Recent Advances in Quantum Integrable Systems

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On the completeness of solutions of Bethe’s equations

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Introduction

spin-1/2 periodic XXX chain:

$$H = \frac{1}{4} \sum_{n=1}^{N} (\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - 1), \quad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1$$
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H |\lambda_1, \ldots, \lambda_M \rangle = E |\lambda_1, \ldots, \lambda_M \rangle
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|\lambda_1, \ldots, \lambda_M\rangle = \prod_{k=1}^{M} B(\lambda_k)|0\rangle \quad |0\rangle = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^\otimes N
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Bethe states
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\[ \left( \lambda_k + \frac{i}{2} \right)^N \prod_{\substack{j=1 \atop j \neq k}}^{M} (\lambda_k - \lambda_j - i) = \left( \lambda_k - \frac{i}{2} \right)^N \prod_{\substack{j=1 \atop j \neq k}}^{M} (\lambda_k - \lambda_j + i) , \] Bethe equations (BE)

\[ k = 1, 2, \ldots, M , \quad M = 0, 1, \ldots, \frac{N}{2} \]
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Are solutions “complete”? i.e. too few, too many, or just right?
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Outline

- Singular solutions: physical & unphysical
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• Completeness/Pauli-principle conjecture
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• Integrable spin $s > 1/2$
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• Conclusions
Singular solutions

Example: \((\lambda_1, \lambda_2) = \left(\frac{i}{2}, -\frac{i}{2}\right)\)
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\]

\[
\left(\lambda_2 + \frac{i}{2}\right)^N (\lambda_2 - \lambda_1 - i) = \left(\lambda_2 - \frac{i}{2}\right)^N (\lambda_2 - \lambda_1 + i)
\]

an exact solution of BE for any value of \(N\)
Singular solutions

Example: \((\lambda_1, \lambda_2) = \left(\frac{i}{2}, -\frac{i}{2}\right)\)  
exact 2-string centered at origin
Singular solutions

Example: \((\lambda_1, \lambda_2) = \left(\frac{i}{2}, -\frac{i}{2}\right)\)

1. Energy is singular:

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E = -\frac{1}{2} \sum_{k=1}^{2} \frac{1}{\lambda_k^2 + \frac{1}{4}} \sim \frac{1}{0} \quad ???
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B(\lambda_1)B(\lambda_2)|0\rangle \sim \frac{0}{0}
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\[
T_a(\lambda) = R_{Na}(\lambda) \cdots R_{1a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad R_{na}(\lambda) = \frac{1}{\left(\lambda + \frac{i}{2}\right)} \begin{pmatrix} 1 & 0 \\ 0 & (\lambda - \frac{i}{2}) \end{pmatrix}
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exact 2-string centered at origin
Singular solutions

Example: \((\lambda_1, \lambda_2) = \left( \frac{i}{2}, -\frac{i}{2} \right)\) exact 2-string centered at origin

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\]

Try naive regulator:
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\lambda_1^{\text{naive}} = \frac{i}{2} + \epsilon, \quad \lambda_2^{\text{naive}} = -\frac{i}{2} + \epsilon
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\lim_{\epsilon \to 0} E = -1 \quad \checkmark
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Try naive regulator:

\[ \lambda_{naive}^1 = \frac{i}{2} + \epsilon, \quad \lambda_{naive}^2 = -\frac{i}{2} + \epsilon \]

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Try naive regulator: \[ \lambda_1^{\text{naive}} = \frac{i}{2} + \epsilon, \quad \lambda_2^{\text{naive}} = -\frac{i}{2} + \epsilon \]

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... but it is not an eigenvector (e.g. for \(N=4\)) ???
Better regulator:

\[ \lambda_1 = \frac{i}{2} + \epsilon + c \epsilon N \]
\[ \lambda_2 = -\frac{i}{2} + \epsilon , \]

[Avdeev, Vladimirov 85; Beisert, Minahan, Staudacher, Zarembo 03]
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How to determine c?
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How to determine \( c \): algebraic Bethe ansatz

transfer matrix 

\[
t(\lambda) = \text{tr}_a T_a(\lambda) = A(\lambda) + D(\lambda)
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t(\lambda) = \text{tr}_a T_a(\lambda) = A(\lambda) + D(\lambda)
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off-shell:

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t(\lambda)|\lambda_1, \ldots , \lambda_M\rangle = \Lambda(\lambda)|\lambda_1, \ldots , \lambda_M\rangle + \sum_{k=1}^{M} F_k(\lambda, \{\lambda\}) B(\lambda) \prod_{j \neq k} B(\lambda_j)|0\rangle
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How to determine \( c \):

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 &\quad \text{“unwanted”}
\end{align*}
\]

\[
F_k(\lambda, \{\lambda\}) = \frac{i}{\lambda - \lambda_k} \left[ \prod_{j \neq k}^{M} \left( \frac{\lambda_k - \lambda_j - i}{\lambda_k - \lambda_j} \right) - \left( \frac{\lambda_k - \frac{i}{2}}{\lambda_k + \frac{i}{2}} \right)^N \prod_{j \neq k}^{M} \left( \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j} \right) \right]
\]

\[ F_k(\lambda, \{\lambda\}) = 0 \iff \text{BE} \]
\[ \lambda_1 = \frac{i}{2} + \epsilon + c \epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon, \]
\( M = 2 : \quad \lambda_1 = \frac{i}{2} + \epsilon + c \epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon, \)

\[
t(\lambda)|\lambda_1, \lambda_2\rangle = \Lambda(\lambda)|\lambda_1, \lambda_2\rangle + F_1(\lambda, \{\lambda\})B(\lambda_2)B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\})B(\lambda_1)B(\lambda)|0\rangle
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Key:

\[ B(\lambda_2) \sim \frac{1}{\epsilon^N}, \quad B(\lambda_1) \sim 1 \]
\( M = 2 \): \( \lambda_1 = \frac{i}{2} + \epsilon + c \epsilon N \), \( \lambda_2 = -\frac{i}{2} + \epsilon \),

t(\lambda)|\lambda_1, \lambda_2\rangle = \Lambda(\lambda)|\lambda_1, \lambda_2\rangle + F_1(\lambda, \{\lambda\})B(\lambda_2)B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\})B(\lambda_1)B(\lambda)|0\rangle

