Separation of variables approach for the exact solution of integrable quantum models[a]: the open XXZ spin chain

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a. Subject related to past and current collaborations with J. Teschner (DESY-Hamburg), N. Grosjean (LPTM-Cery), J.-M. Maillet (ENS-Lyon), S. Faldella (IMB-Dijon), N. Kitanine (IMB-Dijon), D. Levy-Bencheton (IMB-Dijon), V. Terras (LPTMS-Orsay), B. McCoy (YITP-New York).

Main aims:

- To solve exactly lattice integrable quantum models by quantum separation of variables (SOV) characterizing both their spectrum and dynamics (time dependent correlation functions).
- To define a microscopic approach to solve exactly 1+1 dimensional quantum field theories (QFT) by using the SOV solution of their integrable lattice regularizations.

Plan of the seminar:

- Recall of the known results on XXZ spin chain and remark on the existing problems.
- Introduction to our SOV approach:
  - Quantum[1,2] integrability, separation of variables and inverse scattering method.
  - Use of the Hydrogen atom Hamiltonian to present some ideas on SOV.
- Implementation of the quantum separation of variables:
  - SOV for the open XXZ spin chain with general boundary magnetic fields.
  - Completeness of spectrum description by SOV.
  - Equivalent formulation in terms of Baxter like functional equations.
  - First results toward the complete characterization of the model dynamics.
  - Universal characterization of spectrum&dynamics of integrable quantum models by SOV

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[1] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194.
[2] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196.

The spectrum and dynamics of several fundamental integrable quantum models has been analyzed by me and my collaborators developing the SOV approach in the following papers:

G.N., J.Teschner, *The Sine-Gordon model revisited I*, **J. Stat. Mech. 1009 : P09014, 2010**

G.N., *Reconstruction of Baxter Q-operator from Sklyanin SOV for cyclic representations of integrable quantum models*, **Nucl. Phys. B835: 263, 2010**

G.N., *Completeness of Bethe Ansatz by Sklyanin SOV for Cyclic Representations of Integrable Quantum Models*, **JHEP 1103:123,2011**

N.Grosjean, J.-M.Maillet, G.N., *On the form factors of local operators in the lattice sine-Gordon model*,**J.Stat.Mech. (2012) P10006**

N.Grosjean, G.N., *The τ_2-model and the chiral Potts model revisited: completeness of Bethe equations from Sklyanin's SOV method*, **J. Stat. Mech. (2012) P11005**

Since May 2012 on the archive: G.N., *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors*, **Nucl.Phys.B 870: 397 (2013);** *Form factors and complete spectrum of XXX antiperiodic higher spin chains by quantum separation of variables*, **J.Math.Phys. 54, 053516 (2013);** *Antiperiodic dynamical 6-vertex model I: Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic 8-vertex model*, **J. Phys. A: Math. Theor. 46 075003, 2013;** *On the developments of Sklyanin's quantum separation of variables for integrable quantum field theories*, **invited contribution to the Proceedings of the XVIIth INTERNATIONAL CONGRESS ON MATHEMATICAL PHYSICS, August 2012, Aalborg, Danemark**

G.N. *Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and matrix elements of some quasi-local operators*, **J. Stat. Mech. (2012) P10025**

S.Faldella, N.Kitanine, G.N., *Complete spectrum and scalar products for the open spin-1/2 XXZ quantum chains with non-diagonal boundary terms*, **J. Stat. Mech. (2014) P01011.**

S.Faldella, G.N., *SOV approach for integrable quantum models associated to general representations on spin-1/2 chains of the 8-vertex reflection algebra*, **J. Phys. A: Math. Theor. 47 (2014) 115202.**

N.Grosjean, J.-M.Maillet, G.N., *On form factors of local operators in the Bazhanov-Stroganov model and the chiral Potts model*, **accepted for publication at Annales Henri Poincare'.**

N Kitanine, JM Maillet and G Niccoli, *Open spin chains with generic integrable boundaries: Baxter equation and Bethe ansatz completeness from SOV*, **J. Stat. Mech. (2014) P05015.**

• The statement of universality has been verified by implementing this SOV method for several other fundamental models, like the closed XXZ quantum spin chains with the antiperiodic boundary conditions and the open XXZ and XYZ ones with the most general integrable boundary conditions (describing also systems out of equilibrium like PASEP), the dynamical SOS models with antiperiodic boundaries and the 8-vertex models as well as the 6-vertex models in the most general cyclic representations (Bazhanov-Stroganov-model) and the chiral Potts model (central model in statistical mechanics).
The following important universal features appear:

- the eigenvalue and eigenstates of the Hamiltonian of the model are completely characterized by classifying all the solutions (in a model dependent class of functions) to a given set of differences equations (Baxter's second order difference equations for models associated to $U_q(\hat{sl}(2))$ quantum groups in the spectrum of the separate variables),
- the scalar products for eigenstates have the simple form of determinants and the form factors of local operators (which are the basic building blocks to write any correlation function) admit determinant representation given by simple modifications of the scalar product formula.

It is the relative mathematical simplicity with which one can compute exactly the spectrum and dynamics of quantum many-body systems which cannot be solved by other quantum integrable methods and the universality in the representations of the results which make me hope that the SOV method that I am contributing to develop will allow the solutions of advanced models and will become more and more central in the community of quantum integrability.

Recall on the XXZ chain: Spectrum

I) Closed periodic chain:

• The XXZ Heisenberg chain for the quantum description of magnetism:

$$H = \sum_{m=1}^{M} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\,(\sigma_m^z \sigma_{m+1}^z - 1)\right) - \frac{h}{2} \sum_{m=1}^{M} \sigma_m^z,$$

periodic boundary conditions: $\sigma_{M+1} = \sigma_1$, Δ - anisotropy, h- external magnetic field.

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $dim\mathcal{H} = 2^M$. $\sigma_m^{x,y,z}$ local spin operators acting as Pauli matrices in the $\mathcal{H}_m$.

First exact results on the spectrum by Bethe (1931) for the XXX chain, Lieb & Liniger (1963), Yang&Yang (1966), Baxter (1972) for the XYZ chain, Faddeev, Sklyanin, Takhtadjan (1979).

II) Open chain with special boundary magnetic fields (simple case):

• Open XXZ quantum spin chain with z-oriented ($h_\pm$) bounday magnetic fields:

$$H = \sum_{m=1}^{M-1} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\,(\sigma_m^z \sigma_{m+1}^z - 1)\right) - h_- \sigma_1^z - h_+ \sigma_M^z.$$

Scalar diagonal solution to reflection equation introduced by Cherednik (1984). First analysis of the ground state in Bethe Ansatz framework by Alcaraz et al. (1987). Quantum Inverse Scattering and Algebraic Bethe Ansatz formulation introduced by Sklyanin (1988).

