From continuous to the discrete Fourier transform: classical and quantum aspects

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We explain and generalise certain results by Mehta (1987) relating the eigenfunctions of quantum harmonic oscillator with the eigenvectors of the discrete Fourier transform (DFT) . Using far more general construction based on a kind of Poisson summation, we associate the eigenvectors of the DFT with the sums of any absolutely convergent series. This construction in particular provides the overcomplete bases of the DFT eigenvectors by means of Jacobian theta functions or ν-theta functions and leads in a quite natural way to some new identities and addition theorems in the theory of special and q-special functions.

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Spectral theorem for the operator roots of unity

Suppose U is a bounded operator in a Hilbert space,

$$U^n = I, q := e^{\frac{2i\pi}{n}}, n \in Z^+.$$

$$U = \sum_{m=0}^{n-1} q^m P_m, \quad P_m = \frac{1}{n}\sum_{j=0}^{n-1} q^{-jm} U^j.$$

$$\sum_{j=0}^{n-1} P_j = I, \quad P_j^2 = I, \quad UP_j = q^j P_j, \quad P_j P_m = 0, \quad m \neq j.$$

Obviously

$$U^k = \sum_{j=1} q^{jk} P_j.$$

Spectral multiplicities m_j of the eigenvalue q^j of U
In a finite dimensional case we have:

$$m_j := Tr\, P_j = \frac{1}{n} \sum_{k=0}^{n-1} q^{-kj}\, Tr\, U^k.$$

$$f(U) := \sum_{j=1}^{n-1} f(q^j) P_j.$$

Fourier Transform (FT)

$$\tilde{f}(k) = F(f)(k) := \frac{1}{\sqrt{2\pi}} \int_R f(x) e^{ikx}\, dx$$

Inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_R \tilde{f}(k) e^{-ikx}\, dk.$$

Since $(F^2 f)(x) = f(-x)$

$$F^4 = I, \quad (F^3 f)(x) = (F^{-1} f)(x) = \tilde{f}(-x), \tag{1}$$

where I is an identity operator.

Spectral decomposition of F

$$F = P_0 + iP_1 - P_2 - iP_3, \quad P_j = \frac{1}{4}\sum_{k=0}^{3}(-i)^{jk}F^k$$

Therefore any continuous absolutely integrable function $f(x)$ generates the 4 eigenfunctions f_j of F:

$$F(f_j) = i^j f_j, \quad j = 0, 1, 2, 3$$

$$f_j(x) = f(x) + (-i)^j \tilde{f}(x) + (-i)^{2j} f(-x) + (-i)^{3j} \tilde{f}(-x).$$

This allows to generate the eigenfunctions of F quite different from the most known $\psi_n = e^{-\frac{x^2}{2}} H_n(x)$, which are also eigenfunctions of the quantum harmonic oscillator H.

Example:
$f := \exp(-\alpha x^2 + itx)$ with $\alpha > 0, t \in C$

$$F(f)(x) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{(x-t)^2}{4\alpha}}, F^2(f)(x) = f(-x),\ F^3(f)(x) = e^{-\frac{(x+t)^2}{4\alpha}}.$$

Hence we can generate the 2-parametric family of eigenfunctions $f_j(x, \alpha, t) = P_j f$ of the Fourier operator:

$$
\begin{aligned}
f_0 &= e^{-\alpha x^2+itx} + e^{-\alpha x^2-itx} + \frac{1}{\sqrt{2\alpha}}\left(e^{-\frac{(x-t)^2}{4\alpha}} - e^{-\frac{(x+t)^2}{4\alpha}}\right), \\
&= 2\left(e^{-\alpha x^2}\cos tx + \frac{1}{\sqrt{2\alpha}}e^{-\frac{x^2+t^2}{4\alpha}}\cosh\frac{xt}{2\alpha}\right), \\
f_1 &= e^{-\alpha x^2+itx} - e^{-\alpha x^2-itx} - i\frac{1}{\sqrt{2\alpha}}\left(e^{-\frac{(x-t)^2}{4\alpha}} - e^{-\frac{(x+t)^2}{4\alpha}}\right) \\
&= 2i\left(e^{-\alpha x^2}\sin tx + \frac{1}{\sqrt{2\alpha}}e^{-\frac{x^2+t^2}{4\alpha}}\sinh\frac{xt}{2\alpha}\right), \\
f_2 &= e^{-\alpha x^2+itx} + e^{-\alpha x^2-itx} - \frac{1}{\sqrt{2\alpha}}\left(e^{-\frac{(x-t)^2}{4\alpha}} + e^{-\frac{(x+t)^2}{4\alpha}}\right) \\
&= 2\left(e^{-\alpha x^2}\cos tx - \frac{1}{\sqrt{2\alpha}}e^{-\frac{x^2+t^2}{4\alpha}}\cosh\frac{xt}{2\alpha}\right), \\
f_3 &= e^{-\alpha x^2+itx} - e^{-\alpha x^2-itx} + i\frac{1}{\sqrt{2\alpha}}\left(e^{-\frac{(x-t)^2}{4\alpha}} - ie^{-\frac{(x+t)^2}{4\alpha}}\right) \\
&= 2i\left(e^{-\alpha x^2}\sin tx + \frac{1}{\sqrt{2\alpha}}e^{-\frac{x^2+t^2}{4\alpha}}\sinh\frac{xt}{2\alpha}\right).
\end{aligned}
$$