Key:

\[
B(\lambda_2) \sim \frac{1}{\epsilon N}, \quad B(\lambda_1) \sim 1 \quad \text{and} \quad R_{na}(\lambda) = \left[ \frac{1}{(\lambda - \frac{i}{2})} \right] (\lambda - \frac{i}{2}) I_{na} + iP_{na}
\]
$M = 2: \quad \lambda_1 = \frac{i}{2} + \epsilon + c c^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon,$

$t(\lambda) |\lambda_1, \lambda_2\rangle = \Lambda(\lambda) |\lambda_1, \lambda_2\rangle + F_1(\lambda, \{\lambda\}) B(\lambda_2) B(\lambda) |0\rangle + F_2(\lambda, \{\lambda\}) B(\lambda_1) B(\lambda) |0\rangle$

Key:

$B(\lambda_2) \sim \frac{1}{\epsilon^N}, \quad B(\lambda_1) \sim 1$

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$F_1(\lambda, \{\lambda\}) \sim \epsilon^{N+1}, \quad F_2(\lambda, \{\lambda\}) \sim \epsilon$
$M = 2 : \quad \lambda_1 = \frac{i}{2} + \epsilon + c\epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon,$

\[ t(\lambda|\lambda_1, \lambda_2) = \Lambda(\lambda|\lambda_1, \lambda_2) + F_1(\lambda, \{\lambda\}) B(\lambda_2) B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\}) B(\lambda_1) B(\lambda)|0\rangle \]

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\[ t(\lambda) | \lambda_1, \lambda_2 \rangle = \Lambda(\lambda) | \lambda_1, \lambda_2 \rangle + F_1(\lambda, \{\lambda\}) B(\lambda_2) B(\lambda) | 0 \rangle + F_2(\lambda, \{\lambda\}) B(\lambda_1) B(\lambda) | 0 \rangle \]

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“generalized Bethe equations”
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\[ F_1(\lambda, \{\lambda\}) \sim \epsilon^{N+1}, \quad F_2(\lambda, \{\lambda\}) \sim \epsilon \]

\[ F_1(\lambda, \{\lambda\}) = \left( \frac{c + 2i^{-N} - i}{\lambda - \frac{i}{2}} \right) \epsilon^N + O(\epsilon^{N+1}), \quad F_2(\lambda, \{\lambda\}) = \left( \frac{2i - i^N c}{\lambda + \frac{i}{2}} \right) + O(\epsilon) \]
\[ M = 2 : \quad \lambda_1 = \frac{i}{2} + \epsilon + c \epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon, \]

\[ t(\lambda) | \lambda_1, \lambda_2 \rangle = \Lambda(\lambda) | \lambda_1, \lambda_2 \rangle + F_1(\lambda, \{\lambda\}) B(\lambda_2) B(\lambda) |0\rangle + F_2(\lambda, \{\lambda\}) B(\lambda_1) B(\lambda) |0\rangle \]

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F_1(\lambda, \{\lambda\}) = \left( \frac{c + 2i^{-(N+1)}}{\lambda - \frac{i}{2}} \right) \epsilon^N + O(\epsilon^{N+1}), \quad F_2(\lambda, \{\lambda\}) = \left( \frac{2i - i^{-N} c}{\lambda + \frac{i}{2}} \right) + O(\epsilon)
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  t(\lambda)\langle \lambda_1, \lambda_2 \rangle = \Lambda(\lambda)\langle \lambda_1, \lambda_2 \rangle + F_1(\lambda, \{\lambda\})B(\lambda_2)B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\})B(\lambda_1)B(\lambda)|0\rangle
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  F_1(\lambda, \{\lambda\}) \sim \epsilon^{N+1}, \quad F_2(\lambda, \{\lambda\}) \sim \epsilon
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"generalized Bethe equations"

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\]

\[\Rightarrow\]

N even: \( c = 2i(-1)^{N/2} \)
\( M = 2 : \quad \lambda_1 = \frac{i}{2} + \epsilon + c \epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon, \)

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t(\lambda)|\lambda_1, \lambda_2\rangle = \Lambda(\lambda)|\lambda_1, \lambda_2\rangle + F_1(\lambda, \{\lambda\})B(\lambda_2)B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\})B(\lambda_1)B(\lambda)|0\rangle
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\[
\implies \quad \text{N even: } \quad c = 2i(-1)^{N/2}
\]

**N odd:** no solution for \( c \)!
\[ M = 2 : \quad \lambda_1 = \frac{i}{2} + \epsilon + c\epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon, \]

\[ t(\lambda)|\lambda_1, \lambda_2\rangle = \Lambda(\lambda)|\lambda_1, \lambda_2\rangle + F_1(\lambda, \{\lambda\})B(\lambda_2)B(\lambda)|0\rangle + F_2(\lambda, \{\lambda\})B(\lambda_1)B(\lambda)|0\rangle \]

Key:

\[ B(\lambda_2) \sim \frac{1}{\epsilon^N}, \quad B(\lambda_1) \sim 1 \]

Need

\[ F_1(\lambda, \{\lambda\}) \sim \epsilon^{N+1}, \quad F_2(\lambda, \{\lambda\}) \sim \epsilon \]

“generalized Bethe equations”

\[ F_1(\lambda, \{\lambda\}) = \left( c + 2i^{-(N+1)} \right) \frac{\lambda - \frac{i}{2}}{\lambda - \frac{i}{2}} \epsilon^N + O(\epsilon^{N+1}), \quad F_2(\lambda, \{\lambda\}) = \left( 2i - i^{-N}c \right) \frac{\lambda + \frac{i}{2}}{\lambda + \frac{i}{2}} + O(\epsilon) \equiv 0 \]

\[ \Rightarrow \]

N even: \[ c = 2i(-1)^{N/2} \]

N odd: no solution for c !

