal{Q}_r(0) \equiv H_r \qquad \text{with} \qquad [H_r, H_s] = 0 \qquad r, s = 2, 3, \ldots$$

One-loop/Heisenberg charges generated by leading boost operator: [Tetel'man 82]

$$H_{r+1} = [\mathcal{B}_2, H_r], \qquad\qquad B_n = \sum_k k\, \mathcal{Q}_{n,k}.$$

Construct higher-loop charges using higher boost operators: [Bargheer, Beisert FL, 08/09]

$$\frac{d}{dg}\mathcal{Q}_r(g) = \tau_s[\mathcal{B}_s(g), \mathcal{Q}_r(g)], \qquad \Rightarrow \qquad [\mathcal{Q}_r(g), \mathcal{Q}_t(g)] = 0.$$

Solve perturbatively for generator $T_{\text{Boost}}(g)$:

$$\mathcal{Q}_r(g) = T_{\text{Boost}}(g)\, H_r\, T^{-1}_{\text{Boost}}(g) + \text{wrapping}$$

- T_{Boost} singular on periodic chains $\rightarrow$ **no similarity transformation**.

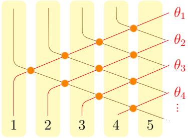
Two Perturbative Definitions of Higher-Loop Spin Chains II

II. Inhomogeneous Spin Chains

Get Higher Loop Spectrum from inhomogeneous Bethe Ansatz: [Beisert, Dippel Staudacher, 04]

$$\prod_{j=1}^{L} \frac{u_k - \theta_j(g) + \frac{i}{2}}{u_k - \theta_j(g) - \frac{i}{2}} = \prod_{j=1;j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad \theta_j^{\mathsf{BDS}}(g) = 2g \sin \frac{2\pi j}{L}.$$

Generate inhomogeneous charges from Heisenberg charges using inhomogeneous Corner Transfer Matrix: [Baxter 76][Jiang,Kostov FL,Serban, 14]

$T_\theta(u) =$ 

,

$\bullet \sim$ R-matrix $\check{R}(u,\theta)$

$$\mathcal{Q}_2(\theta) = T_\theta(0)\, H_2\, T_\theta^{-1}(0) + \text{wrapping}$$

- Works at leading orders, no general proof yet!
- Compare also relation of leading boost $\mathcal{B}_2$ to homogeneous CTM. [Thacker 86]
- T_θ singular on periodic chains $\rightarrow$ **no similarity transformation**.

S-Operator

Two singular transformations T_{Boost} and T_θ generate the same twist of the Bethe equations!

$$\mathcal{Q}_2(g) \simeq T_{\text{Boost}}(g) H_2 T^{-1}_{\text{Boost}}(g) \qquad \mathcal{Q}_2(\theta) \simeq T_\theta(0)\, H_2\, T_\theta^{-1}(0)$$

$\Rightarrow$ Inhomogeneous and Boost-Deformed Chains are related by similarity transformation S up to wrapping order: [Bargheer Beisert, FL, 09][Jiang, Kostov FL, Serban, 14]

Unitary S-Operator $\quad S = T_{\text{Boost}} \times T_\theta^{-1} \quad \Rightarrow \quad \mathcal{Q}_2(g) = S\mathcal{Q}_2(\theta)S^{-1}$

- S is well-defined on periodic spin chains as opposed to T_{Boost} and T_θ!

Three-Point Functions

Correlator of three eigenstates of the dilatation operator in three $\mathfrak{su}(2)$ sectors:

States:	$\mathcal{O}_1(x_1)$	$\mathcal{O}_2(x_2)$	$\mathcal{O}_3(x_3)$
Made of Scalars:	Z, X	$\bar{Z}, \bar{X}$	$Z, \bar{X}$
Bethe state:	$\vert\mathbf{u}_1\rangle$	$\vert\mathbf{u}_2\rangle$	$\vert\mathbf{u}_3\rangle$

Conformal symmetry:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}(g^2)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

