Recent Advances
in
Quantum Integrable Systems

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On the completeness of solutions of Bethe's equations

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Introduction

spin-1/2 periodic XXX chain:

$$H = \frac{1}{4} \sum_{n=1}^{N} \left(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - 1 \right) \,, \qquad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1$$

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$$\cancel{\left(\lambda_2+\frac{i}{2}\right)}^N (\lambda_2-\lambda_1-i) = \left(\lambda_2-\frac{i}{2}\right)^N \cancel{(\lambda_2-\lambda_1+i)}$$

an exact solution of BE for any value of N

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exact 2-string
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... but it is not an eigenvector (e.g. for N=4) ???

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[Avdeev, Vladimirov 85;
Beisert, Minahan, Staudacher, Zarembo 03]

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$$F_k(\lambda, \{\lambda\}) = \frac{i}{\lambda - \lambda_k}\left[\prod_{j\neq k}^{M}\left(\frac{\lambda_k - \lambda_j - i}{\lambda_k - \lambda_j}\right) - \left(\frac{\lambda_k - \frac{i}{2}}{\lambda_k + \frac{i}{2}}\right)^N \prod_{j\neq k}^{M}\left(\frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j}\right)\right]$$

$F_k(\lambda, \{\lambda\}) = 0 \;\Leftrightarrow\;$ BE

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"generalized Bethe equations"

$$F_1(\lambda,\{\lambda\}) = \underbrace{\left(\frac{c+2i^{-(N+1)}}{\lambda-\frac{i}{2}}\right)}_{\equiv 0}\epsilon^N + O(\epsilon^{N+1})\,, \quad F_2(\lambda,\{\lambda\}) = \underbrace{\left(\frac{2i-i^{-N}c}{\lambda+\frac{i}{2}}\right)}_{\equiv 0} + O(\epsilon)$$

$\Rightarrow$

N even: $\quad c = 2i(-1)^{N/2}$

N odd: no solution for c !

Although $\pm i/2$ satisfies BE, does NOT correspond to eigenstate of t(λ) !

$$M = 2: \qquad \lambda_1 = \frac{i}{2} + \epsilon + c\,\epsilon^N\,, \qquad \lambda_2 = -\frac{i}{2} + \epsilon\,,$$

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General singular solution:

$$\{\frac{i}{2}\,, -\frac{i}{2}\,, \lambda_3\,, \ldots\,, \lambda_M\} \qquad\qquad \lambda_3\,, \ldots\,, \lambda_M \quad \textbf{distinct} \quad \neq \pm\frac{i}{2}$$

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Regularization + generalized Bethe equations $\Rightarrow$

$$c = -\frac{2}{i^{N+1}} \prod_{j=3}^{M} \frac{\lambda_j - \frac{3i}{2}}{\lambda_j + \frac{i}{2}}\,, \qquad c = 2i^{N+1} \prod_{j=3}^{M} \frac{\lambda_j + \frac{3i}{2}}{\lambda_j - \frac{i}{2}}$$

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Consistency $\Rightarrow$

$$\boxed{\left[-\prod_{k=3}^{M}\left(\frac{\lambda_k+\frac{i}{2}}{\lambda_k-\frac{i}{2}}\right)\right]^{N}=1}$$

Condition for the singular solution to be physical

Completeness/Pauli-principle conjecture

$\mathcal{N}(N, M) \equiv$ # solutions $\{\lambda_1\,, \ldots\,, \lambda_M\}$ of BE with finite pairwise distinct roots

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Bethe states are SU(2) highest-weight states:

$$S^{+}|\lambda_1\,, \ldots\,, \lambda_M\rangle = 0\,, \qquad S^{\pm} = S^{x} \pm i S^{y} \qquad\qquad \vec{S} = \frac{1}{2}\sum_{n=1}^{N} \vec{\sigma}_n$$

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$$M$$

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$$\mathcal{N}(N,M) \stackrel{?}{=} \binom{N}{M} - \binom{N}{M-1}$$

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Our conjecture:

$$\boxed{\mathcal{N}(N,M) - \mathcal{N}_s(N,M) + \mathcal{N}_{sp}(N,M) = \binom{N}{M} - \binom{N}{M-1}}$$

$\mathcal{N}_s(N,M) \equiv$ # singular solutions

$\mathcal{N}_{sp}(N,M) \equiv$ # singular solutions that are physical

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Physical meaning: BE have "too many" solutions. But, after discarding unphysical singular solutions, remain with just right #

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To check conjecture,
must find ALL solutions of BE with pairwise distinct roots.