Although \( \pm i/2 \) satisfies BE, does NOT correspond to eigenstate of \( t(\lambda) \) !
$M = 2 : \quad \lambda_1 = \frac{i}{2} + \epsilon + c\epsilon^N, \quad \lambda_2 = -\frac{i}{2} + \epsilon,$

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F_1(\lambda, \{\lambda\}) = \left( \frac{c + 2i/(-N+1)}{\lambda - i/2} \right)e^N + O(\epsilon^{N+1}), \quad F_2(\lambda, \{\lambda\}) = \left( \frac{2i - i^{-N}c}{\lambda + i/2} \right) + O(\epsilon)
\]

\[
\Rightarrow \quad N \text{ even: } \quad c = 2i(-1)^{N/2}
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\[
N \text{ odd: no solution for } c ! \quad \text{unphysical singular solution}
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\[
\implies \\
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Although \( \pm i/2 \) satisfies BE, does NOT correspond to eigenstate of \( t(\lambda) \)!
General singular solution:

\[ \left\{ \frac{i}{2}, -\frac{i}{2}, \lambda_3, \ldots, \lambda_M \right\} \]

\[ \lambda_3, \ldots, \lambda_M \text{ distinct } \neq \pm \frac{i}{2} \]
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\[
\left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right)^{N-1} \left( \frac{\lambda_k - \frac{3i}{2}}{\lambda_k + \frac{3i}{2}} \right) = \prod_{\substack{j \neq k \atop j=3}}^{M} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \quad k = 3, \ldots, M.
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Regularization + generalized Bethe equations \[ \Rightarrow \]

\[
c = -\frac{2}{i^{N+1}} \prod_{j=3}^{M} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}}, \quad c = 2i^{N+1} \prod_{j=3}^{M} \frac{\lambda_j + \frac{3i}{2}}{\lambda_j - \frac{i}{2}}
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Consistency \( \Rightarrow \)

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\left[-\prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right) \right]^N = 1
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Condition for the singular solution to be physical
Completeness/Pauli-principle conjecture

\[ \mathcal{N}(N, M) \equiv \# \text{ solutions } \{\lambda_1, \ldots, \lambda_M\} \text{ of BE with finite pairwise distinct roots} \]
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Bethe states are SU(2) highest-weight states:

\[ S^+|\lambda_1, \ldots, \lambda_M\rangle = 0, \quad S^\pm = S^x \pm iS^y \]

\[ \bar{S} = \frac{1}{2} \sum_{n=1}^{N} \bar{\sigma}_n \]
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\[ n_s = \left( \frac{N}{2} - s \right) - \left( \frac{N}{2} - s - 1 \right) \]

multiplicity of spin \( s \) rep
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\[M = 0, 1, \ldots, \frac{N}{2} \quad \checkmark\]
Naive conjecture:

\[ \mathcal{N}(N, M) \triangleq \binom{N}{M} - \binom{N}{M - 1} \]

# solutions with M pairwise distinct roots
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Wrong!

Assumes (incorrectly) that every solution of BE with pairwise distinct roots produces eigenstate of \( t(\lambda) \)
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Our conjecture:

\[
\mathcal{N}(N, M) - \mathcal{N}_s(N, M) + \mathcal{N}_{sp}(N, M) = \binom{N}{M} - \binom{N}{M - 1}
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\( \mathcal{N}_s(N, M) \equiv \text{# singular solutions} \)

\( \mathcal{N}_{sp}(N, M) \equiv \text{# singular solutions that are physical} \)

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\left[ - \prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right) \right]^N = 1
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Physical meaning: BE have “too many” solutions. But, after discarding unphysical singular solutions, remain with just right #
Two-way nature of this conjecture:

1. For every highest-weight eigenstate of $t(\lambda)$, there exists a solution of BE with pairwise distinct roots
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Previous studies have focused only on 1.
Two-way nature of this conjecture:

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Previous studies have focused only on 1.

To check conjecture, must find ALL solutions of BE with pairwise distinct roots.

For $N>5$, brute force is not an option...
Homotopy continuation
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Toy example:

Want to solve \( x^2 - 5x + 6 = 0 \)
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Homotopy continuation

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Start from a problem whose solutions we know \( x^2 - 1 = 0 \) “start system”

and deform it to the problem that we want to solve:

\[
H(x, t) \equiv (1 - t)(x^2 - 5x + 6) + t(x^2 - 1) \quad \text{“homotopy”}
\]

& consider \( H(x, t) = 0 \quad 0 \leq t \leq 1 \)
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Track to \( t = 0 \) by solving (numerically)

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\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial t} = 0
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Get \( x = 3, 2 \)
Works very well for systems of polynomial equations

\[ f_1(\lambda_1, ..., \lambda_M) = 0 \]
\[ \vdots \]
\[ f_M(\lambda_1, ..., \lambda_M) = 0 \]
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Bezout bound:

\# finite solutions is at most

\[ D \equiv \deg(f_1) \cdot \deg(f_2) \cdots \deg(f_M) \]
Works very well for systems of polynomial equations $f_1(\lambda_1, \ldots, \lambda_M) = 0$

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**Bezout bound:**

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By choosing start system with $D$ solutions ("total degree" homotopy), can find all the solutions
Various software packages are available - highly parallelizable!