Since F is linear operator , the linear combinations of the functions above, corresponding to the different values of parameters α and t, are also producing the eigenfunctions. We can even integrate them with respect to t and α with some densities $\rho(\alpha, t)$, with compact support, vanishing for $\alpha \leq \epsilon, \epsilon > 0$. In this way we can generate eigenfunctions of F depending on any number of the functional parameters $\rho_i(\alpha, t)$.

Creation and annihilation operators

$$a := -\frac{d}{d\,x} + x, \quad a^* := \frac{d}{d\,x} + x,$$

$$F\,a = ia\,F, \quad F\,a^* = -ia^*$$

For any eigenfunction $f \in S$ of F , corresponding to the eigenvalue i^k, we have :

$$F(a^j\,f) = i^{k+j} a^j f, \qquad F((a^*)^j f) = i^{k-j}\,(a^*)^j f.$$

Schur matrix and DFT

$$\Phi_{jk}(n) = \frac{1}{\sqrt{n}} q^{jk}, \quad j,k = 0,\ldots,n-1,\; q = e^{\frac{2i\pi}{n}}, \quad \Phi_{jk} = \Phi_{kj},\; \Phi^{-1} = \bar{\Phi}.$$

The DFT (discrete Fourier transform) $f \to \tilde{f}$ describes the action of Φ on the finite sequences ,(vectors), $\mathbf{f} = (f_0, \ldots, f_{n-1})$:

$$\tilde{f}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_j\, e^{\frac{2\pi ijk}{n}}, \quad \tilde{f}_{k\pm n} = \tilde{f}_k$$

Inversion formula for the DFT and Parceval identity

$$f_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{f}_k \, e^{-\frac{2\pi ijk}{n}} \quad \text{or } \mathbf{f} = \Phi^{-1}\tilde{\mathbf{f}}.$$

$$\sum_{j=0}^{n-1} |f_j|^2 = \sum_{j=0}^{n-1} |\tilde{f}_j|^2.$$

Square of the DFT matrix

$$\Phi^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

In other words,

$$\Phi^2_{jm} = \delta_{n-j,m} = \Phi^2_{mj}, \; (\Phi^2 f)_j = f_{n-j}, \Phi^4 = I, \quad \Phi^3 = \Phi^{-1} = \Phi^* = \bar{\Phi}.$$

$$\operatorname{Tr} \Phi^2 = \frac{3 + (-1)^n}{2}$$

The quantity $G = \sqrt{n}\,\mathrm{Tr}\,\Phi =$ is the famous Gauss sum:

$$\frac{1}{\sqrt{n}} G(n) = \mathrm{Tr}\,F = \frac{1}{\sqrt{n}} \sum_{s=0}^{n-1} q^{s^2} = \frac{1+(-i)^n}{1-i}.$$

Spectral decomposition of the DFT operator

$$\Phi = \sum_{j=0}^{3} i^j \, p_j, \quad p_j = \frac{1}{4} \sum_k (-i)^{jk} \Phi^k,$$

$$\sum_{j=0}^{3} p_j = I, \quad p_j \, p_k = p_j \, \delta_{jk}.$$

The spectral projectors p_j are real symmetric matrices!

Generating eigenvectors of DFT from any finite sequence
It follows from the spectral decomposition above, that any finite sequence sequence $f_j, j = 0, \ldots, n-1$ generates the eigenvector of $\Phi : \Phi\mathbf{v} = i^k\mathbf{v}$ via a formula

$$v_j(k) = f_j + (-i)^k\tilde{f}_j + (-1)^k f_{n-j} + (-i)^{3k}\tilde{f}_{-j}.$$

Of course for the sequences f_j extended periodically we can replace f_{n-j} by f_{-j} .