Recall on the XXZ chain: Quantum dynamics (correlation functions)

Free fermion $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa, Zamolodchikov ...

First attempts using Bethe ansatz for general anisotropy since 1984 : Izergin, Korepin,...

- Q-deformed KZ equations: (Zero External Magnetic Field)

  $\hookrightarrow$ **Massive case, Infinite Chain: 1992 Jimbo, Miki, Miwa and Nakayashiki**

  $\hookrightarrow$ **Massive case, Half-Infinite Chain: 1995 Jimbo, Kedem, Kojima, Konno, and Miwa**

  $\hookrightarrow$ **Conjectured for Massless case, Infinite Chain: 1996 Jimbo and Miwa**

- In the framework of algebraic Bethe ansatz (ABA)

  $\hookrightarrow$ **Periodic Chain: Zero&Non-Zero External Magnetic Field, Massive&Massless.**

  ***Lyon Group*: 1999 Kitanine, Maillet, Terras**

  $\hookrightarrow$ **Open diagonal chain: Zero&Non-Zero External Magnetic Field, Massive&Massless.**

  ***Lyon Group*: 2007 Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras.**

- Quantum Transfer Matrix approach for the periodic chain at non-zero temperature

  $\hookrightarrow$ ***Wuppertal Group*: 2004 Gohmann, Hasenclever, Klumper, Seel, Boos, Suzuki.**

Several other important developments on the quantum dynamics by several authors, some of them attending this conference.

Recall on the XXZ chain: Spectrum

III) Open chain with general boundary magnetic fields (complicate case):

The Hamiltonian of the open chain with most general integrable boundary magnetic fields:

$$H = \sum_{i=1}^{\mathsf{N}-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh\eta \sigma_i^z \sigma_{i+1}^z) + (\frac{\sigma_1^z \cosh\zeta_- + 2\kappa_-(\sigma_1^x \cosh\tau_- + i\sigma_1^y \sinh\tau_-)}{\sinh\zeta_-}$$

$$+ \frac{\sigma_1^z \cosh\zeta_+ + 2\kappa_+(\sigma_1^x \cosh\tau_+ + i\sigma_1^y \sinh\tau_+)}{\sinh\zeta_+}) \sinh\eta$$

IIIa) Boundary terms satisfy one constraint:

• First results on the spectrum by fusion of transfer matrices by Nepomechie (2001). First numerical verification of completeness of the spectrum description by Nepomechie & Ravanini (2003).

• First construction of eigenstates generalized ABA by using Baxter's like gauge transformations by Cao, Lin, Shi, Wang (2002).

• Coordinate Bethe ansatz: N. Crampé, E. Ragoucy, D. Simon (2010-2011).

IIIb) No constrains are imposed (completely general boundary magnetic fields):

Generalized ABA does not apply several other approaches for the spectrum:

• Eigenvalues&eigenstates characterization by q-Onsager algebra Baseilhac&Koizumi (2005).

• Eigenvalues characterization for the anisotropy parameter at roots of units by Murgan, Nepomechie, Shi (2006).

• Quantum separation of variables[3] for the open XXZ chain with non z-oriented boundary magnetic fields by N. (2012) in the case of one triangular K-matrix[4], Faldella, Kitanine, N. (2013), Kitanine, Maillet, N. (2014) for completely general K-matrices.

• Eigenvalues analysis by an ansatz on T-Q functional equations of inhomogeneous type by Li, Cao, Yang, Shi, Wang (2013). First numerical verification of the completeness for the open XXX chain by Nepomechie (2013) and a generalized ABA construction of Bethe states for short open XXX quantum chain in this T-Q functional equations inhomogeneous framework by Belliard, Crampe' (2013).

• Further other methods: W. Galleas (2007), V. Pasquier (2014) ...

• SOV seems to solve problems which appear in others approaches:

– On the spectrum: pure functional methods based on fusion of transfer matrices and T-Q functional equations do not allow to construct eigenstates and to distinguish admissible and inadmissible Bethe equation solutions. Problem of completeness in Bethe ansatz approaches.

– On the dynamics: a simple scalar product formula of determinant form is missing in generalized ABA framework as soon as we have non z-oriented boundary magnetic fields.

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[3] **The eigenvalues problem in the open XXX chain have been previously analyzed in the functional version of the SOV by Frahm, Grelik, Seel, Wirth (2008).**

[4] **For two upper triangular K-matrices a generalized algebraic Bethe ansatz by Belliard, Ragoucy, Crampe' (2012) for XXX spin chain and Pimenta, A. Lima-Santos (2013) for XXZ spin chain.**

- **Quantum integrability:** (H Hamiltonian, $\mathcal{H}$ quantum space of model)
  $\exists\ \mathsf{T}(\lambda) \in \mathsf{End}(\mathcal{H})$: i) $[\mathsf{T}(\lambda), \mathsf{T}(\lambda')]=0 \quad \forall \lambda, \lambda'$, ii) $[\mathsf{T}(\lambda), H]=0 \quad \forall \lambda \in \mathbb{C}$,

Quantum separation of variables (SOV): introduction

- Let $\mathsf{Y}_n \in \mathsf{End}(\mathcal{H})$ and $P_n \in \mathsf{End}(\mathcal{H})$ be N couples of canonical conjugate operators:

$$[\mathsf{Y}_n, \mathsf{Y}_m] = [P_n, P_m] = 0, \qquad [\mathsf{Y}_n, P_m] = \delta_{n,m}/2\pi i \;\; \forall (n,m) \in \{1, .., \mathsf{N}\}^2,$$

where $\{\mathsf{Y}_1, \ldots, \mathsf{Y}_\mathsf{N}\}$ are simultaneous diagonalizable operators with simple spectrum.

Definition: The Y_n define the quantum separate variables for the spectral problem of $\mathsf{T}(\lambda)$ if and only if $\exists$N separate relations of the form:

$$F_n(\mathsf{Y}_n, P_n, \mathsf{T}(\mathsf{Y}_n)) = 0 \qquad \forall n \in \{1, .., \mathsf{N}\};$$

quantum analogue of the classical ones in the Hamilton-Jacobi's approach.