For N>5, brute force is not an option...

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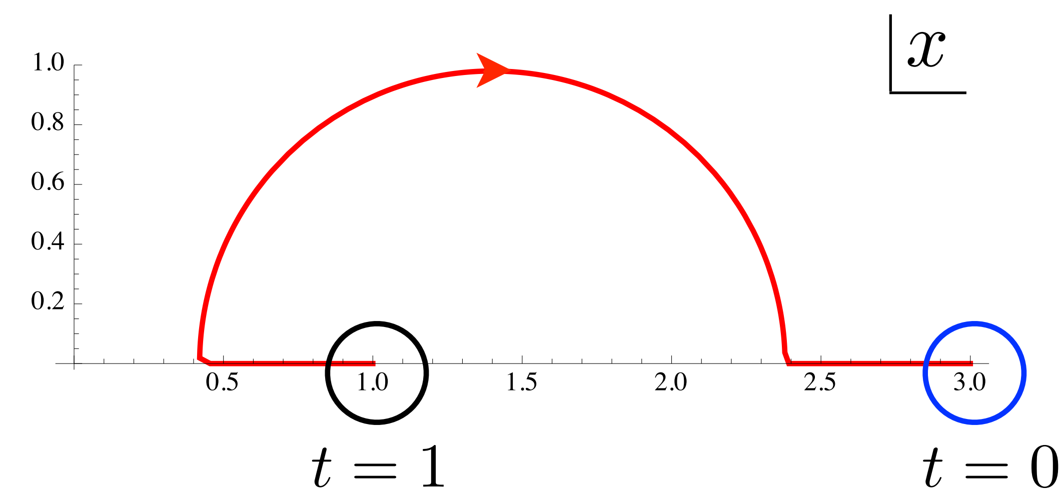

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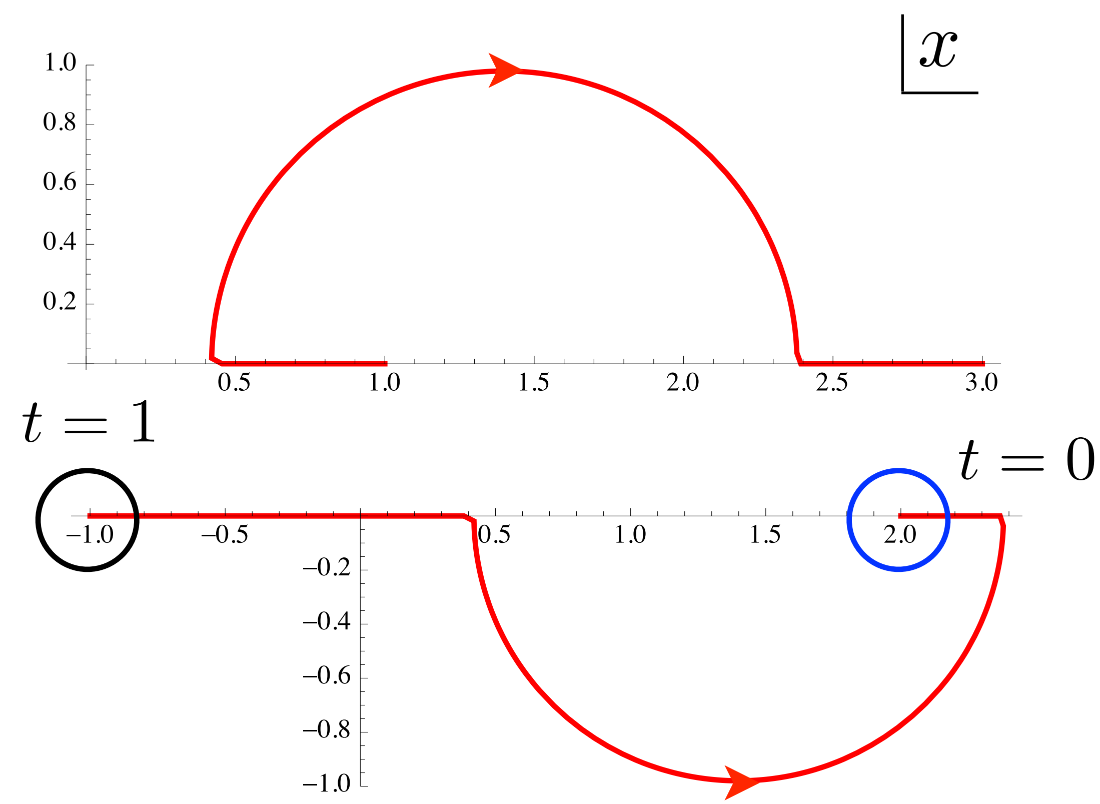