Spectral multiplicities of $\Phi(n)$

$$\begin{array}{lclccc}
n = 4m+2 & \Rightarrow & m(1) = m+1 & m(i) = m & m(-1) = m+1 & m(-i) = m, \\
n = 4m & \Rightarrow & m(1) = m+1 & m(i) = m & m(-1) = m & m(-i) = m-1 \\
n = 4m+1 & \Rightarrow & m(1) = m+1 & m(i) = m & m(-1) = m & m(-i) = m \\
n = 4m+3 & \Rightarrow & m(1) = m+1 & m(i) = m+1 & m(-1) = m+1 & m(-i) = m
\end{array}$$

Characteristic polynomial $P(z)$ and determinant of Φ

$$P(z) = \prod_{k=0}^{k=3} (z - i^k)^{m_k}, \ \det F = \prod_{k=1}^{k=3} i^{km_k} = i^{\frac{n(n-1)}{2} + (n-1)^2} = i^{\frac{3n^2-5n}{2}+1},$$

The explicit formulas for spectral projectors p_j given above provide the complete set of eigenvectors of Φ: we have to take m_0 first columns of p_0, m_1 first nontrivial, (since first columns of p_1 and p_3 are formed by zeros), columns of p_1, m_2 first columns of p_2 and m_3 first nontrivial columns of p_3 to get it. The completeness follows from the relation $\sum p_j = I$. The eigenvector $\mathbf{v}(k, m)$, equal to the m-th column of p_k, has the components $v_j(m, k)$ (equal to the matrix elements $p_{mj} = p_{jm}$ of p_k), :

$$4v_j(k, m) = \delta_{jm} + (-1)^k \delta_{n-j,m} + (-i)^k \frac{q^{jm}}{\sqrt{n}} + (-i)^{3k} \frac{q^{-jm}}{\sqrt{n}}$$

$$\Phi(2) := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad , F^2 = I,\ p_1 = p_3 = 0.$$

Orthogonal matrix O_2 reducing F to the form $\mathrm{diag}[1, -1] = O^{-1}FO$ is

$$O_2 = \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{-1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

n=4 The DFT matrix Φ is

$$\Phi = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Here $p_3 = 0$, since $m(-i) = m_3 = 0$.
O_4 : $O_4^{-1}\Phi\, O_4 = \mathrm{diag}[1, 1, -1, i]$

$$O_4 = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & -1 & 0 \\ 1 & 0 & 1 & \sqrt{2} \\ -1 & \sqrt{2} & 1 & 0 \\ 1 & 0 & 1 & -\sqrt{2} \end{pmatrix}.$$

Generalized Poisson summation formula
Let E be the space of the continuous functions absolutely integrable together with their Fourier transform and $f \in E$. Poisson summation formula:

$$ab \sum_{r \in Z} f(b(ar + x)) = \sqrt{2\pi} \sum_{m \in Z} \tilde{f}(2\pi m(ab)^{-1}) e^{-\frac{2\pi i m x}{a}}.$$

In order to create the DFT structure in the RHS of the Poisson formula , we specify a, b and x and write m as follows :

$$a = n, x = j, b = \sqrt{\frac{2\pi}{n}} j = 0, \dots, n-1;$$

$$m = nr + k, r \in Z, k = 0, \dots, n-1.$$

The Poisson formula now takes a form:

$$\sum_{r \in Z} f\left(\sqrt{\frac{2\pi}{n}}(nr + j)\right) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[\sum_{r \in Z} \tilde{f}\left(\sqrt{\frac{2\pi}{n}}(nr + k)\right)\right] q^{-jk}.$$

Consider a following mapping M :

$$M : E \to C^n, \;\; M(f) := \mathbf{v}, v_j = \sum_{r \in Z} f\left(\sqrt{\frac{2\pi}{n}}(nr + j)\right) ; j = 0, \ldots, n{-}1.$$

we can interpret Poisson formula as an intertwining relation between FT and DFT. Namely the following statement holds .

Theorem

The mapping M is intertwining F and Φ :

$$\Phi\, M = M\, F.$$

The mapping M is surjective, it maps any eigenvector of F to some eigenvector of Φ. Any continuous, absolutely integrable eigenfunction ψ of F generates the eigenvector $M(\psi)$ of Φ, corresponding to the same eigenvalue :

$$F(\psi) = i^k \psi \rightarrow \Phi\, M(\psi) = i^k\, M(\psi), \quad k = 0, 1, 2, 3.$$

In particular case, when f is any eigenfunction of H, this connection was found by Mehta (1987)

Let $\sum_{m\in Z} g_m$ be any absolutely convergent series and

$$g(x) := \sum_{m\in Z} g_m q^{mx}, \quad , q := \exp\frac{2i\pi}{n}, \quad f_j := \sum_{r\in Z} g_{rn+j}.$$

Obviously $f_j = f_{j+n}$.