Integrability:

$$C_{123}(g^2) = \frac{\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\rangle}{(\langle \mathbf{u}_1|\mathbf{u}_1\rangle\langle \mathbf{u}_2|\mathbf{u}_2\rangle\langle \mathbf{u}_3|\mathbf{u}_3\rangle)^{1/2}}$$ [Escobedo,Gromov Sever,Vieira,10]

Scalar Products (of one on-shell and one off-shell Bethe state):

$${}_{\text{loop}}\langle \mathbf{u}| = \langle \mathbf{u}, \boldsymbol{\theta}|S^{-1} \qquad\qquad |\mathbf{u}\rangle_{\text{loop}} = S|\mathbf{u}, \boldsymbol{\theta}\rangle$$

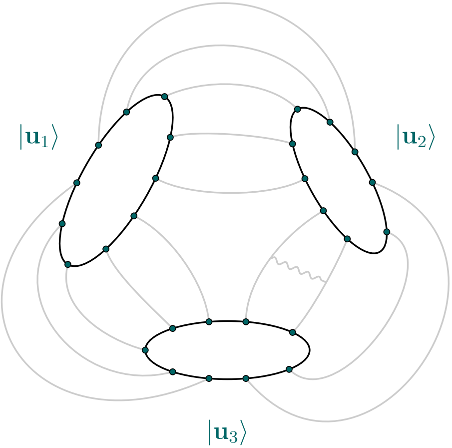
$\Rightarrow$ Use *Slavnov-Determinant-Formula* for scalar products: [Slavnov 89][Jiang, Kostov FL, Serban, 14]

$${}_{\text{loop}}\langle \mathbf{u}|\mathbf{v}\rangle_{\text{loop}} = \langle \mathbf{u}, \boldsymbol{\theta}|\mathbf{v}, \boldsymbol{\theta}\rangle \simeq A_{\mathbf{u}\cup\mathbf{v},\boldsymbol{\theta}}, \qquad A_{\mathbf{u},\boldsymbol{\theta}} = \mathrm{Det}(\mathbb{I} - K)$$

$$K_{jk} = \frac{iE_j}{u_j - u_k + i}, \qquad E_j = \frac{Q_\theta(u_j - \frac{i}{2})}{Q_\theta(u_j + \frac{i}{2})} \prod_{k=1;k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k}, \qquad Q_\theta(u) = \prod_{j=1}^{L}(u - \theta_j).$$

Find Structure Constants $\langle \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3} \rangle$

General idea: [Okuyama Tseng, 04] [Roiban Volovich, 04] [Alday,David Gava,Narain, 05] [Escobedo,Gromov Sever,Vieira, 10]



Tree-level: [Slavnov 89] [Escobedo, Gromov Sever,Vieira, 10] [Foda 11]

- Solved $\rightarrow$ Slavnov-determinants

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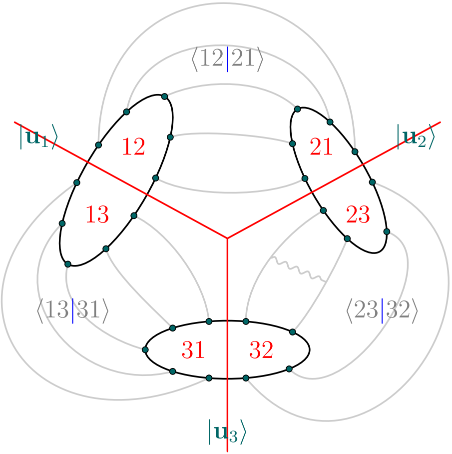

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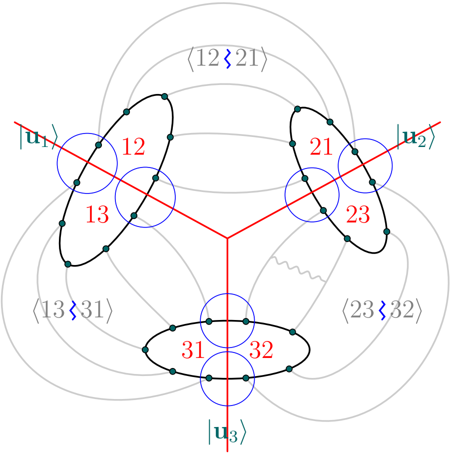