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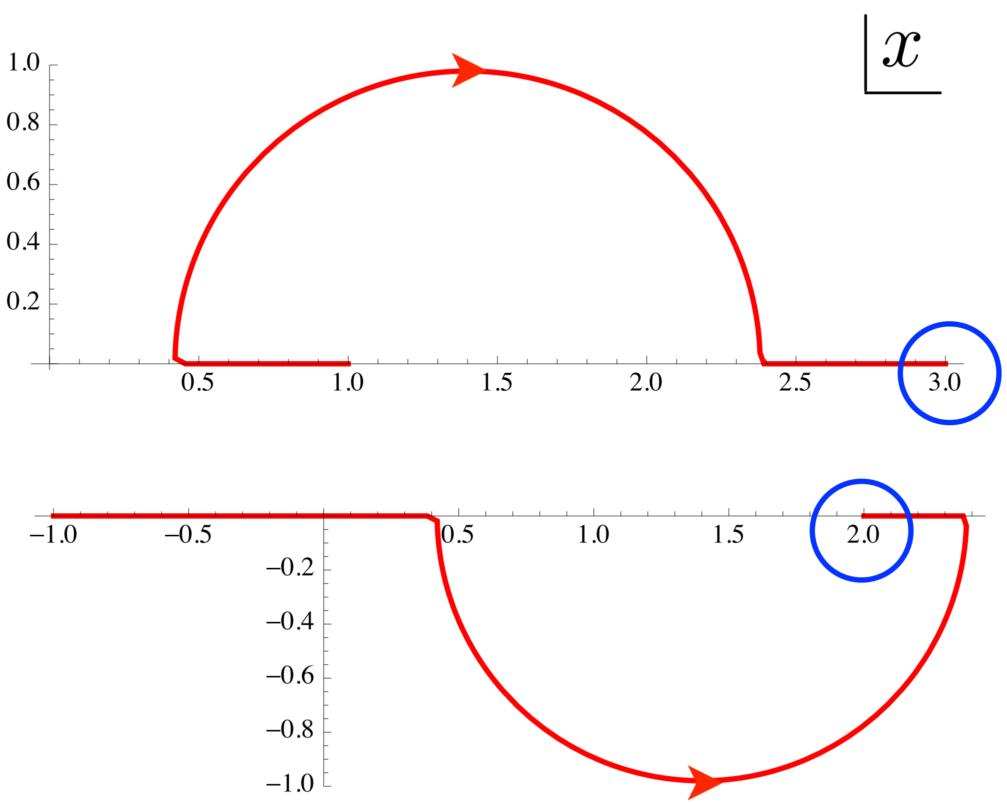
Get $x = 3, 2$



Works very well for systems of polynomial equations

$$\begin{aligned} f_1(\lambda_1, ..., \lambda_M) &= 0 \\ &\vdots \\ f_M(\lambda_1, ..., \lambda_M) &= 0 \end{aligned}$$

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finite solutions is at most

$$D \equiv deg(f_1) \cdot deg(f_2) \cdots deg(f_M)$$

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Want distinct roots, so can restrict

$$0 \le j_1 < j_2 < j_3 < \cdots < j_M \le N+M-3$$

N=8, M=4:

number	Bethe roots $\{\lambda_k\}$		
1	± 0.5250121022236669	± 0.1294729463749287	
2	0.5570702385744416	0.1470126111961413	-0.3520414248852914 ± 0.5005581696433306I
3*	± 0.5I	-0.2930497652740115 ± 0.5002695484553508I	
4*	± 0.5I	0.09053461122303935	0.4866819617430914
5*	± 0.5I	-0.04929340793103601+1.631134975618312I	-0.2430919428911911-0.06188079036780695I
6*	± 0.5I	0.6439488581706157-0.1197616885579488I	0.05986712277687283+1.571716944714331I
7*	± 0.5I	0.04929340793103601+1.631134975618312I	0.2430919428911911-0.06188079036780695I
8**	± 0.5I	± 0.5638252623934961	
9	0.2205600072920844	-0.6691229228815117	0.2242814577947136 ± 1.002247276506607I
10*	± 0.5I	0.1695810016454493	-0.522716443014433
11*	± 0.5I	-0.05986712277687283+1.571716944714331I	-0.6439488581706157-0.1197616885579488I
12*	± 0.5I	1.653144833689466I	-0.050307293346599I
13*	± 0.5I	0.522716443014433	-0.1695810016454493
14*	± 0.5I	0.050307293346599I	-1.653144833689466I
15*	± 0.5I	-0.4866819617430914	-0.09053461122303935
16*	± 0.5I	0.04929340793103601-1.631134975618312I	0.2430919428911911+0.06188079036780695I
17*	± 0.5I	0.2930497652740115 ± 0.5002695484553508I	
18**	± 0.5I	± 0.1424690678305666	
19**	± 0.5I	± 1.556126503577051I	
20*	± 0.5I	3.517084291308099I	1.508105736964082I
21	0.2443331937711654	-0.08378710739142802	-0.08027304318986867 ± 1.005588273959932I
22*	± 0.5I	0.05986712277687283-1.571716944714331I	0.6439488581706157+0.1197616885579488I
23	0.1211861779691729	-0.5716111771864383	0.2252124996086327 ± 0.5000288621635332I
24		± 0.4632647275890309 ± 0.5022938535699026I	
25	-0.1470126111961413	-0.5570702385744416	0.3520414248852914 ± 0.5005581696433306I
26*	± 0.5I	-0.05986712277687283-1.571716944714331I	-0.6439488581706157+0.1197616885579488I
27	0.08378710739142802	-0.2443331937711654	0.08027304318986867 ± 1.005588273959932I
28*	± 0.5I	-0.2430919428911911+0.06188079036780695I	-0.04929340793103601-1.631134975618312I
29	± 1.025705081230743I	± 0.0413091275245562	
30*	± 0.5I	-1.508105736964082I	-3.517084291308099I
31	0.5716111771864383	-0.1211861779691729	-0.2252124996086327 ± 0.5000288621635332I
32	-0.2205600072920844	0.6691229228815117	-0.2242814577947136 ± 1.002247276506607I

* singular unphysical

** singular physical

N=8, M=4:

number	Bethe roots $\{\lambda_k\}$		
1	$\pm$ 0.5250121022236669	$\pm$ 0.1294729463749287	
2	0.5570702385744416	0.1470126111961413	-0.3520414248852914 $\pm$ 0.5005581696433306I
3*	$\pm$ 0.5I	-0.2930497652740115 $\pm$ 0.5002695484553508I	
4*	$\pm$ 0.5I	0.09053461122303935	0.4866819617430914
5*	$\pm$ 0.5I	-0.04929340793103601+1.631134975618312I	-0.2430919428911911-0.06188079036780695I
6*	$\pm$ 0.5I	0.6439488581706157-0.1197616885579488I	0.05986712277687283+1.571716944714331I
7*	$\pm$ 0.5I	0.04929340793103601+1.631134975618312I	0.2430919428911911-0.06188079036780695I
8**	$\pm$ 0.5I	$\pm$ 0.5638252623934961	
9	0.2205600072920844	-0.6691229228815117	0.2242814577947136 $\pm$ 1.002247276506607I
10*	$\pm$ 0.5I	0.1695810016454493	-0.522716443014433
11*	$\pm$ 0.5I	-0.05986712277687283+1.571716944714331I	-0.6439488581706157-0.1197616885579488I
12*	$\pm$ 0.5I	1.653144833689466I	-0.050307293346599I
13*	$\pm$ 0.5I	0.522716443014433	-0.1695810016454493
14*	$\pm$ 0.5I	0.050307293346599I	-1.653144833689466I
15*	$\pm$ 0.5I	-0.4866819617430914	-0.09053461122303935
16*	$\pm$ 0.5I	0.04929340793103601-1.631134975618312I	0.2430919428911911+0.06188079036780695I
17*	$\pm$ 0.5I	0.2930497652740115 $\pm$ 0.5002695484553508I	
18**	$\pm$ 0.5I	$\pm$ 0.1424690678305666	
19**	$\pm$ 0.5I	$\pm$ 1.556126503577051I	
20*	$\pm$ 0.5I	3.517084291308099I	1.508105736964082I
21	0.2443331937711654	-0.08378710739142802	-0.08027304318986867 $\pm$ 1.005588273959932I
22*	$\pm$ 0.5I	0.05986712277687283-1.571716944714331I	0.6439488581706157+0.1197616885579488I
23	0.1211861779691729	-0.5716111771864383	0.2252124996086327 $\pm$ 0.5000288621635332I
24		$\pm$ 0.4632647275890309 $\pm$ 0.5022938535699026I	
25	-0.1470126111961413	-0.5570702385744416	0.3520414248852914 $\pm$ 0.5005581696433306I
26*	$\pm$ 0.5I	-0.05986712277687283-1.571716944714331I	-0.6439488581706157+0.1197616885579488I
27	0.08378710739142802	-0.2443331937711654	0.08027304318986867 $\pm$ 1.005588273959932I
28*	$\pm$ 0.5I	-0.2430919428911911+0.06188079036780695I	-0.04929340793103601-1.631134975618312I
29	$\pm$ 1.025705081230743I	$\pm$ 0.0413091275245562	
30*	$\pm$ 0.5I	-1.508105736964082I	-3.517084291308099I
31	0.5716111771864383	-0.1211861779691729	-0.2252124996086327 $\pm$ 0.5000288621635332I
32	-0.2205600072920844	0.6691229228815117	-0.2242814577947136 $\pm$ 1.002247276506607I