- The separate relations are used to solve the spectral problem of $\mathsf{T}(\lambda)$:

$$F_n(y_n, \frac{i}{2\pi}\frac{\partial}{\partial y_n}, t(y_n))\Psi_t(y_1, ..., y_\mathsf{N}) = 0, \quad \text{with } \Psi_t(y_1, ..., y_\mathsf{N}) = \langle y_1, ..., y_\mathsf{N}|t\rangle,$$

where y_n, $t(\lambda)$ and $\langle y_1, ..., y_\mathsf{N}|$, $|t\rangle$ are eigenvalues and eigenstates of Y_n and $\mathsf{T}(\lambda)$, respectively, then:

$$|t\rangle = \sum_{\text{Over spectrum of } \{\mathsf{Y}_n\}} \Psi_t(y_1, ..., y_\mathsf{N})|y_1, ..., y_\mathsf{N}\rangle$$

$$\mathsf{SOV} \Longrightarrow \Psi_t(y_1, ..., y_\mathsf{N}) = \prod_{n=1}^{\mathsf{N}} \mathsf{Q}_t^{(n)}(y_n),$$

where $Q_t^{(n)}(\lambda)$ is a solution of the separate equation in y_n.

Quantum description for the Hydrogen atom: integrability and separate variables

By choosing spherical coordinates in the position representation the Hamiltonian reads:

$$\langle r, \theta, \varphi | H = [-(\partial^2/\partial r^2)(\hbar^2 r/2m) + \mathbf{L}^2/2mr^2 - e^2/r]\langle r, \theta, \varphi|,$$

where $\mathbf{L}$ is the angular momentum, vector differential operator in θ and φ only.

This quantum system is integrable as $H_3 = H$, $H_2 = \mathbf{L}^2$ and $H_1 = \mathbf{L}_z$ form a C.S.C.O. of conserved charges, $[H, \mathbf{L}^2] = [H, \mathbf{L}_z] = [\mathbf{L}^2, \mathbf{L}_z] = 0$ and the separate relations read:

$$F_n(y_n, \frac{i}{2\pi}\frac{\partial}{\partial y_n}, h_3(k,l), h_2(l), h_1(m))\Psi_{k,l,m}(r, \theta, \varphi) = 0, \quad y_3 = r, y_2 = \theta, y_1 = \varphi.$$

where $h_3(k,l) = E_I/(k+l)^2$, $h_2(l) = l(l+1)\hbar^2$, $h_1(m) = m\hbar$ and:

$$F_3(r, \frac{i}{2\pi}\frac{\partial}{\partial r}, h_3(k,l), h_2(l)) \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}r + \frac{h_2(l)}{2mr^2} - \frac{e^2}{r} - h_3(k,l),$$

$$F_2(\theta, \frac{i}{2\pi}\frac{\partial}{\partial \theta}, h_2(l), h_1(m)) \equiv -\hbar^2\frac{\partial^2}{\partial \theta^2} - \frac{\hbar^2}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{h_1^2(m)}{\sin^2\theta} - h_2(l),$$

$$F_1(\varphi, \frac{i}{2\pi}\frac{\partial}{\partial \varphi}, h_1(m)) \equiv -i\hbar\frac{\partial}{\partial\varphi} - h_1(m),$$

and the wavefunctions are separated in the eigenvalues of the separate operators (r, θ and φ):

$$\Psi_{k,l,m}(r, \theta, \varphi) \equiv \langle r, \theta, \varphi | \Psi_{k,l,m}\rangle = R_{k,l}(r)Y_l^m(\theta, \varphi) \text{ with } Y_l^m(\theta, \varphi) = F_l^m(\theta)e^{im\varphi},$$

where $Y_{l,m}(\theta, \varphi)$ are the spherical harmonic functions.

- **Characterization of integrability by quantum inverse scattering method (QISM)**

Integrable quantum model with Hamiltonian $H \in \text{End}(\mathcal{H})$ and quantum space $\mathcal{H} \equiv \otimes_{n=1}^{\mathsf{N}} \mathcal{H}_{\mathsf{n}}$ (tensor product of local quantum spaces $\mathcal{H}_{\mathsf{n}}$) is characterized by the monodromy matrix $\mathsf{M}_a(\lambda) \in \text{End}(\mathbb{C}^M \otimes \mathcal{H})$, whose matrix elements are operators on $\mathcal{H}$:

$$\boxed{\begin{array}{c} R_{ab}(\lambda-\mu)\, \mathsf{M}_a(\lambda)\mathsf{M}_b(\mu) = \mathsf{M}_b(\mu)\mathsf{M}_a(\lambda)\, R_{ab}(\lambda-\mu) \\ \text{Yang-Baxter equation} \\ \\ \underbrace{\mathsf{T}(\lambda) = \text{Tr}\, \mathsf{M}(\lambda)}_{\text{Transfer matrix}},\ \ [H, \mathsf{T}(\lambda)] = [\mathsf{T}(\mu), \mathsf{T}(\lambda)] = 0. \end{array}}$$

The elements $(\mathsf{M}_a(\lambda))_{i,j} \in \text{End}(\mathcal{H})$ are generators of the Yang-Baxter algebra and Yang-Baxter algebra representation theory is associated to that of quantum (super)groups $U_q(sl(n,m))$, with $n+m=M$.

In the case $M=2$, $\mathsf{M}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$ with$A(\lambda), B(\lambda), C(\lambda)$ and$D(\lambda) \in \text{End}(\mathcal{H})$, a part the trace the determinant is the other spectral invariant and its quantum analog:

$$\det_q \mathsf{M}(\lambda) = A(\lambda)\, D(\lambda-\eta) - B(\lambda)\, C(\lambda-\eta)$$

is a central element of the Yang-Baxter algebra associated to $U_q(sl(2))$, $q \equiv e^\eta$.

- **Sklyanin's approach for SOV characterization** If $B(\lambda)$ is diagonalizable and with simple spectrum then the quantum separate variables Y_n for the spectral problem of the transfer matrix $\mathsf{T}(\lambda)$ are defined by the zeros operators of $B(\lambda)$.

Introduction to SOV & SOV characterization of T-spectrum: "Yang-Baxter case"

- SOV-representation is defined in the basis formed by the B-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_\mathsf{N}|\}$ parametrized by the zeros of $B(\lambda)$:

$$\langle \mathbf{y}|\, B(\lambda) = b_\mathbf{y}(\lambda)\, \langle \mathbf{y}|\, , \qquad b_\mathbf{y}(\lambda) \equiv b_0 \prod_{n=1}^{\mathsf{N}} \sinh\left(\lambda - y_n\right) \, .$$

By the Yang-Baxter commutations relations, it holds:

$$\langle y_1, \ldots, y_k, \ldots, y_N|A(y_k) = a(y_k)\langle y_1, \ldots, y_k - \eta, \ldots, y_\mathsf{N}|,$$

$$\langle y_1, \ldots, y_k, \ldots, y_N|D(y_k) = d(y_k)\langle y_1, \ldots, y_k + \eta, \ldots, y_\mathsf{N}|,$$

$$d(\lambda - \eta)a(\lambda) = \det{}_q M(\lambda).$$

- Eigenvalues $t(\lambda)$ and wave functions $\Psi_t(y_1, \ldots., y_\mathsf{N}) \equiv \langle y_1, \ldots, y_\mathsf{N}|t\rangle$ are characterized by:

$$a(y_k)\Psi_t(y_1, .., y_k - \eta, .., y_\mathsf{N}) + d(y_k)\Psi_t(y_1, .., y_k + \eta, .., y_\mathsf{N}) = t(y_k)\Psi_t(y_1, .., y_\mathsf{N}).$$

These equations follow by computing the matrix elements $\langle y_1, \ldots., y_\mathsf{N}|\mathsf{T}(y_k)|t\rangle$ and lead to the factorized form:

$$\Psi_t(y_1, \ldots., y_\mathsf{N}) = \prod_{j=1}^{\mathsf{N}} Q_t(y_j),$$

where $Q_t(\lambda)$ is a solution of the Baxter equation.

