Lagrangian multiform theory and integrability

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(joint work with Sarah Lobb, Pavlos Xenitidis and Sikarin Yoo-Kong)

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The problem

Multidimensional consistency: We know that many "integrable" equations, discrete and continuous possess the property of multidimenional consistency.

- continuous: commuting flows, higher symmetries & master symmetries, hierarchies;
- discrete: consistency-around-the-cube, Bäcklund transforms, higher continuous symmetries, commuting discrete flows

In all these cases we can think of the *dependent* variable a (possibly vector-valued) function of many (discrete and continuous) variables

$$u = u(n, m, h, \ldots; x, t_1, t_2, \ldots)$$

on which we can impose many equations simultaneously, and it is the *compatibility* of those equations that makes the integrability manifest.

Key question: How to capture the property of multidimensional consistency within a Lagrange formalism?

Main problem: We note that the conventional variational principle, through the EL equations, only produces one equation per component of the dependent variables, but not an entire system of compatible equations on one and the same dependent variable!

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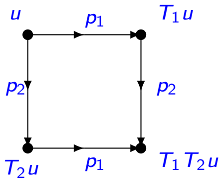
Multidimensional consistency on the lattice

quadrilateral PΔEs on the 2D lattice:

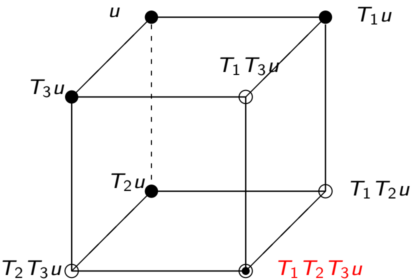
$$\mathbf{Q}(u, T_1 u, T_2 u, T_1 T_2 u; {\color{red}p_1, p_2}) = 0$$

notation of shifts on the elementary quadrilateral on a rectangular lattice:

$$u := u(n_1, n_2), \;\; T_1 u = u(n_1+1, n_2)$$
$$T_2 u := u(n_1, n_2+1), \;\; T_1 T_2 u = u(n_1+1, n_2+1)$$



Consistency-around-the cube:



Verifying consistency: Values at the black disks are initial values, values at open circles are uniquely determined from them, but there are three different ways to compute $T_1 T_2 T_3 u$.

Conventional variational formalism: discrete Euler-Lagrange equations

Define an action functional:

$$S[u(n_1,n_2)] = \sum_{n_1,n_2\in\mathbb{Z}} \mathscr{L}(u, T_1u, T_2u; p_1, p_2)\ .$$

Following the usual *least-action principle*, the lattice equations for u are determined by the demand that S attains a minimum under local variations $u(n_1,n_2) \to u(n_1,n_2) + \delta u(n_1,n_2)$. Thus,

$$\begin{aligned}\delta S &= \sum_{n_1,n_2\in\mathbb{Z}} \Big\{ \frac{\partial}{\partial u}\mathscr{L}(u, T_1u, T_2u; p_1, p_2)\delta u + \frac{\partial}{\partial T_1 u}\mathscr{L}(u, T_1u, T_2u; p_1, p_2)\delta(T_1u) \\ &\qquad + \frac{\partial}{\partial T_2 u}\mathscr{L}(u, T_1u, T_2u; p_1, p_2)\delta(T_2u)\Big\} = 0\end{aligned}$$

Setting $\delta(T_iu) = T_i\delta u$, and resumming each of the terms we get:

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(ignoring boundary terms) and since δu is arbitrary the discrete Euler-Lagrange (EL) equation follow:

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Closure relation and Lagrangian multiform structure

Closure property: *Multidimensionally consistent systems of lattice equations, possess Lagrangians which obey the following relation* [1]*:*

$$\Delta_1\mathscr{L}(u, T_2u, T_3u; p_2, p_3) + \Delta_2\mathscr{L}(u, T_3u, T_1u; p_3, p_1) + \Delta_3\mathscr{L}(u, T_1u, T_2u; p_1, p_2) = 0$$

on the solutions of the equations.
Here $\Delta_i = T_i - \mathrm{id}$ denotes the difference operator, i.e.. on functions f of $u = u(n_1, n_2, n_3)$ we have: $\Delta_i f(u) = f(T_iu) - f(u)$.