* singular unphysical

** singular physical

Solving the T-Q equation

Solving the T-Q equation

The eigenvalues Λ(λ) of t(λ) are polynomials

$$\Lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j$$

and satisfy

$$\Lambda(\lambda)\, Q(\lambda) = \left(\lambda + \frac{i}{2}\right)^N Q(\lambda - i) + \left(\lambda - \frac{i}{2}\right)^N Q(\lambda + i)$$

$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j\,, \qquad Q_M = 1$$

Solving the T-Q equation

The eigenvalues $\Lambda(\lambda)$ of $t(\lambda)$ are polynomials

$$\Lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j \tag{1}$$

and satisfy

$$\Lambda(\lambda)\, Q(\lambda) = \left(\lambda + \frac{i}{2}\right)^N Q(\lambda - i) + \left(\lambda - \frac{i}{2}\right)^N Q(\lambda + i) \tag{2}$$

$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j \,, \qquad Q_M = 1 \tag{3}$$

Can solve (2) numerically for both $\Lambda(\lambda)$ and $Q(\lambda)$;
then, by finding the zeros of $Q(\lambda)$, obtain all the Bethe roots!

[Baxter 01]

Solving the T-Q equation

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and satisfy

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$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j \,, \qquad Q_M = 1 \qquad (3)$$

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[Baxter 01]

Substitute (1) & (3) into (2);
equate coefficients of equal powers of λ; solve.

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$$\Lambda(\lambda) = \sum_{j=0}^{N} T_j \lambda^j \qquad (1)$$

and satisfy

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$$Q(\lambda) = \prod_{m=1}^{M} (\lambda - \lambda_m) = \sum_{j=0}^{M} Q_j \lambda^j \,, \qquad Q_M = 1 \qquad (3)$$

Can solve (2) numerically for both $\Lambda(\lambda)$ and $Q(\lambda)$;
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[Baxter 01]

Substitute (1) & (3) into (2);
equate coefficients of equal powers of λ; solve.

Up to N=9 on this laptop

Results

M \ N	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}\,, \mathcal{N}_s\,, \mathcal{N}_{sp}\,; \mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp})$$

Results

$N \backslash M$	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}, \mathcal{N}_s, \mathcal{N}_{sp}; \mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp})$$

solutions

Results

M \ N	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
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6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}\,,\mathcal{N}_s\,,\mathcal{N}_{sp}\,;\mathcal{N}-\mathcal{N}_s+\mathcal{N}_{sp})$$

solutions

singular

$$\{\pm\frac{i}{2}\,,\ldots\}$$

Results

$M \backslash N$	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
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6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}\,, \mathcal{N}_s\,, \mathcal{N}_{sp}\,; \mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp})$$

solutions

singular

$$\{\pm\frac{i}{2}\,, \ldots\}$$

singular physical

$$\left[-\prod_{k=3}^{M}\left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}}\right)\right]^N = 1$$

Results

M \ N	1	2	3	$\boxed{4}$	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
$\boxed{8}$	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	$\boxed{(32,\ 21,\ 3;\ 14)}$			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
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13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}\,, \mathcal{N}_s\,, \mathcal{N}_{sp}\,; \mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp})$$

solutions

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$$\{\pm\frac{i}{2}\,, \ldots\}$$

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$$\left[-\prod_{k=3}^{M}\left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}}\right)\right]^N = 1$$

Results

$N \backslash M$	1	2	3	$\boxed{4}$	5	6	7
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12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}\,, \mathcal{N}_s\,, \mathcal{N}_{sp}\,; \underbrace{\mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp}}_{\displaystyle \| \atop \displaystyle \binom{N}{M} - \binom{N}{M-1}})$$

perfect agreement with conjecture!