- **Quantum inverse scattering formulation of open integrable quantum models:**

The boundary monodromy matrix $\mathcal{U}_{-}(\lambda)$ satisfy the following Reflection algebra:

$$R_{12}(\lambda-\mu)\,\mathcal{U}_{-,1}(\lambda)\,R_{12}(\lambda+\mu)\,\mathcal{U}_{-,2}(\mu)=\mathcal{U}_{-,2}(\mu)\,R_{12}(\lambda+\mu)\,\mathcal{U}_{-,1}(\lambda)\,R_{12}(\lambda-\mu),$$

where R is for example the 6-vertex trigonometric R-matrix and $\mathcal{U}_{-}(\lambda)$ is defined by:

$$\mathcal{U}_{-}(\lambda)=M_0(\lambda)K_{-}(\lambda)\hat{M}_0(\lambda)=\left(\begin{array}{cc}\mathcal{A}_{-}(\lambda) & \mathcal{B}_{-}(\lambda)\\ \mathcal{C}_{-}(\lambda) & \mathcal{D}_{-}(\lambda)\end{array}\right),\ \hat{M}(\lambda)=(-1)^{\mathsf{N}}\,\sigma_0^y\,M^{t_0}(-\lambda)\,\sigma_0^y$$

where $M_0(\lambda)$ is a solution to the Yang-Baxter equation and

$$K_{\pm}(\lambda)\equiv\left(\begin{array}{cc}a_{\pm}(\lambda) & b_{\pm}(\lambda)\\ c_{\pm}(\lambda) & d_{\pm}(\lambda)\end{array}\right)\equiv\frac{\left(\begin{array}{cc}\sinh(\lambda+\zeta_{\pm}\pm\eta/2) & \kappa_{\pm}e^{\tau_{\pm}}\sinh(2\lambda\pm\eta)\\ \kappa_{\pm}e^{-\tau_{\pm}}\sinh(2\lambda\pm\eta) & \sinh(\zeta_{\pm}\mp\eta/2-\lambda)\end{array}\right)}{\sinh\zeta_{\pm}},$$

then the transfer matrix:

$$\mathsf{T}(\lambda)\equiv\mathrm{tr}_0\{K_{+}(\lambda)\mathcal{U}_{-}(\lambda)\}=a_{+}(\lambda)\,\mathcal{A}_{-}(\lambda)+d_{+}(\lambda)\,\mathcal{D}_{-}(\lambda)+c_{+}(\lambda)\,\mathcal{B}_{-}(\lambda)+b_{+}(\lambda)\,\mathcal{C}_{-}(\lambda)$$

defines a one parameter family of conserved charges for a class of the integrable quantum models. The open XXZ quantum spin chains are examples of integrable quantum model in this class. I have introduced the SOV for open XXZ quantum spin chains (arXiv.org 07/2012) for this triangular conditions. Here, the separate variables are defined by $\mathcal{B}_{-}(\lambda)$, the general case for XXZ and XYZ has been developed in collaboration with Faldella, Kitanine & Maillet in the 2013-2014.

Introduction to SOV & SOV characterization of T-spectrum: "Reflection algebra case"

- SOV-representation is defined in the basis formed by the B-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_{\mathsf{N}}|\}$ parametrized by the zeros of $\mathcal{B}_-((\lambda)$:

$$\langle \mathbf{y}|\, \mathcal{B}_-(\lambda) = b_{-,\mathbf{y}}(\lambda)\, \langle \mathbf{y}|, \quad b_{-,\mathbf{y}}(\lambda) \equiv b_-(\lambda) \prod_{n=1}^{\mathsf{N}} \sinh(\lambda - y_n) \prod_{n=1}^{\mathsf{N}} \sinh(\lambda + y_n)\, .$$

By the reflection algebra commutations relations, it holds:

$$\langle y_1, \ldots, y_k, \ldots, y_N|\mathcal{A}_-(\pm y_k) = \mathsf{a}_-(\pm y_k)\langle y_1, \ldots, y_k \mp \eta, \ldots, y_{\mathsf{N}}|,$$

$$\mathcal{D}_-(\lambda) = (\sinh(2\lambda - \eta)\mathcal{A}_-(-\lambda) + \sinh \eta \mathcal{A}_-(\lambda))/ \sinh 2\lambda$$

$$\frac{\det_q \mathcal{U}_-(\lambda)}{\sinh(2\lambda - \eta)} = \mathcal{A}_-(\lambda + \tfrac{\eta}{2})\mathcal{A}_-(\tfrac{\eta}{2} - \lambda) + \mathcal{B}_-(\lambda + \tfrac{\eta}{2})\mathcal{C}_-(\tfrac{\eta}{2} - \lambda) = \mathsf{a}_-(\tfrac{\eta}{2} + \lambda)\mathsf{a}_-(\tfrac{\eta}{2} - \lambda).$$

- In the triangular case $(b_+(\lambda) = 0)$ the transfer matrix reads:

$$\mathsf{T}(\lambda) \equiv \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\} = \bar{a}_+(\lambda)\, \mathcal{A}_-(\lambda) + \bar{a}_+(-\lambda)\, \mathcal{A}_-(-\lambda) + c_+(\lambda)\, \mathcal{B}_-(\lambda).$$

defined $\mathbf{A}(\lambda) \equiv \bar{a}_+(\lambda)\mathsf{a}_-(\lambda)$ then the eigenvalues and eigenstates are characterized by:

$$\mathbf{A}(y_k)\Psi_t(y_1, .., y_k - \eta, .., y_{\mathsf{N}}) + \mathbf{A}(-y_k)\Psi_t(y_1, .., y_k + \eta, .., y_{\mathsf{N}}) = t(y_k)\Psi_t(y_1, .., y_{\mathsf{N}}).$$