Discrete Poisson summation formula

$$\sum_{j=0}^{n-1} q^{jl} \sum_{m\in Z} g_{nm+j} = \sum_{k\in Z} g_k\, q^{kl} = g(l), \quad l = 0, \ldots, n-1;$$
$$\sum_{m\in Z} g_{nm+j} = \frac{1}{n} \sum_{l=0}^{n-1} g(l)\, q^{-lj}, \qquad q = e^{\frac{2\pi i}{n}}.$$

Of course the second formula is immediate consequence of the first obtained by applying the inversion formula for DFT to its LHS . To prove the first formula we set $r = nm + j$, which implies that $j = r - nm, q^{jl} = q^{rl-lnm} = q^{rl}$. Now it is clear that a double sum in the LHS of can be written as a single sum which is exactly the same as the RHS which completes the proof.

Theorem

Let $\sum_{p\in Z} g_p$ is any absolutely converging series. Then vector $v(k)$ with the components $v_j(k)$,

$$v_j(k) = \sum_{m\in Z}(g_{nm+j}+(-1)^k g_{nm-j})+\frac{(-i)^k}{\sqrt{n}}\sum_{m\in Z}(g_m+(-1)^k g_{-m})e^{\frac{2\pi imj}{n}},$$

is an eigenvector of the DFT :
$\Phi v = i^k v.$

We can compare now the explicit expressions for the eigenvectors of Φ coming from spectral theorem with those obtained from the formula above. Assuming that g_n are the functions depending on some parameters, we can expect that for the functions represented by the RHS of we can get in this way some functional identities. We will discuss here only the identities for generalized theta functions (ν-theta functions) although there are many other applications.

ν-theta functions

The function $\theta(x,\tau,\nu)$, which we will shortly call ν-theta function, is defined by the formula

$$\theta(x,\tau,\nu) = \sum_{m=-\infty}^{\infty} e^{\pi i\tau m^{2\nu}+2\pi imx}, \qquad \nu\in Z^{+}, \operatorname{Im}\tau > 0$$

It is obvious that $\theta(x, \tau, \nu)$ is an entire function of x satisfying the relation

$$\theta(x+1, \tau, \nu) = \theta(x, \tau, \nu).$$

$\theta(x, \tau, \nu)$ reduces to the usual theta function $\theta(x, \tau)$ when $\nu = 1$.

ν-theta function satisfies the following PDE

$$2\,(2\pi)^{2\nu-1}(-1)^{\nu}\,\frac{\partial\theta}{\partial\tau} = i\,\frac{\partial^{2\nu}\theta}{\partial x^{2\nu}}.$$

ν-theta function $\theta_{a,b}(x,\tau,\nu)$ with characterstics $[a,b]$ is introduced by the formulas

$$\theta_{a,b}(x,\tau,\nu) = \sum_{n \in Z} \exp[\pi i \tau (n+a)^{2\nu} + 2\pi i (n+a)(x+b)].$$

This function also gives the solution to a same parabolic PDE.

In the case $\nu = 1$, $\theta_{a,b}(x, \tau, \nu)$ reduces to the usual theta function with characteristics $\theta_{a,b}(x, \tau)$,
simply connected with $\theta(x, \tau)$:

$$\theta(x, \tau) = \theta(x + a\tau + b, \tau) \exp[\pi i a^2 \tau + 2\pi i a(x + b)].$$

Suppose that

$$g_m = \exp\left(\frac{\pi i \tau m^{2\nu}}{n^{2\nu}} + \frac{2\pi i m x}{n}\right).$$

We obviously have the formula

$$\sum_{m \in Z} g_{mn+j} = \theta_{\frac{j}{n},0}(x, \tau, \nu)$$

Proposition

For any τ with $Im\tau > 0$ vector $v(\tau, x, k, \nu)$ with the components v_j,

$$v_j(x,\tau,k) = \theta_{\frac{j}{n},0}(x,\tau,\nu) + (-1)^k \theta_{\frac{-j}{n},0}(x,\tau,\nu)$$
$$+\frac{1}{\sqrt{n}}\left[(-i)^k \theta\left(\frac{j+x}{n}, \frac{\tau}{n^{2\nu}}, \nu\right) + (-i)^{3k}\theta\left(\frac{x-j}{n}, \frac{\tau}{n^{2\nu}}, \nu\right)\right].$$

is the eigenvector of the DFT :

$$\Phi(n)v(k) = i^k v(k).$$

This result already implies interesting functional relations between the ν-theta functions.