solutions

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$$\{\pm\frac{i}{2}, \ldots\}$$

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7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
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12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

$$(\mathcal{N}\,, \mathcal{N}_s\,, \mathcal{N}_{sp}\,; \underbrace{\mathcal{N} - \mathcal{N}_s + \mathcal{N}_{sp}}_{||})$$

$$\binom{N}{M} - \binom{N}{M-1}$$

perfect agreement with conjecture!

solutions

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$$\{\pm \frac{i}{2}\,, \ldots\}$$

singular physical

$$\left[- \prod_{k=3}^{M} \left(\frac{\lambda_k + \frac{i}{2}}{\lambda_k - \frac{i}{2}} \right) \right]^N = 1$$

BE have "too many" solutions. But, after discarding unphysical singular solutions, remain with just right #

Remarks:

- Many unphysical singular solutions are not self-conjugate

  $\therefore$ do not obey string hypothesis

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- Many unphysical singular solutions are not self-conjugate

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N=8, M=4

number	Bethe roots $\{\lambda_k\}$		
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3*	$\pm$ 0.5I	-0.2930497652740115 $\pm$ 0.5002695484553508I	
4*	$\pm$ 0.5I	0.09053461122303935	0.4866819617430914
5*	$\pm$ 0.5I	-0.04929340793103601+1.631134975618312I	-0.2430919428911911-0.06188079036780695I
6*	$\pm$ 0.5I	0.6439488581706157-0.1197616885579488I	0.05986712277687283+1.571716944714331
7*	$\pm$ 0.5I	0.04929340793103601+1.631134975618312I	0.2430919428911911-0.06188079036780695I
8**	$\pm$ 0.5I	$\pm$ 0.5638252623934961	
9	0.2205600072920844	-0.6691229228815117	0.2242814577947136 $\pm$ 1.002247276506607I
10*	$\pm$ 0.5I	0.1695810016454493	-0.522716443014433
11*	$\pm$ 0.5I	-0.05986712277687283+1.571716944714331	-0.6439488581706157-0.1197616885579488I
12*	$\pm$ 0.5I	1.653144833689466I	-0.0503072933465991
13*	$\pm$ 0.5I	0.522716443014433	-0.1695810016454493
14*	$\pm$ 0.5I	0.0503072933465991	-1.653144833689466I
15*	$\pm$ 0.5I	-0.4866819617430914	-0.09053461122303935
16*	$\pm$ 0.5I	0.04929340793103601-1.631134975618312I	0.2430919428911911+0.06188079036780695I
17*	$\pm$ 0.5I	0.2930497652740115 $\pm$ 0.5002695484553508I	
18**	$\pm$ 0.5I	$\pm$ 0.1424690678305666	
19**	$\pm$ 0.5I	$\pm$ 1.556126503577051I	
20*	$\pm$ 0.5I	3.517084291308099I	1.508105736964082I
21	0.2443331937711654	-0.08378710739142802	-0.08027304318986867 $\pm$ 1.005588273959932I
22*	$\pm$ 0.5I	0.05986712277687283-1.571716944714331	0.6439488581706157+0.1197616885579488I
23	0.1211861779691729	-0.5716111771864383	0.2252124996086327 $\pm$ 0.5000288621635332I
24		$\pm$ 0.4632647275890309 $\pm$ 0.5022938535699026I	
25	-0.1470126111961413	-0.5570702385744416	0.3520414248852914 $\pm$ 0.5005581696433306I
26*	$\pm$ 0.5I	-0.05986712277687283-1.571716944714331	-0.6439488581706157+0.1197616885579488I
27	0.08378710739142802	-0.2443331937711654	0.08027304318986867 $\pm$ 1.005588273959932I
28*	$\pm$ 0.5I	-0.2430919428911911+0.06188079036780695I	-0.04929340793103601-1.631134975618312I
29	$\pm$ 1.025705081230743I	$\pm$ 0.0413091275245562	
30*	$\pm$ 0.5I	-1.508105736964082I	-3.517084291308099I
31	0.5716111771864383	-0.1211861779691729	-0.2252124996086327 $\pm$ 0.5000288621635332I
32	-0.2205600072920844	0.6691229228815117	-0.2242814577947136 $\pm$ 1.002247276506607I

Remarks:

- Many unphysical singular solutions are not self-conjugate

  ∴ do not obey string hypothesis

N=8, M=4

number	Bethe roots $\{\lambda_k\}$		
1	± 0.5250121022236669	± 0.1294729463749287	
2	0.5570702385744416	0.1470126111961413	-0.3520414248852914 ± 0.5005581696433306I
3*	± 0.5I	-0.2930497652740115 ± 0.5002695484553508I	
4*	± 0.5I	0.09053461122303935	0.4866819617430914
5*	± 0.5I	-0.04929340793103601+1.631134975618312I	-0.2430919428911911-0.06188079036780695I
6*	± 0.5I	0.6439488581706157-0.1197616885579488I	0.05986712277687283+1.571716944714331
7*	± 0.5I	0.04929340793103601+1.631134975618312I	0.2430919428911911-0.06188079036780695I
8**	± 0.5I	± 0.5638252623934961	
9	0.2205600072920844	-0.6691229228815117	0.2242814577947136 ± 1.002247276506607I
10*	± 0.5I	0.1695810016454493	-0.522716443014433
11*	± 0.5I	-0.05986712277687283+1.571716944714331	-0.6439488581706157-0.1197616885579488I
12*	± 0.5I	1.653144833689466I	-0.050307293346599I
13*	± 0.5I	0.522716443014433	-0.1695810016454493
14*	± 0.5I	0.050307293346599I	-1.653144833689466I
15*	± 0.5I	-0.4866819617430914	-0.09053461122303935
16*	± 0.5I	0.04929340793103601-1.631134975618312I	0.2430919428911911+0.06188079036780695I
17*	± 0.5I	0.2930497652740115 ± 0.5002695484553508I	
18**	± 0.5I	± 0.1424690678305666	
19**	± 0.5I	± 1.556126503577051I	
20*	± 0.5I	3.517084291308099I	1.508105736964082I
21	0.2443331937711654	-0.08378710739142802	-0.08027304318986867 ± 1.005588273959932I
22*	± 0.5I	0.05986712277687283-1.571716944714331	0.6439488581706157+0.1197616885579488I
23	0.1211861779691729	-0.5716111771864383	0.2252124996086327 ± 0.5000288621635332I
24		± 0.4632647275890309 ± 0.5022938535699026I	
25	-0.1470126111961413	-0.5570702385744416	0.3520414248852914 ± 0.5005581696433306I
26*	± 0.5I	-0.05986712277687283-1.571716944714331	-0.6439488581706157+0.1197616885579488I
27	0.08378710739142802	-0.2443331937711654	0.08027304318986867 ± 1.005588273959932I
28*	± 0.5I	-0.2430919428911911+0.06188079036780695I	-0.04929340793103601-1.631134975618312I
29	± 1.025705081230743I	± 0.0413091275245562	
30*	± 0.5I	-1.508105736964082I	-3.517084291308099I
31	0.5716111771864383	-0.1211861779691729	-0.2252124996086327 ± 0.5000288621635332I
32	-0.2205600072920844	0.6691229228815117	-0.2242814577947136 ± 1.002247276506607I

OK, since only solutions corresponding to eigenstates need to be self-conjugate

[Vladimirov 86]

- For odd N, most singular solutions are unphysical

- For odd N, most singular solutions are unphysical

N \ M	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
$\boxed{5}$	(4,0,0; 4)	(6, 1, $\boxed{0}$; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
$\boxed{7}$	(6,0,0; 6)	(15, 1, $\boxed{0}$; 14)	(20, 6, $\boxed{0}$; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
$\boxed{9}$	(8,0,0; 8)	(28, 1, $\boxed{0}$; 27)	(54, 8, 2; 48)	(69, 27, $\boxed{0}$; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
$\boxed{11}$	(10,0,0; 10)	(45, 1, $\boxed{0}$; 44)	(120, 10, $\boxed{0}$; 110)	(209, 44, $\boxed{0}$; 165)	(252, 120, $\boxed{0}$; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
$\boxed{13}$	(12, 0, 0; 12)	(66, 1, $\boxed{0}$; 65)	(220, 12, $\boxed{0}$; 208)	(494, 65, $\boxed{0}$; 429)	(792, 220, $\boxed{0}$; 572)	(919, 490, $\boxed{0}$, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

- For odd N, most singular solutions are unphysical

N \ M	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

noted earlier: $\pm i/2$ is unphysical for ALL odd N

- For odd N, most singular solutions are unphysical

M \ N	1	2	3	4	5	6	7
2	(1,0,0; 1)						
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4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

noted earlier: $\pm i/2$ is unphysical for ALL odd N

Exception N=9, M=3: there are 2 singular physical solutions

- For odd N, most singular solutions are unphysical

N \ M	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

noted earlier: $\pm i/2$ is unphysical for ALL odd N

Exception N=9, M=3: there are 2 singular physical solutions

periodic: N = 9+6n, n=0,1,2,...