These equations lead to factorized wavefunctions by Baxter equation solutions $Q_t(y_j)$:

$$\Psi_t(y_1, \ldots, y_{\mathsf{N}}) = \prod_{j=1}^{N} Q_t(y_j).$$

Derivation of SOV-representations of $\mathcal{A}_{-}(\lambda)$ in the Reflection algebra case

a) Yang-Baxter commutation relation for the 6-vertex R-matrix:

$$\mathcal{A}_{-}(\mu)\mathcal{B}_{-}(\lambda)=\frac{\sinh(\lambda-\mu+\eta)\sinh(\mu+\lambda-\eta)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\mathcal{B}_{-}(\lambda)\mathcal{A}_{-}(\mu)$$

$$+\frac{\sinh(2\lambda-\eta)\sinh\eta}{\sinh(\mu-\lambda)\sinh 2\lambda}\mathcal{B}_{-}(\mu)\mathcal{A}_{-}(\lambda)-\frac{\sinh\eta}{\sinh(\lambda+\mu)\sinh 2\lambda}\mathcal{B}_{-}(\mu)\tilde{\mathcal{D}}_{-}(\lambda)$$

b) The centrality of the quantum determinant: $(\epsilon \pm 1)$

$$\frac{\det_q \mathcal{U}_{-}(\lambda-\eta/2)}{\sinh(2\lambda-2\eta)}=\mathcal{A}_{-}(\lambda)\mathcal{A}_{-}(\eta-\lambda)+\mathcal{B}_{-}(\lambda)\mathcal{C}_{-}(\eta-\lambda)=\mathsf{a}_{-}(\lambda)\mathsf{a}_{-}(\eta-\lambda)$$

- Action of a) for $\mu = y_k$ on the B-eigenstate $\langle y_1, .., y_\mathsf{N}|$:

$$b_{-,y_1,..,y_k-\eta,..,y_\mathsf{N}}(\lambda)\langle y_1,\ldots,y_k,\ldots,y_N|\mathcal{A}_{-}(y_k)=\left(\langle y_1,\ldots,y_k,\ldots,y_\mathsf{N}|\mathcal{A}_{-}(y_k)\right)\mathcal{B}_{-}(\lambda)$$

where:

$$b_{-,y_1,..,y_k-\eta,..,y_\mathsf{N}}(\lambda)\equiv\frac{\sinh(\lambda-y_k+\eta)\sinh(y_k+\lambda-\eta)}{\sinh(\lambda-y_k)\sinh(\lambda+y_k)}b_{-,y_1,..,y_k,..,y_\mathsf{N}}(\lambda).$$

- The Simplicity of $\mathcal{B}_{-}$-spectrum, the Reflection algebra $\&$ quantum determinant so imply:

$$\langle y_1,\ldots,y_k,\ldots,y_N|\mathcal{A}_{-}(y_k)=\mathsf{a}_{-}(y_k)\langle y_1,\ldots,y_k-\eta,\ldots,y_\mathsf{N}|,$$

$$\langle y_1,\ldots,y_k-\eta,\ldots,y_N|\mathcal{A}_{-}(\eta-y_k)=\mathsf{a}_{-}(\eta-y_k)\langle y_1,\ldots,y_k,\ldots,y_\mathsf{N}|.$$

- **Summary on SOV method:** If the monodromy matrix $\mathsf{M}(\lambda)$ (or $\mathcal{U}_-(\lambda)$) belongs to End($\mathbb{C}^{M=2} \otimes \mathcal{H}$) and $B(\lambda) = (M_a(\lambda))_{1,2}$ (or $\mathcal{B}_-(\lambda) = (\mathcal{U}_-(\lambda))_{1,2}$) is a one parameter family of simultaneously diagonalizable operators with simple spectrum, then the Y_n, operators zeros of $B(\lambda)$ (or $\mathcal{B}_-(\lambda)$), are the quantum separate variables for the spectral problem of $\mathsf{T}(\lambda) = \mathrm{tr}_0\{M(\lambda)\}$ (or $\mathsf{T}(\lambda) \equiv \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}$, with $K_+(\lambda)$ triangular matrix):

T $(\lambda)\,|t\rangle = t(\lambda)|t\rangle$, $|t\rangle$ **eigenvector of $\mathsf{T}(\lambda)$,** $t(\lambda)$ **eigenvalue of $\mathsf{T}(\lambda)$,**

$$\Updownarrow$$

SOV representation: $|t\rangle = \sum_{\{y\}} \prod_{j=1}^{N} Q_t(y_j)|y_1, \ldots, y_N\rangle, \qquad Q_t(y_j) \in \mathbb{C},$

Baxter's equation: $t(y_j)Q_t(y_j) = a(y_j)Q_t(y_j - \eta) + d(y_j)Q_t(y_j + \eta).$

The Baxter's equation is a direct consequence of the Yang-Baxter equation or Reflection equations and it is the quantum analogue of the spectral curve computed in the zeros of the classical $B(\lambda)$.

- **Motivations to use quantum separation of variables (SOV):**

The SOV method allows to solve the problems which appear in other more traditional methods, like Bethe ansatz and Baxter's Q-operator, giving:
a) the proof of completeness of the spectrum description,
b) the analysis of a larger class of integrable quantum models,
c) more symmetrical approach to classical and quantum integrability.

- **Quantum inverse scattering formulation of the open XXZ quantum spin-1/2 chain:**

The transfer matrix $\mathcal{T}(\lambda) \equiv \text{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}$ defines a one parameter family of conserved charges and generates the open XXZ spin chain Hamiltonian with general integrable boundaries:

$$H = \sum_{i=1}^{\mathsf{N}-1}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cosh\eta\,\sigma_i^z\sigma_{i+1}^z) + \frac{\sigma_1^z\cosh\zeta_- + 2\kappa_-(\sigma_1^x\cosh\tau_- + i\sigma_1^y\sinh\tau_-)}{\sinh^{-1}\eta\sinh\zeta_-}$$

$$+ \frac{\sigma_1^z\cosh\zeta_+ + 2\kappa_+(\sigma_1^x\cosh\tau_+ + i\sigma_1^y\sinh\tau_+)}{\sinh^{-1}\eta\sinh\zeta_+} = \frac{2(\sinh\eta)^{1-2\mathsf{N}}\frac{d}{d\lambda}\mathcal{T}(\lambda)_{|\lambda=\eta/2}}{\text{tr}\{K_-(\eta/2)\}\,\text{tr}\{K_+(\eta/2)\}} + \text{constant.}$$

Here $M_0(\lambda) = R_{0\mathsf{N}}(\lambda - \xi_\mathsf{N} - \eta/2)\dots R_{01}(\lambda - \xi_1 - \eta/2)$, R the 6-vertex R-matrix.