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f_1(\lambda_1, ..., \lambda_M) &= 0 \\
& \vdots \\
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We used “Bertini” on a cluster (176 cores) to solve
BE up to N=14, M=7 (1.2 hour)
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\lambda_k^{N+M-2} &= 1 \\
k &= 1, 2, \ldots, M
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\[\Rightarrow \]
\[
\lambda_k = \omega^{j_k} \, , \quad \omega = e^{2\pi i/(N+M-2)} \, , \quad j_k = 0, 1, \cdots, N + M - 3
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\[ \lambda_k = \omega^{j_k}, \quad \omega = e^{2\pi i/(N+M-2)}, \quad j_k = 0, 1, \cdots, N + M - 3 \]

Want distinct roots, so can restrict
\[ 0 \leq j_1 < j_2 < j_3 < \cdots < j_M \leq N + M - 3 \]
The poles to cancel, the condition for which is precisely the Beth equations (1.3). Indeed, dividing both sides of (4.3) by \( Q_j \) and then, by finding the zeros of

\[ \lambda_j \pm \frac{1}{C_j} \]

we determine all the solutions of the Bethe equations \((4.4)\), in contradiction with the fact that the transfer matrix eigenvalues satisfy the celebrated T-Q equation \([5, 6, 7, 8]\).

Moreover, the transfer matrix eigenvalues are independent of \( \lambda_j \), and must therefore be regular. The only way out of this paradox is for the poles to cancel, the condition for which is precisely the Beth equations (1.3).

Interestingly, it is possible to solve the T-Q equation (4.3) numerically for both \( \lambda_j \), and then, by finding the zeros of

\[ \lambda_j \pm \frac{1}{C_j} \],

we determine all the solutions of the Bethe equations (1.3). The Bethe roots \( \{\lambda_k\} \) for both \( N=8, M=4 \):

<table>
<thead>
<tr>
<th>number</th>
<th>Bethe roots ( {\lambda_k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>\pm 0.5250121022236669 \pm 0.1294729463749287</td>
</tr>
<tr>
<td>2</td>
<td>0.5570702385744416 0.1470126111961413</td>
</tr>
<tr>
<td>3</td>
<td>\pm 0.5 0.1470126111961413 0.5002695484553508I</td>
</tr>
<tr>
<td>4</td>
<td>\pm 0.5I 0.5000288621635332I 0.5000288621635332I</td>
</tr>
<tr>
<td>5</td>
<td>\pm 0.5I \pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>6</td>
<td>\pm 0.5I \pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>7</td>
<td>\pm 0.5I \pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>8</td>
<td>\pm 0.5I \pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>9</td>
<td>0.220560007920844 -0.66912292881517</td>
</tr>
<tr>
<td>10</td>
<td>\pm 0.5I 0.1695810016454493</td>
</tr>
<tr>
<td>11</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>12</td>
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<tr>
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<td>\pm 0.5I \pm 0.5I</td>
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<tr>
<td>14</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>15</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>16</td>
<td>\pm 0.5I \pm 0.5I</td>
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<tr>
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<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>19</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>20</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>21</td>
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</tr>
<tr>
<td>22</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>23</td>
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</tr>
<tr>
<td>24</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>25</td>
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<tr>
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<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>27</td>
<td>0.08378710739142802 -0.2430919428911911</td>
</tr>
<tr>
<td>28</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
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<td>29</td>
<td>\pm 0.5I \pm 0.5I</td>
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<td>30</td>
<td>\pm 0.5I \pm 0.5I</td>
</tr>
<tr>
<td>31</td>
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</tr>
<tr>
<td>32</td>
<td>-0.220560007920844 0.66912292881517</td>
</tr>
</tbody>
</table>

* singular unphysical

** singular physical
is for the poles to cancel, the condition for which is precisely the Beth
LHS is a polynomial in
here by
Furthermore, it can be shown that the eigenvalues of the transfer
that are unphysical are labeled by
Table 1: the Bethe equations (1.3)
where the coefficients
are independent of
and must therefore be regular. The only way out of this paradox
of degree
of degree
\( \lambda_j \)
\( Q \)
\( T \)
\( \lambda_j \)
\( N \)
\( M \)
\( * \)
\( ** \)
Interestingly, it is possible to solve the T-Q equation (4.3) numerically
** singular unphysical
** singular physical

<table>
<thead>
<tr>
<th>number</th>
<th>Bethe roots ( { \lambda_k } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \pm 0.525012102223669 \pm 0.1294729463749287 )</td>
</tr>
<tr>
<td>2</td>
<td>( 0.5570702385744416 \pm 0.1470126111961413 )</td>
</tr>
<tr>
<td>3*</td>
<td>( \pm 0.5I ) ( -0.2930497652740115 \pm 0.5002695484553081 )</td>
</tr>
<tr>
<td>4*</td>
<td>( \pm 0.5I ) ( 0.0905346112303935 )</td>
</tr>
<tr>
<td>5*</td>
<td>( \pm 0.5I ) ( -0.04929340793103601+1.631134975618312I )</td>
</tr>
<tr>
<td>6*</td>
<td>( \pm 0.5I ) ( 0.6439488581706157-0.1197616885579488I )</td>
</tr>
<tr>
<td>7*</td>
<td>( \pm 0.5I ) ( 0.4929340793103601+1.631134975618312I )</td>
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<tr>
<td>8**</td>
<td>( \pm 0.5I ) ( 0.5638252623934961 )</td>
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<td>13*</td>
<td>( \pm 0.5I ) ( 0.522716443014433 )</td>
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<td>14*</td>
<td>( \pm 0.5I ) ( 0.5030729336599I )</td>
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<td>( \pm 0.5I ) ( 0.4929340793103601-1.631134975618312I )</td>
</tr>
<tr>
<td>17*</td>
<td>( \pm 0.5I ) ( 0.2930497652740115 \pm 0.5002695484553508I )</td>
</tr>
<tr>
<td>18**</td>
<td>( \pm 0.5I ) ( 0.1424690678305666 )</td>
</tr>
<tr>
<td>19**</td>
<td>( \pm 0.5I ) ( 1.556126503577051I )</td>
</tr>
<tr>
<td>20*</td>
<td>( \pm 0.5I ) ( 3.51708429130899I )</td>
</tr>
<tr>
<td>21</td>
<td>( 0.2443331937711654 \pm 0.639488581706157 \pm 0.1197616885579488I )</td>
</tr>
<tr>
<td>22*</td>
<td>( \pm 0.5I ) ( 0.5005581696433306I )</td>
</tr>
<tr>
<td>23</td>
<td>( 0.5000288621635332I )</td>
</tr>
<tr>
<td>24</td>
<td>( \pm 0.5I ) ( 0.4632647275890309 \pm 0.5002938535699026I )</td>
</tr>
<tr>
<td>25</td>
<td>( -0.147012611961413 \pm 0.5570702385744416 )</td>
</tr>
<tr>
<td>26*</td>
<td>( \pm 0.5I ) ( -0.0598671277687283-1.57171694471433I )</td>
</tr>
<tr>
<td>27</td>
<td>( 0.08378710739142802 \pm 0.486819617430914 )</td>
</tr>
<tr>
<td>28*</td>
<td>( \pm 0.5I ) ( -0.2430919428911911 \pm 0.6188079036780695I )</td>
</tr>
<tr>
<td>29</td>
<td>( \pm 1.025705081230743I \pm 0.5002695484553508I )</td>
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<tr>
<td>30*</td>
<td>( \pm 0.5I ) ( -0.5005581696433306I )</td>
</tr>
<tr>
<td>31</td>
<td>( 0.571611171861635332I )</td>
</tr>
<tr>
<td>32</td>
<td>( -0.2205600072920844 \pm 0.669122928815117 )</td>
</tr>
</tbody>
</table>
Solving the T-Q equation
Solving the $T$-$Q$ equation