Expect other exceptions for higher M

- Few of the singular solutions are physical $\mathcal{N}_s \gg \mathcal{N}_{sp}$

$N \backslash M$	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

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$N \backslash M$	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

- For M ~ N/2, # unphysical singular ≳ # highest-weight states

$$\mathcal{N}_s - \mathcal{N}_{sp} \gtrsim \binom{N}{M} - \binom{N}{M-1}$$

$N \backslash M$	1	2	3	4	5	6	7
2	(1,0,0; 1)						
3	(2,0,0; 2)						
4	(3,0,0; 3)	(2, 1, 1; 2)					
5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
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9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

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M \ N	1	2	3	4	5	6	7
2	(1,0,0; 1)						
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6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
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12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

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5	(4,0,0; 4)	(6, 1, 0; 5)					
6	(5,0,0; 5)	(9, 1, 1; 9)	(9, 5, 1; 5)				
7	(6,0,0; 6)	(15, 1, 0; 14)	(20, 6, 0; 14)				
8	(7,0,0; 7)	(20, 1, 1; 20)	(34, 7, 1; 28)	(32, 21, 3; 14)			
9	(8,0,0; 8)	(28, 1, 0; 27)	(54, 8, 2; 48)	(69, 27, 0; 42)			
10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

Suggests that $\mathcal{N} \stackrel{?}{=} \binom{N}{M} - \binom{N}{M-1}$ is wrong also for N→∞

But there are “proofs” of $\mathcal{N} \stackrel{?}{=} \binom{N}{M} - \binom{N}{M-1}$ for N$\longrightarrow\infty$

[Bethe 31, Kirillov 85, Faddeev 96]

But there are “proofs” of $\mathcal{N} \stackrel{?}{=} \binom{N}{M} - \binom{N}{M-1}$ for N$\longrightarrow\infty$

[Bethe 31, Kirillov 85, Faddeev 96]

Suggests that these proofs are not quite right

• Remarkable new conjectures

[Kirillov & Sakamoto 14]

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10	(9,0,0; 9)	(35, 1, 1; 35)	(83, 9, 1; 75)	(122, 36, 4; 90)	(122, 84, 4; 42)		
11	(10,0,0; 10)	(45, 1, 0; 44)	(120, 10, 0; 110)	(209, 44, 0; 165)	(252, 120, 0; 132)		
12	(11,0,0; 11)	(54, 1, 1; 54)	(163, 10, 1; 154)	(325, 55, 5; 275)	(456, 163, 4; 297)	(452, 330, 10; 132)	
13	(12, 0, 0; 12)	(66, 1, 0; 65)	(220, 12, 0; 208)	(494, 65, 0; 429)	(792, 220, 0; 572)	(919, 490, 0, 429)	
14	(13, 0, 0; 13)	(77, 1, 1; 77)	(285, 13, 1; 273)	(709, 78, 6; 637)	(1281, 286, 6; 1001)	(1701, 715, 15; 1001)	(1701, 1287, 15; 429)

Example: N and M $\geq$ 4 both even

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Similar expressions for certain other values of N and M

Based on relation to “rigged configurations” [Kirillov & Reshetikhin 86]

Integrable spin s > 1/2

[Takhtajan 82; Babujian 83]

BE

$$(\lambda_k + is)^N \prod_{\substack{j=1 \\ j\neq k}}^{M} (\lambda_k - \lambda_j - i) = (\lambda_k - is)^N \prod_{\substack{j=1 \\ j\neq k}}^{M} (\lambda_k - \lambda_j + i)\,,$$

$$k = 1\,,2\,,\ldots\,,M\,, \qquad M = 0\,,1\,,\ldots\,,sN$$

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Condition to be physical: other roots $\lambda_{2s+2}\,,\ldots\,,\lambda_M$ should obey

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7	(6,0,0,0; 6)	(21,0,0,0; 21)	(49,1,1,0; 49)	(90,6,0,0; 84)	(123,21,3,0; 105)	(141,50,0,0; 91)	(120,90,6,0; 36)	
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maybe accounts for RSOS structure ? [Reshetikhin 1991]

$$\underbrace{\mathbf{s} \otimes \cdots \otimes \mathbf{s}}_{N} = \bigoplus_{S=0}^{sN} n(N,S)\,\mathbf{S}$$

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6	2	1.5 , 1.5
6	2	-1.5 , -1.5

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Questions

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Thank you!