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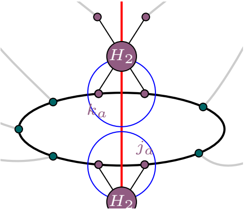
One Loop: Two Types of Loop-Insertions

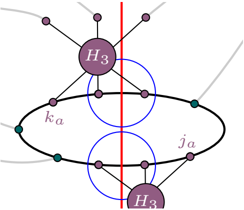
1. To the correlator: [Okuyama Tseng, 04][Roiban Volovich, 04][Alday,David Gava,Narain, 05]



Insertion of the Heisenberg-Hamiltonian (one-loop Dilatation Operator) at the splitting points:

$$\mathbb{I}_a = 1 - g^2(H_{2,k_a} + H_{2,j_a}), \quad a=1,2,3$$

2. To the eigenstates: [Jiang,Kostov FL,Serban,14]



Insertion from the S-operator (transformation of eigenstates) at the splitting points:

$$\delta S_a = 1 - g^2(H_{3,k_a} + H_{3,j_a}), \quad a=1,2,3$$

Combine Things

Skip some nontrivial steps: [Escobedo,Gromov Sever, Vieira,10][Foda 11][Jiang,Kostov FL,Serban,14]

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle = \sum_{i_1,\ldots,i_{L_{12}}=\uparrow,\downarrow} \langle \mathbf{u}_2, \boldsymbol{\theta}_2 | \delta S_2^{-1}\, \mathbb{I}_2 \, | i_1 \ldots i_{L_{12}} \underbrace{\uparrow \ldots \uparrow}_{L_{23}} \rangle$$

$$\times \langle i_1 \ldots i_{L_{12}} \underbrace{\downarrow \ldots \downarrow}_{L_{13}} | \, \mathbb{I}_1 \, \delta S_1 | \mathbf{u}_1, \boldsymbol{\theta}_1 \rangle$$

$$\times \langle \underbrace{\uparrow \ldots \uparrow}_{L_{23}} \underbrace{\downarrow \ldots \downarrow}_{L_{13}} | \mathbb{I}_3 \, \delta S_3 | \mathbf{u}_3, \boldsymbol{\theta}_3 \rangle$$

Two steps to get simple form:

1. Rewrite insertions $\mathbb{I}_a$ and δS_a in terms of derivatives $\partial_k = \partial / \partial \theta_k$
2. Fix θ to coupling-dependent BDS-values $\theta_{a,\ell}^{\text{BDS}}(g) = 2g \sin \frac{2\pi\ell}{L_a}$

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle = (1 + g^2 \hat{\Delta}) F_{123}(\boldsymbol{\theta}) + \mathcal{O}(g^3) \Big|_{\theta = \theta^{\text{BDS}}}$$

[Jiang,Kostov FL,Serban,14]

$$F_{123}(\boldsymbol{\theta}) = A_{\mathbf{u}_2 \cup \mathbf{u}_1, \boldsymbol{\theta}_{12}} + A_{\mathbf{u}_3, \boldsymbol{\theta}_{13}} \qquad \hat{\Delta} = \hat{\Delta}_{12} + \hat{\Delta}_{03} \qquad \delta E_r^{ab} = E_r^b - E_r^a$$

$$\hat{\Delta}_{ab} A_{\mathbf{u}_a \cup \mathbf{v}_b, \boldsymbol{\theta}_{bc}} = \left(\partial_1^b \partial_2^b - i \delta E_2^{ab} \, \partial_1^b + i \delta E_3^{ab} - \tfrac{1}{2} (\delta E_2^{ab})^2 \right) A_{\mathbf{u}_a \cup \mathbf{v}_b, \boldsymbol{\theta}_{bc}}$$

- Agrees with [Gromov Vieira, 12], but simpler → Can take thermodynamical limit.