Let e_k are the eigenvectors of $\Phi(n)$: $\Phi(n)e_k = i^k e_k$. We obviously have the relation:

$$\Phi(n)(e_0 + e_1 + e_2 + e_3) = e_0 + ie_1 - e_2 - ie_3$$

Applying this relation to the eigenvectors of $\Phi(n)$ given by the series above we get the following statement:

Proposition In the case $\nu = 1$ the result formulated below is well known and describes the connection between two bases in the space of entire functions with prescribed quasi periodicity properties (Mumford , Tata lectures on Thetas).
The ν-theta functions

$$\theta\left(\frac{x+k}{n}, \frac{\tau}{n^{2\nu}}, \nu\right), \ \theta_{j/n,0}(x,\tau,\nu), \ k, j = 0, \ldots, n-1$$

are connected by the discrete Fourier transform :

$$\theta_{j/n,0}(x,\tau,\nu) = \frac{1}{n}\sum_{k=0}^{n-1} q^{-jk}\theta\left(\frac{x+k}{n}, \frac{\tau}{n^{2\nu}}, \nu\right),$$
$$\theta\left(\frac{x+k}{n}, \frac{\tau}{n^{2\nu}}, \nu\right) = \sum_{j=0}^{n-1} q^{jk}\theta_{j/n,0}(x,\tau,\nu).$$

In particular, taking $j = 0$ $k = 0$ in the formulas above we we obtain that

$$\theta(x, \tau, \nu) = \frac{1}{n} \sum_{k=0}^{n-1} \theta\left(\frac{x+k}{n}, \frac{\tau}{n^{2\nu}}, \nu\right),$$
$$\theta\left(\frac{x}{n}, \frac{\tau}{n^{2\nu}}, \nu\right) = \sum_{j=0}^{n-1} \theta_{j/n,0}(x, \tau, \nu)$$

Unitarity of the discrete Fourier transform implies now the following relations between the ν-theta functions (Parseval identity):

$$\sum_{k=0}^{n-1}\left|\theta\left(\frac{x+k}{n},\frac{\tau}{n^{2\nu}},\nu\right)\right|^2 = n\sum_{k=0}^{n-1}|\theta_{k/n,0}(x,\tau,\nu)|^2$$

In the simplest case $\nu = 1, n = 2$ we have two eigenvalues of $\Phi(2)$,- $\lambda_1 = 1$ and $\lambda_2 = -1$, and two eigenvectors $v1, v(-1)$ The generaal formula expressing the components of these eigenvectors by means of Jacobi theta functions reads, $(j = 0, 1)$:

$$v_j(1) = \theta_{j/2,0}(x,\tau)+\theta_{-j/2,0}(x,\tau)+\frac{1}{\sqrt{2}}\left[\theta\left(\frac{x+j}{2},\frac{\tau}{4}\right)+\theta\left(\frac{x-j}{2},\frac{\tau}{4}\right)\right],$$

$$v_j(-1) = \theta_{j/2,0}(x,\tau)+\theta_{-j/2,0}(x,\tau)-\frac{1}{\sqrt{2}}\left[\theta\left(\frac{x+j}{2},\frac{\tau}{4}\right)+\theta\left(\frac{x-j}{2},\frac{\tau}{4}\right)\right]$$

Since $\Phi(2)(v(1)+v(2)) = v(1) - v(2)$ the formulas above imply two identities:

$$\theta(x,\tau)+\theta_{1/2,0} = \theta\left(\frac{x}{2},\frac{\tau}{4}\right), \theta(x,\tau)-\theta_{1/2,0}(x,\tau) = \theta\left(\frac{x+1}{2},\frac{\tau}{4}\right).$$

Suppose that $\Phi v(x,\tau) = v(x,\tau)$. $v(x+\tau/2,\tau)$ and $v(y+1,\tau)$ are again eigenvectors , corresponding to the same eigenvalue $\lambda_1 = 1$. Therefore

$$\det\left[v(x+\tau/2,\tau), v(y+1,\tau)\right] = 0.$$

This gives a quadratic relation between theta functions allowing to prove extended and classical quartic Riemann identities.

THANK YOU FOR YOUR ATTENTION !