Let us use the notation: $\sinh\alpha_\pm\cosh\beta_\pm = (2\kappa_\pm)^{-1}\sinh\zeta_\pm$, $\cosh\alpha_\pm\sinh\beta_\pm = (2\kappa_\pm)^{-1}\cosh\zeta_\pm$, so we have the following expression for the central quantum determinant:

$$\det{}_q\mathcal{U}_-(\lambda) = \det{}_q K_-(\lambda)\prod_{\epsilon=\pm1}\det{}_q M_0(\epsilon\lambda) = \sinh(2\lambda-\eta)\prod_{\epsilon=\pm1}\mathsf{A}_-(\epsilon\lambda+\eta/2).$$

Here: $\det_q M(\lambda) = a(\lambda+\eta/2)d(\lambda-\eta/2)$, $\det_q K_\pm(\lambda) = \sinh(2\lambda\pm\eta)\prod_{\epsilon=\pm1} g_\pm(\epsilon\lambda+\eta/2)$

with $\mathsf{A}_-(\lambda) = g_-(\lambda)a(\lambda)d(-\lambda)$, $d(\lambda) = a(\lambda-\eta)$, $a(\lambda) = \prod_{n=1}^N\sinh(\lambda-\xi_n)$, and $g_\pm(\lambda) = \sinh(\lambda+\alpha_\pm\pm\eta/2)\cosh(\lambda\mp\beta_\pm\pm\eta/2)\sinh^{-1}\alpha_\pm\cosh^{-1}\beta_\pm$.

- **Separation of variable representation for open XXZ quantum spin chain**

Theorem (N. 2012) a $\mathcal{B}_-(\lambda)$-eigenbasis of $\mathcal{H}$ is defined by the states:

$$\langle h_1, \ldots, h_\mathsf{N}| \equiv \frac{1}{\mathrm{N}}\langle 0| \prod_{n=1}^{\mathsf{N}} \left(\frac{\mathcal{A}_-(\eta/2-\xi_n)}{\mathsf{A}_-(\eta/2-\xi_n)}\right)^{h_n}, \quad \langle 0| \equiv \otimes_{n=1}^{\mathsf{N}} \langle 1, n|, \quad h_n \in \{0,1\},$$

where N is a normalization and it holds $\langle h_1, \ldots, h_\mathsf{N}|\mathcal{B}_-(\lambda) = \mathrm{B}_{-,\mathbf{h}}(\lambda)\langle h_1, \ldots, h_\mathsf{N}|$, with:

$$\mathrm{B}_{-\mathbf{h}}(\lambda) \equiv \mathrm{B}_0(\lambda) a_\mathbf{h}(\lambda) a_\mathbf{h}(-\lambda), \quad a_\mathbf{h}(\lambda) \equiv \prod_{n=1}^{\mathsf{N}} \sinh(\lambda - \xi_n - (h_n - \frac{1}{2})\eta), \quad \mathbf{h} \equiv (h_1, \ldots, h_\mathsf{N}).$$

Then the action of the remaining reflection algebra generators follows by:

$$\langle \mathbf{h}|\mathcal{A}_-(\lambda) = \sum_{a=1}^{2\mathsf{N}} \frac{\sinh(2\lambda-\eta)\sinh(\lambda+\zeta_a^{(h_a)})}{\sinh(2\zeta_a^{(h_a)}-\eta)\sinh 2\zeta_a^{(h_a)}} \prod_{\substack{b=1 \\ b\neq a \bmod \mathsf{N}}}^{\mathsf{N}} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(h_a)} - \cosh 2\zeta_b^{(h_b)}} \mathsf{A}_-(\zeta_a^{(h_a)})$$

$$\times \langle \mathbf{h}|\mathsf{T}_a^{-\varphi_a} + (-1)^{\mathsf{N}} \det_q M(0)\cosh(\lambda-\eta/2) \prod_{b=1}^{\mathsf{N}} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh\eta - \cosh 2\zeta_b^{(h_b)}} \langle \mathbf{h}|$$

$$+ (-1)^{\mathsf{N}} \coth\zeta \det_q M(i\pi/2)\sinh(\lambda-\eta/2) \prod_{b=1}^{\mathsf{N}} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh\eta + \cosh 2\zeta_b^{(h_b)}} \langle \mathbf{h}|,$$

Theorem extended to the gauge transformed boundary operators by Faldella, Kitanine N. 2013.

Baxter's Gauge transformations for open XXZ spin chains

Gauge transformations has been introduced in the 8-vertex model by Baxter 1972 and Faddeev Takhtadjan 1979. First used in their trigonometric form by Cao et al (2003) to diagonalize the boundary matrices and define a generalization of ABA.

Definition of gauge transformations:

$$\bar{G}(\lambda|\beta) = (X(\lambda|\beta), Y(\lambda|\beta)), \quad \tilde{G}(\lambda|\beta) = (X(\lambda|\beta+1), Y(\lambda|\beta-1))$$

where we have defined:

$$X(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha+\beta)\eta]} \\ 1 \end{pmatrix}, \qquad Y(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha-\beta)\eta]} \\ 1 \end{pmatrix},$$

Transfer matrix: $$\mathcal{T}(\lambda) = \mathsf{a}_+(\lambda|\beta-1)\mathcal{A}_-(\lambda|\beta) + \mathsf{a}_+(-\lambda|\beta-1)\mathcal{A}_-(-\lambda|\beta) + K_+^{(L)}(\lambda|\beta-1)_{21}\mathcal{B}_-(\lambda|\beta-2) + K_+^{(L)}(\lambda|\beta-1)_{12}\mathcal{C}_-(\lambda|\beta+2)$$

written in terms of the elements of the gauge transformed boundary monodromy matrix:

$$\mathcal{U}_-(\lambda|\beta) = e^{-\lambda+\eta/2}\tilde{G}^{-1}(\lambda-\eta/2|\beta)\,\mathcal{U}_-(\lambda)\,\tilde{G}(\eta/2-\lambda|\beta) = \begin{pmatrix} \mathcal{A}_-(\lambda|\beta+2) & \mathcal{B}_-(\lambda|\beta) \\ \mathcal{C}_-(\lambda|\beta+2) & \mathcal{D}_-(\lambda|\beta) \end{pmatrix}$$

For ABA one needs the gauge transformed K_+ and K_- to be diagonal and triangular, respectively; instead for SOV one only need the gauge transformed K_+ to be triangular.