The eigenvalues $\Lambda(\lambda)$ of $t(\lambda)$ are polynomials

$$\Lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j$$

and satisfy

$$\Lambda(\lambda) Q(\lambda) = \left(\lambda + \frac{i}{2}\right)^N Q(\lambda - i) + \left(\lambda - \frac{i}{2}\right)^N Q(\lambda + i)$$

$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j, \quad Q_M = 1$$
Solving the T-Q equation

The eigenvalues \( \Lambda(\lambda) \) of \( t(\lambda) \) are polynomials

\[
\Lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j
\]

and satisfy

\[
\Lambda(\lambda) Q(\lambda) = \left( \lambda + \frac{i}{2} \right)^N Q(\lambda - i) + \left( \lambda - \frac{i}{2} \right)^N Q(\lambda + i)
\]  

\[
Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j, \quad Q_M = 1
\]

Can solve (2) numerically for both \( \Lambda(\lambda) \) and \( Q(\lambda) \); then, by finding the zeros of \( Q(\lambda) \), obtain all the Bethe roots!

[Baxter 01]
Solving the T-Q equation

The eigenvalues $\Lambda(\lambda)$ of $t(\lambda)$ are polynomials

$$\Lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j$$  \hspace{1cm} (1)

and satisfy

$$\Lambda(\lambda) \cdot Q(\lambda) = \left(\lambda + \frac{i}{2}\right)^N Q(\lambda - i) + \left(\lambda - \frac{i}{2}\right)^N Q(\lambda + i)$$  \hspace{1cm} (2)

$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j, \hspace{1cm} Q_M = 1$$  \hspace{1cm} (3)

Can solve (2) numerically for both $\Lambda(\lambda)$ and $Q(\lambda)$; then, by finding the zeros of $Q(\lambda)$, obtain all the Bethe roots!

Substitute (1) & (3) into (2); equate coefficients of equal powers of $\lambda$; solve.

[Baxter 01]
Solving the T-Q equation

The eigenvalues $\lambda(\lambda)$ of $t(\lambda)$ are polynomials

$$\lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j \quad (1)$$

and satisfy

$$\lambda(\lambda) Q(\lambda) = \left(\lambda + \frac{i}{2}\right)^{\lambda} Q(\lambda - i) + \left(\lambda - \frac{i}{2}\right)^{\lambda} Q(\lambda + i) \quad (2)$$

$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j , \quad Q_M = 1 \quad (3)$$

Can solve (2) numerically for both $\lambda(\lambda)$ and $Q(\lambda)$; then, by finding the zeros of $Q(\lambda)$, obtain all the Bethe roots! [Baxter 01]

Substitute (1) & (3) into (2);

equate coefficients of equal powers of $\lambda$; solve.

Up to $N=9$ on this laptop
Results

<table>
<thead>
<tr>
<th>$\mathcal{M}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(1,0,0; 1)</td>
<td>(2, 1, 1; 2)</td>
<td>(9, 5, 1; 5)</td>
<td>(32, 21, 3; 14)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(2,0,0; 2)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
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<td>(3,0,0; 3)</td>
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</tr>
<tr>
<td>5</td>
<td>(4,0,0; 4)</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(5,0,0; 5)</td>
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<td></td>
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<tr>
<td>7</td>
<td>(6,0,0; 6)</td>
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<tr>
<td>8</td>
<td>(7,0,0; 7)</td>
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<td></td>
</tr>
<tr>
<td>9</td>
<td>(8,0,0; 8)</td>
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<tr>
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</tr>
<tr>
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<tr>
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<td>(11,0,0; 11)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>(12, 0, 0; 12)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>14</td>
<td>(13, 0, 0; 13)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

$\mathcal{N} = \mathcal{N}_s + \mathcal{N}_{sp}$; $\mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp}$

Remarkably, the quantities $\mathcal{N}$ in all the entries of Table 2 coincide with $\mathcal{N}_M$, in perfect agreement with the conjecture (2.12). Although this conjecture was motivated from consideration of a physical model (1.1), it can be viewed solely as a statement about the solutions of the polynomial equations (1.3) and (2.11), which begs for a proof.

It is easy to see that the number of solutions for $\mathcal{M} = 1$ is $\mathcal{N}_1$, $\mathcal{N}(\mathcal{N}, 1) = \mathcal{N}_1$.