Comparison With String Computation

$\mathcal{N} = 4$ super Yang–Mills theory dual to IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Consider two limits on the two sides of the gauge/string duality:

Thermodynamical Limit (Gauge Theory):

- State of length L with M excitations.
- Take $L \to \infty$, $M \to \infty$, $\frac{M}{L}$ finite, $\lambda' = \frac{g^2}{L^2} \ll 1$.

Frolov–Tseytlin Limit (String Theory): [Frolov Tseytlin, 02]

- Rotating string on S^3 with angular momentum J
- $g^2 \to \infty$, $J \to \infty$, $\lambda' = \frac{g^2}{J^2} \ll 1$

Spectrum: Known that first two orders in λ' agree in gauge & string theory.

Three-point functions?: One-loop three-point function requires two-loop eigenfunction of dilatation operator.
$\Rightarrow$ Expect match at first order in λ'.

Thermodynamical vs Frolov–Tseytlin Limit

Gauge Theory: [Kostov 12²][Kostov Matsuo, 12] [Jiang,Kostov FL,Serban, 14][Bettelheim Kostov, 14]

suppressed by $1/L$ since localized at splitting points

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle_{\text{Gauge}} = F_{123}(\boldsymbol{\theta}) + g^2 \hat{\Delta} F_{123}(\boldsymbol{\theta}) + \mathcal{O}(g)^4$$

$$\simeq \left[\oint \frac{du}{2\pi} \mathrm{Li}_2 \left[e^{ip_1(u) + ip_2(u) - iq_3(u)} \right] \right]_1 + \left[\sim \right]_2$$

String Theory: [Kazama Komatsu, 13] also: [Janik, 11 Wereszczynski]

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle_{\text{String}} \simeq \left[\sim \right]_1 + \left[\sim \right]_2 \quad \rightarrow \quad \text{Integrand agrees} \quad \checkmark$$

$$+ \left[\sim \right]_3 + \left[\sim \right]_4 \quad \rightarrow \quad \text{Should vanish for agreement.} \quad ?$$

Do contours match for first term? Does the second term vanish?

Summary & Outlook

Summary:

Asymptotic Spectrum:

Inhomogeneous Bethe Ansatz $\rightarrow$ Fix $\theta = \theta(g)$

Three-Point Functions (1 Loop):

Inhomogeneous Correlators $\rightarrow$ Fix $\theta = \theta(g)$ & $\hat{\Delta}$ acts on splitting pts

$\rightarrow$ *Matches string theory result in Frolov–Tseytlin limit.*

Future Three-Point Puzzles:

Two loops: Use above method $\rightarrow$ Recursion for $\hat{\Delta}$?

Asymptotic Spin Chain	Generator	Relate Eigenstates by
Inhomogeneous Chain	T_θ	$S = T_{\text{Boost}} \times T_\theta^{-1}$
$\mathcal{N} = 4$ SYM $\mathfrak{su}(2)$ — Boost	T_{Boost}	

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Higher Loops: How to include Dressing Phase?

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Inhomogeneous Chain		T_θ	$S = T_{\text{Boost}} \times T_\theta^{-1}$
$\mathcal{N} = 4$ SYM $\mathfrak{su}(2)$	Boost	T_{Boost}	
	Dressing	T_{Dressing}	$S = T_{\text{Dressing}} \times$?
?		?	

Integrability Puzzles

- Role of higher boost operators?
- . . . and their generalization to bilocal charges (dressing phase)?
- More boosts for other models? See e.g. applications to open boundaries [FL 12] or the XXZ chain [Beisert, Fiévet de Leeuw, FL, 13]
- Study inhomogeneous Corner Transfer Matrix as generator.
- Condensed matter model with dressing phase? Dynamic inhomogeneities?
- Similar structures in other long-range spin chain models (e.g. Inozemtsev)?

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