Here we have used the following notations:

$$\begin{aligned} K_{+}^{(L)}(\lambda|\beta-1)_{12} &= \tilde{Y}(\eta/2-\lambda|\beta)K_{+}(\lambda)Y(\lambda-\eta/2|\beta-2), \\ \tilde{Y}(\lambda|\beta) &= e^{(\lambda+(\alpha+2)\eta)}\left(-1, e^{-[\lambda+(\alpha-\beta)\eta]}\right)/(2\sinh\beta\eta) \end{aligned}$$

and

$$\mathsf{a}_{+}(\lambda|\beta) = \frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda \sinh(\beta-1)\eta \sinh\zeta_{+}}\Big[\sinh\zeta_{+}\cosh(\lambda-\eta/2)\sinh(\lambda+\eta/2+\beta\eta)$$
$$-\cosh\zeta_{+}\sinh(\lambda-\eta/2)\cosh(\lambda+\eta/2+\beta\eta)-\kappa_{+}\sinh(2\lambda-\eta)\sinh(\tau_{+}+\alpha\eta+2\eta)\Big].$$

From the explicit expression of $K_{+}^{(L)}(\lambda|\beta-1)_{12}$, the condition required to use SOV:

$$K_{+}^{(L)}(\lambda|\beta-1)_{12}=0$$

is equivalent to the following choice of one of the two gauge parameters:

$$(\alpha-\beta+2)\eta = -\tau_{+}+(-1)^{k}(\alpha_{+}-\beta_{+})+i\pi k.$$

For the characterization of the SOV representation it is important to remark that:

$$\det_q K_{+}(\lambda-\eta/2) = \sinh(2\lambda+2\eta)\mathsf{a}_{+}(\lambda|\beta-1)\mathsf{a}_{+}(-\lambda+\eta|\beta-1)$$

once the gauge parameter β is fixed to satisfy the SOV condition $K_{+}^{(L)}(\lambda|\beta-1)_{12}=0$.

- **Separation of variables characterization of open XXZ spin chain transfer matrix**

Theorem (N. 2012; Faldella, Kitanine, N. 2013) Let $\Sigma_{\mathcal{T}}$ be the set of the eigenvalue functions of the transfer matrix $\mathcal{T}(\lambda)$, then any $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ has the form:

$$\tau(\lambda) = \sum_{a=1}^{\mathsf{N}} \frac{\cosh^2 2\lambda - \cosh^2 \eta}{\cosh^2 2\zeta_a^{(0)} - \cosh^2 \eta} \prod_{\substack{b=1 \\ b\neq a}}^{\mathsf{N}} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_b^{(0)}} \tau(\zeta_a^{(0)}),$$

$$+ (\cosh 2\lambda + \cosh \eta) \prod_{b=1}^{\mathsf{N}} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh \eta - \cosh 2\zeta_b^{(0)}} \det{}_q M(0)$$

$$+ (-1)^{\mathsf{N}}(\cosh 2\lambda - \cosh \eta) \prod_{b=1}^{\mathsf{N}} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh \eta + \cosh 2\zeta_b^{(0)}} \coth \zeta_- \coth \zeta_+ \det{}_q M(i\pi/2).$$

$\mathcal{T}(\lambda)$ has simple spectrum and $\Sigma_{\mathcal{T}}$ coincides with the set of solutions to:

$$\tau(\xi_n - \eta/2)\tau(\xi_n + \eta/2) = \frac{\det_q K_+(\xi_n) \det_q \mathcal{U}_-(\xi_n)}{\sinh(\eta + 2\xi_n) \sinh(\eta - 2\xi_n)}, \quad \forall n \in \{1, \ldots, \mathsf{N}\},$$

and defined $\eta_n^{(h_n)} \equiv \cosh 2\zeta_n^{(h_n)}$ and $\zeta_n^{(h_n)} \equiv \xi_n + (h_n - \frac{1}{2})\eta$ it holds:

$$|\tau\rangle = \sum_{h_1,\ldots,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} Q_\tau(\zeta_n^{(h_n)}) \prod_{1\leq b<a\leq \mathsf{N}} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) \langle h_1, \ldots, h_{\mathsf{N}}|,$$

is the unique right $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ characterized by:

$$Q_\tau(\zeta_a^{(1)})/Q_\tau(\zeta_a^{(0)}) = \tau(\zeta_a^{(0)})/\mathbf{A}(\zeta_a^{(1)}), \quad \mathbf{A}(\lambda) \equiv (-1)^N \frac{\sinh(2\lambda + \eta)}{\sinh 2\lambda} g_+(\lambda) g_-(\lambda) a(\lambda) d(-\lambda).$$

- **TQ-functional equation equivalent to the separation of variables characterization**

Theorem (Kitanine, Maillet, N. 2013) Let us introduce the following function of the boundary parameters:

$$F_0 = \frac{2\kappa_+\kappa\left(\cosh(\tau_+ - \tau_-) - \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- - (\mathsf{N}+1)\eta)\right)}{\sinh\zeta_+ \sinh\zeta_-},$$

and then the function:

$$F(\lambda) = F_0\left(\cosh^2 2\lambda - \cosh^2\eta\right)\prod_{b=1}^{\mathsf{N}}\prod_{i=0}^{1}(\cosh 2\lambda - \cosh 2\zeta_b^{(i)}),$$

and let the inhomogeneities $\{\xi_1, \ldots, \xi_{\mathsf{N}}\}$ and the boundary parameters be generics, then $\mathcal{T}(\lambda)$ has simple spectrum and $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ if and only if there exists one and only one function:

$$Q(\lambda) = 2^{\mathsf{N}}\prod_{a=1}^{\mathsf{N}}\left(\cosh 2\lambda - \cosh 2\lambda_a\right),$$

such that: $$\tau(\lambda)Q(\lambda) = \mathbf{A}(\lambda)Q(\lambda-\eta) + \mathbf{A}(-\lambda)Q(\lambda+\eta) + F(\lambda)$$

$$\det_q K_+(\lambda)\det_q \mathcal{U}_-(\lambda) = \mathbf{A}(\lambda+\eta/2)\mathbf{A}(-\lambda+\eta/2)\sinh(2\lambda+\eta)\sinh(2\lambda-\eta).$$

Toward quantum dynamics: definition and problems to solve

Definition Form factors $\langle t'|\mathcal{O}_n|t\rangle$ are the matrix elements of a local operator $\mathcal{O}_n$ between the eigencovector $\langle t'|$ and the eigenvector $|t\rangle$ of $\mathsf{T}(\lambda)$.

The form factors are the "elementary objects" w.r.t. any time dependent correlation function can be expanded by using the decomposition of the identity in the transfer matrix eigenbasis:

$$\langle t'|\mathcal{O}_n(\theta_1)\mathcal{O}_m(\theta_2)|t''\rangle = \sum_{t\in\Sigma_{\mathsf{T}}} \frac{\langle t'|\mathcal{O}_n|t\rangle\langle t|\mathcal{O}_m|t''\rangle}{\langle t|t\rangle} e^{(h_{t'}-h_t)\theta_1+(h_t-h_{t''})\theta_2}, \forall n < m \in \{1, \ldots, \mathsf{N}\}$$

where $h_{t'}$ and $h_{t''}$ are the Hamiltonian eigenvalues on the eigenstates $|t''\rangle$ and $|t'\rangle$ and by definition of time evolution operator, it holds $\mathcal{O}_n(\theta) \equiv e^{iH\theta}\mathcal{O}_n e^{-iH\theta}$.