Moreover, for $\mathcal{M} = 2$, we observe $\mathcal{N}(\mathcal{N}, 2) = \mathcal{N}_2 + 3\mathcal{N}_1 + \mathcal{N}_1^2$.

It would be interesting to formulate conjectures (not to mention proofs) for $\mathcal{N}(\mathcal{N}, \mathcal{M})$ for $\mathcal{M} \geq 3$.

Several remarks about the singular solutions are in order:

(i) Inspection of Table 1 and the supplemental tables [64] shows that many (but not all) of the unphysical singular solutions (i.e., those solutions labeled by a single star $\star$) are self-conjugate. This does not violate any theorems, since only solutions corresponding to eigenstates of the Hamiltonian are required to be invariant under complex conjugation [69]. Such solutions definitely do not obey the string hypothesis, since string configurations are (by definition) self-conjugate.

(ii) For odd values of $\mathcal{N}$, it appears from Table 2 that most singular solutions are unphysical; i.e., $\mathcal{N}_{sp}(\mathcal{N}, \mathcal{M}) = 0$ for some value of $\mathcal{M}$ if $\mathcal{N}$ is odd. An exception is the case $\mathcal{N} = 9, \mathcal{M} = 3$, for which $\mathcal{N}_{sp}(9, 3) = 2$; and this repeats with a periodicity of 6: $\mathcal{N}_{sp}(15, 3) = 2$, etc. We expect that similar exceptions occur for higher values of $\mathcal{M}$.
Results

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
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<td>(1, 0, 0; 1)</td>
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<td>3</td>
<td>(3, 0, 0; 3)</td>
<td>(2, 1, 1; 2)</td>
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<td>4</td>
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</tr>
<tr>
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<td>5</td>
<td>(5, 0, 0; 5)</td>
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<td>(9, 5, 1; 5)</td>
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<td>(32, 21, 3; 14)</td>
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<td>(28, 1, 0; 27)</td>
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<td>(1701, 1287, 15; 429)</td>
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\((N, N_s, N_{sp}; N - N_s + N_{sp})\)

# solutions
Results

<table>
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<th>$N$</th>
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<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
</tr>
</thead>
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<td>$(1,0,0; 1)$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>$(9, 5, 1; 5)$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td>5</td>
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<td>$(34, 7, 1; 28)$</td>
<td>$(32, 21, 3; 14)$</td>
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<tr>
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<td>7</td>
<td>$(5,0,0; 5)$</td>
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<td>$(122, 36, 4; 90)$</td>
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$(\mathcal{N}, \mathcal{N}_s, \mathcal{N}_{sp}; \mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp})$

# solutions

# singular

$\{\pm \frac{i}{2}, \ldots\}$
Results

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<tr>
<th>M</th>
<th>N</th>
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Remarkably, the quantities $N_{\text{phys}}$ in all the entries of Table 2 coincide with $N + N_{\text{sp}}$ in perfect agreement with the conjecture (2.12). Although this conjecture was motivated from consideration of a physical model (1.1), it can be viewed solely as a statement about the solutions of the polynomial equations (1.3) and (2.11), which begs for a proof.

It is easy to see that the number of solutions for $M = 1$ is $N_{1}$, in perfect agreement with the conjecture (2.12). Although this conjecture was motivated from consideration of a physical model (1.1), it can be viewed solely as a statement about the solutions of the polynomial equations (1.3) and (2.11), which begs for a proof.

Moreover, for $M = 2$, we observe

\[
N_{2} = 1 + 2N_{1} + 3N_{3} + 1 + (N_{1}^2).
\]

(5.1)

It would be interesting to formulate conjectures (not to mention proofs) for $N_{(N,M)}$ for $M \geq 3$.

Several remarks about the singular solutions are in order:

(i) Inspection of Table 1 and the supplemental tables [64] shows that many (but not all) of the unphysical singular solutions (i.e., those solutions labeled by a single star $\star$) are self-conjugate. This does not violate any theorems, since only solutions corresponding to eigenstates of the Hamiltonian are required to be invariant under complex conjugation [69]. Such solutions definitely do not obey the string hypothesis, since string configurations are (by definition) self-conjugate.

(ii) For odd values of $N$, it appears from Table 2 that most singular solutions are unphysical; i.e., $N_{\text{sp}} = 0$ for most values of $M$ if $N$ is odd. An exception is the case $N = 9, M = 3$, for which $N_{\text{sp}}(9,3) = 2$; and this repeats with a periodicity of 6:

\[
N_{\text{sp}}(15,3) = 2, \quad \text{etc.}
\]

We expect that similar exceptions occur for higher values of $M$.

\[
(\mathcal{N}, \mathcal{N}_{s}, \mathcal{N}_{\text{sp}}; \mathcal{N} - \mathcal{N}_{s} + \mathcal{N}_{\text{sp}})
\]

\[
\# \text{ solutions}
\]

\[
\# \text{ singular}
\]

\[
\{ \pm \frac{i}{2}, \ldots \}
\]

\[
\# \text{ singular physical}
\]

\[
\left[ - \prod_{k=3}^{M} \left( \frac{\lambda_{k} + \frac{i}{2}}{\lambda_{k} - \frac{i}{2}} \right) \right]^{\mathcal{N}} = 1
\]
### Results

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\[\begin{align*}
\left( N, N_s, N_{sp} ; N - N_s + N_{sp} \right) \\
\text{# solutions} \\
\text{# singular} \\
\{ \pm \frac{i}{2}, \ldots \} \\
\text{# singular physical} \\
\left[ - \prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right)^N \right] = 1
\end{align*}\]
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$$\left( N, N_s, N_{sp} \right)$$

$$\left( N - N_s + N_{sp} \right)$$

Perfect agreement

with conjecture!

$$\begin{pmatrix} N \end{pmatrix} - \begin{pmatrix} N \\ M \end{pmatrix}$$

$$\left[ - \prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right) \right]^{N} = 1$$
## Results

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\[
(N, N_s, N_{sp} ; N - N_s + N_{sp})
\]

perfect agreement with conjecture!

\[
\begin{align*}
\text{# solutions} & \uparrow \\
\text{# singular} \uparrow & \quad \iff \\
\text{# singular physical} & \quad \left( \frac{N}{M} \right) - \left( \frac{N}{M-1} \right)
\end{align*}
\]

\[
\left[ - \prod_{k=3}^{M} \left( \frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right) \right]^N = 1
\]

BE have “too many” solutions. But, after discarding unphysical singular solutions, remain with just right #
Remarks:

• Many unphysical singular solutions are not self-conjugate
  • do not obey string hypothesis
Remarks:

- Many unphysical singular solutions are not self-conjugate, do not obey string hypothesis.

N=8, M=4

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</table>
Remarks:

- Many unphysical singular solutions are not self-conjugate
  - do not obey string hypothesis

### Table 1: Bethe roots \( \{ \lambda_k \} \)

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\( N=8, M=4 \)

OK, since only solutions corresponding to eigenstates need to be self-conjugate

[Vladimirov 86]
• For odd N, most singular solutions are unphysical
For odd $N$, most singular solutions are unphysical

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For odd $N$, most singular solutions are unphysical

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noted earlier: $\pm i/2$ is unphysical for ALL odd $N$
For odd $N$, most singular solutions are unphysical

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noted earlier: $\pm i/2$ is unphysical for ALL odd $N$

Exception $N=9, M=3$: there are 2 singular physical solutions
For odd N, most singular solutions are unphysical

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noted earlier:  \( \pm i/2 \) is unphysical for ALL odd N

Exception \( N=9, M=3 \): there are 2 singular physical solutions

periodic: \( N = 9+6n, \ n=0,1,2,... \)

Expect other exceptions for higher M
Few of the singular solutions are physical $N_s \gg N_{sp}$

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### Few of the singular solutions are physical
\[ \mathcal{N}_s \gg \mathcal{N}_{sp} \]

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### For M ~ N/2, # unphysical singular \(\geq\) # highest-weight states
\[ \mathcal{N}_s - \mathcal{N}_{sp} \geq \binom{N}{M} - \binom{N}{M - 1} \]
 Few of the singular solutions are physical \( \mathcal{N}_s \gg \mathcal{N}_{sp} \)

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For \( M \sim N/2 \), \# unphysical singular \( \gg \# \) highest-weight states

\[
\mathcal{N}_s - \mathcal{N}_{sp} \gtrsim \left( \frac{N}{M} \right) - \left( \frac{N}{M-1} \right)
\]

Suggests that \( \mathcal{N} \approx \left( \frac{N}{M} \right) - \left( \frac{N}{M-1} \right) \) is wrong also for \( N \to \infty \)
But there are “proofs” of $\mathcal{N} \equiv \binom{N}{M} - \binom{N}{M-1}$ for $N \to \infty$

[Bethe 31, Kirillov 85, Faddeev 96]
But there are “proofs” of \( \mathcal{N} = \binom{N}{M} - \binom{N}{M-1} \) for \( N \to \infty \)

[Bethe 31, Kirillov 85, Faddeev 96]

Suggests that these proofs are not quite right
Remarkable new conjectures

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6 & (5,0,0; 5) & (9, 1, 1; 9) & & & & & \\
7 & (6,0,0; 6) & (15, 1, 0; 14) & & & & & \\
8 & (7,0,0; 7) & (20, 1, 1; 20) & & & & & \\
9 & (8,0,0; 8) & (28, 1, 0; 27) & & & & & \\
10 & (9,0,0; 9) & (35, 1, 1; 35) & & & & & \\
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13 & (12,0,0; 12) & (66, 1, 0; 65) & & & & & \\
14 & (13,0,0; 13) & (77, 1, 1; 77) & & & & & \\
\hline
\end{tabular}
\end{table}

Example: N and M ≥ 4 both even

\[
\mathcal{N}(N, M) = \binom{N - 1}{M} - \binom{\frac{N-2}{2}}{\frac{M-2}{2}}
\]

\[
\mathcal{N}_s(N, M) = \binom{N - 1}{M} - \binom{N}{M} + \binom{N}{M - 1}
\]

\[
\mathcal{N}_{sp}(N, M) = \binom{\frac{N-2}{2}}{\frac{M-2}{2}}
\]
Remarkable new conjectures

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Example: N and M ≥ 4 both even

\[ N(N, M) = \binom{N - 1}{M} - \binom{N - 2}{M - 2} \]

\[ N_s(N, M) = \binom{N - 1}{M} - \binom{N}{M} + \binom{N}{M - 1} \]

\[ N_{sp}(N, M) = \binom{N - 2}{M - 2} \]

Similar expressions for certain other values of N and M
Remarkable new conjectures

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
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Example: N and M $\geq 4$ both even

$$\mathcal{N}(N, M) = \binom{N - 1}{M} - \binom{\frac{N-2}{2}}{M-2}$$

$$\mathcal{N}_s(N, M) = \binom{N - 1}{M} - \binom{N}{M} + \binom{N}{M - 1}$$

$$\mathcal{N}_{sp}(N, M) = \binom{\frac{N-2}{2}}{M-2}$$

Similar expressions for certain other values of N and M

Based on relation to “rigged configurations”
Integrable spin $s > 1/2$

\[
\text{BE} \quad (\lambda_k + is)^N \prod_{\substack{j \neq k \quad j=1}}^M (\lambda_k - \lambda_j - i) = (\lambda_k - is)^N \prod_{\substack{j \neq k \quad j=1}}^M (\lambda_k - \lambda_j + i),
\]

\[
k = 1, 2, \ldots, M, \quad M = 0, 1, \ldots, sN
\]