Two difficult problems to solve:

i) Reconstruction of local operators $\mathcal{O} \in \mathrm{End}(\mathcal{H})$ in terms of the quantum separate variables.

$\hookrightarrow$ Computation of the action of local operators $\mathcal{O}$ on the eigenvector $|t\rangle$.

ii) Scalar product $\langle t'|t\rangle$ under the form of determinant.

Steps i) and ii) allow us to get in a determinant form the form factors of a basis of local operators.

Toward quantum dynamics: scalar products

Sklyanin's measure for open chain The action of a left -eigenstate on a right -eigenstate reads:
Niccoli (2012)

$$\langle \mathbf{h}|\mathbf{h}'\rangle = \frac{\delta_{\mathbf{h},\mathbf{h}'}}{V(\eta_n^{(h_1)},\dots,\eta_{\mathsf{N}}^{(h_{\mathsf{N}})})} = \prod_{1\leq b<a\leq \mathsf{N}} \frac{\delta_{\mathbf{h},\mathbf{h}'}}{\eta_a^{(h_a)}-\eta_b^{(h_b)}}.$$

Scalar products for open chain Let $\langle\alpha|$ and $|\beta\rangle$ be a covector and vector of separate forms:
N. (2012), Faldella, Kitanine, N. (2013)

$$\langle\alpha| = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} \alpha_a(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq \mathsf{N}} (\eta_a^{(h_a)}-\eta_b^{(h_b)})\langle h_1,\dots,h_{\mathsf{N}}|,$$

$$|\beta\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} \beta_a(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq \mathsf{N}} (\eta_a^{(h_a)}-\eta_b^{(h_b)})|h_1,\dots,h_{\mathsf{N}}\rangle,$$

in the $\mathcal{B}$-eigenbasis, then the action of $\langle\alpha|$ on $|\beta\rangle$ reads:

$$\langle\alpha|\beta\rangle = \det_{\mathsf{N}} ||\mathcal{M}_{a,b}^{(\alpha,\beta)}|| \quad \text{with} \quad \mathcal{M}_{a,b}^{(\alpha,\beta)} \equiv \sum_{h=0}^{1} \alpha_a(\zeta_a^{(h)})\beta_a(\zeta_a^{(h)})(\eta_a^{(h)})^{(b-1)}$$

Toward quantum dynamics: Some results for open chain and universal form

Some matrix elements Let $\langle\tau|$ and $|\tau'\rangle$ be a generic couple of left and right $\mathcal{T}$-eigenstates:

Niccoli (2012)

$$\langle\tau|\sigma_n^-\cdots\sigma_{\mathsf{N}}^-|\tau'\rangle=\frac{1}{V(\eta_n^{(0)},\dots,\eta_{\mathsf{N}}^{(0)})}\prod_{n\leq a<b\leq\mathsf{N}}\frac{\sinh(\xi_a+\xi_b-\eta)}{\sinh(\xi_a+\xi_b)}\det_{2\mathsf{N}-n}||\Sigma_{a,b}^{(n,\tau,\tau')}||$$

$$\times\prod_{a=n}^{\mathsf{N}}\frac{\tau'(\zeta_a^{(1)})Q_\tau(\zeta_a^{(1)})Q_{\tau'}(\zeta_a^{(1)})\sinh 2\xi_a}{\det_q\mathcal{U}_-(\xi_a)},$$

where:

$$\Sigma_{a,b}^{(n,\tau,\tau')}\equiv\mathcal{M}_{a,b}^{(\tau,\tau')}\quad\text{for } a\in\{1,\dots,n-1\},\ b\in\{1,\dots,2\mathsf{N}-n\},$$

$$\Sigma_{a,b}^{(n,\tau,\tau')}\equiv\left(\eta_a^{(0)}\right)^{(b-1)}\quad\text{for } a\in\{n,\dots,\mathsf{N}\},\ b\in\{1,\dots,2\mathsf{N}-n\},$$

$$\Sigma_{a,b}^{(n,\tau,\tau')}\equiv\left(\eta_a^{(1)}\right)^{(b-1)}\quad\text{for } a\in\{\mathsf{N}+1,\dots,2\mathsf{N}-n\},\ b\in\{1,\dots,2\mathsf{N}-n\}.$$

Universality form (First derived for sine-Gordon model with Grosjean and Maillet 2012) For all the integrable quantum model so far considered there exists a basis of operators $\mathbb{B}_{\mathcal{H}}$ in End($\mathcal{H}$) such that for any $\mathsf{O}\in\mathbb{B}_{\mathcal{H}}$ the matrix elements on transfer matrix eigenstates read:

$$\langle t'|\mathsf{O}|t\rangle=\det_{\mathsf{M_O}}||\Phi_{a,b}^{(\mathsf{O},t',t)}||,\quad\Phi_{a,b}^{(\mathsf{O},t',t)}\equiv\sum_{c=0}^{p-1}F_{\mathsf{O},b}(y_a^{(c)})Q_t(y_a^{(c)})Q_{t'}(y_a^{(c)})(y_a^{(c)})^{2b-1},$$

where the integer $\mathsf{M_O}$ and the coefficients $F_{\mathsf{O},b}(y_a^{(c)})$ characterize the operator O.

Related projects: There are two lines of research that I am developing simultaneously in the use of the SOV method for the exact spectrum and dynamics characterization.

I) To complete the exact characterization of the dynamics for the models already analyzed: Computation of correlation functions.

Research Groups:	Projects:
• ENS, Lyon, France: J.-M.Maillet et al.	Antiperiodic XXZ spin chains and sine-Gordon model.
• YITP, New-York, USA: B.McCoy et al.	XYZ chains with general integrable boundaries and chiral Potts model.
• LPTM, Orsay, France: V.Terras et al.	Dynamical 6-vertex and closed XYZ chains
• IMB, Dijon, France: N.Kitanine et al.	Open integrable quantum systems and out of equilibrium statistical mechanics: PASEP.

II) Analysis of spectrum and dynamics by SOV of more advanced integrable quantum models:

• ENS, Lyon, France: J.-M.Maillet et al. DESY, Hamburg, Germany: J.Teschner	Spin chains associated to higher rank (super)quantum groups spectrum and dynamics, toward solution of nonlinear sigma models by SOV-method.

These last models should lead to the SOV tools for the solution of the Hubbard model.