[Takhtajan 82; Babujian 83]
Integrable spin $s > 1/2$
Integrable spin $s > 1/2$

\[ (\lambda_k + is)^N \prod_{j \neq k}^{M} (\lambda_k - \lambda_j - i) = (\lambda_k - is)^N \prod_{j \neq k}^{M} (\lambda_k - \lambda_j + i), \]

\[ k = 1, 2, \ldots, M, \quad M = 0, 1, \ldots, sN \]

- singular solutions contain exact string

\[ is, i(s - 1), \ldots, -i(s - 1), -is \]

[Takhtajan 82; Babujian 83]
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exact $(2s+1)$-string centered at origin

[Takhtajan 82; Babujian 83]
Integrable spin $s > 1/2$

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- singular solutions contain exact string

\[ is, i(s - 1), \ldots, -i(s - 1), -is \]

exact $(2s+1)$-string centered at origin

Condition to be **physical**: other roots $\lambda_{2s+2}, \ldots, \lambda_M$ should obey

\[
\left[ (-1)^{2s} \prod_{k=2s+2}^{M} \left( \frac{\lambda_k + is}{\lambda_k - is} \right) \right]^N = 1
\]
• some solutions with repeated roots are physical! “strange”
some solutions with repeated roots are physical! "strange"

\[ s = 1: \quad i, -i, 0, 0, \ldots \]
some solutions with repeated roots are physical! “strange”

$s=1$: $i, -i, 0, 0, \ldots$ repeated!
• some solutions with repeated roots are physical! “strange”

\[ s=1: \quad i, -i, 0, 0, \ldots \]

Condition to be **physical**: other roots \( \lambda_5, \ldots, \lambda_M \) should obey

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\prod_{j=5}^{M} \left( \frac{\lambda_j + 2i}{\lambda_j - 2i} \right) = (-1)^N
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</tbody>
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# of strange solutions
• some solutions with repeated roots are physical! “strange”

\[ s=1: \quad i, -i, 0, 0, \ldots \]

Condition to be physical: other roots \( \lambda_5, \ldots, \lambda_M \) should obey

\[
\prod_{j=5}^{M} \left( \frac{\lambda_j + 2i}{\lambda_j - 2i} \right) = (-1)^N
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</tbody>
</table>

\# of strange solutions

maybe accounts for RSOS structure? [Reshetikhin 1991]
\[ s \otimes \cdots \otimes s = \bigoplus_{N}^{sN} \bigoplus_{S=0}^{n(N, S)} S \]
The equation is:

\[ s \otimes \cdots \otimes s = \bigoplus_{N}^{sN} n(N, S) S \]

multiplicity of spin S rep (known)
spin of Bethe state with $M$ roots: $S = sN - M$

$multiplicity of spin S rep (known)$
spin-s conjecture:

\[ s \otimes \cdots \otimes s = \bigoplus_{S=0}^{sN} n(N, S) S \]

spin of Bethe state with \( M \) roots: \( S = sN - M \)

spin-s conjecture:

\[ \mathcal{N}(N, M) - \mathcal{N}_s(N, M) + \mathcal{N}_{sp}(N, M) + \mathcal{N}_{\text{strange}}(N, M) = n(N, sN - M) \]
spin-s conjecture:

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\]

spin of Bethe state with \(M\) roots: \(S = sN - M\)

spin-s conjecture:

\[
\mathcal{N}(N, M) - \mathcal{N}_s(N, M) + \mathcal{N}_{sp}(N, M) + \mathcal{N}_{\text{strange}}(N, M) = n(N, sN - M)
\]

BE have “too many” solutions. After discarding unphysical singular solutions, there may not remain enough solutions with pairwise distinct roots to account for all states.
\[
\mathbf{s} \otimes \cdots \otimes \mathbf{s} = \bigoplus_{N} n(N, S) \mathbf{S} \quad \text{multiplicity of spin S rep (known)}
\]

spin of Bethe state with \( M \) roots: \( S = sN - M \)

\begin{align*}
\mathcal{N}(N, M) - \mathcal{N}_s(N, M) + \mathcal{N}_{sp}(N, M) + \mathcal{N}_{\text{strange}}(N, M) &= n(N, sN - M)
\end{align*}

BE have “too many” solutions. After discarding unphysical singular solutions, there may not remain enough solutions with pairwise distinct roots to account for all states.

\( s=1 \):

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<td>((8,0,0,0; 8))</td>
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</tbody>
</table>

\[
\begin{pmatrix}
\mathcal{N}, \mathcal{N}_s, \mathcal{N}_{sp}, \mathcal{N}_{\text{strange}}; \mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp} + \mathcal{N}_{\text{strange}}
\end{pmatrix}
\]
Spin of Bethe state with $M$ roots: $S = sN - M$

Spin-s conjecture:

\[
\mathcal{N}(N, M) - \mathcal{N}_s(N, M) + \mathcal{N}_{sp}(N, M) + \mathcal{N}_{\text{strange}}(N, M) = n(N, sN - M)
\]

BE have “too many” solutions. After discarding unphysical singular solutions, there may not remain enough solutions with pairwise distinct roots to account for all states.

$s = 1$:

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<th>$N$</th>
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<td>(7,0,0; 7)</td>
<td>(28,0,0; 28)</td>
<td>(56,0,0; 56)</td>
<td>(84,0,0; 84)</td>
<td>(112,0,0; 112)</td>
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<td>(168,0,0; 168)</td>
<td>(201,0,0; 201)</td>
</tr>
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\[
(N, N_s, N_{sp}, N_{\text{strange}}; N - N_s + N_{sp} + N_{\text{strange}})
\]

Perfect agreement with conjecture!
s=3/2:

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- (non-singular) strange solutions:

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<td>2</td>
<td>$1.5, 1.5$</td>
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<td>6</td>
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<td>$-1.5, -1.5$</td>
</tr>
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Thank you!