• This property suggests that the Lagrangians $\mathscr{L}_{i,j} = \mathscr{L}(u, T_iu, T_ju; p_i, p_j)$ should be considered as *difference forms* (i.e., discrete differential forms) for which the closure property means that these forms are closed, but only for functions u which solve the lattice equation.
• Furthermore, as a consequence of this closedness of the corresponding *Lagrangian 2-form* on solutions of the equations, the corresponding action will be locally invariant under deformations of the underlying geometry of the lattice, i.e. locally independent of the discrete surface in the space of independent variables.
• However, *off-shell*, i.e. for general field configurations (i.e. values of the dependent variable u as a function of the lattice) the action is non-trivial functional of those fields, and also of the lattice-surface on which we evaluate the action.

[1] S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013.

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[1] S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013.

Closure relation and Lagrangian multiform structure

Closure property: *Multidimensionally consistent systems of lattice equations, possess Lagrangians which obey the following relation* [1]*:*

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Example: H1 (lattice potential KdV eq.)

The equation is

$$Q(u, T_1 u, T_2 u, T_1 T_2 u; p_1, p_2) = (u - T_1 T_2 u)(T_1 u - T_2 u) + p_1^2 - p_2^2 = 0$$

The equation in the "3-leg form" is

$$(u + T_1 u) - (u + T_2 u) + \frac{p_1^2 - p_2^2}{u - T_1 T_2 u} = 0$$

The corresponding 3-point Lagrangian is given as[2]

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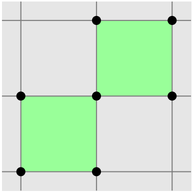

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&\qquad -(u_1 - u_2)u - (p_1^2 - p_2^2)\ln(u_1 - u_2)
\end{aligned}$$

Closure property for H1:

The lagrangian for H1 obeys the following closure relation:

$$\Delta_1 \mathscr{L}(u, T_2 u, T_3 u; p_2, p_3) + \Delta_2 \mathscr{L}(u, T_3 u, T_1 u; p_3, p_1) + \Delta_3 \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) = 0$$

on the solutions of the quadrilateral equation.

Proof: From the explicit form of the Lagrangians we find

$$\begin{aligned}
&\Delta_1 \mathscr{L}(u, u_2, u_3; p_2, p_3) + \Delta_2 \mathscr{L}(u, u_3, u_1; p_3, p_1) + \Delta_3 \mathscr{L}(u, u_1, u_2; p_1, p_2) \\
&= (u_{1,2} - u_{1,3})u_1 + (p_2^2 - p_3^2)\ln(u_{1,2} - u_{1,3}) - (u_2 - u_3)u - (p_2^2 - p_3^2)\ln(u_2 - u_3) \\
&+ (u_{2,3} - u_{1,2})u_2 + (p_3^2 - p_1^2)\ln(u_{2,3} - u_{1,2}) - (u_3 - u_1)u - (p_3^2 - p_1^2)\ln(u_3 - u_1) \\
&+ (u_{1,3} - u_{2,3})u_3 + (p_1^2 - p_2^2)\ln(u_{1,3} - u_{2,3}) - (u_1 - u_2)u - (p_1^2 - p_2^2)\ln(u_1 - u_2)
\end{aligned}$$

wehere we have used the abbreviations: $u_i := T_i u,\ u_{i,j} := T_i T_j u$. Noting that the differences between the double-shifted terms has the form

$$\begin{aligned}
u_{1,2} - u_{1,3} &= \frac{(p_2^2 - p_3^2)u_1 + (p_3^2 - p_1^2)u_2 + (p_1^2 - p_2^2)u_3}{(u_1 - u_2)(u_2 - u_3)(u_3 - u_1)}(u_2 - u_3) \\
&=: A_{1,2,3}(u_2 - u_3)
\end{aligned}$$

where $A_{1,2,3}$ is invariant under permutations of the indices, the expression reduces to

$$\begin{aligned}
&A_{1,2,3}(u_2 - u_3)u_1 + (p_2^2 - p_3^2)\ln\big(A_{1,2,3}(u_2 - u_3)\big) \\
&\qquad - (u_2 - u_3)u - (p_2^2 - p_3^2)\ln(u_2 - u_3) \\
&+ A_{1,2,3}(u_3 - u_1)u_2 + (p_3^2 - p_1^2)\ln\big(A_{1,2,3}(u_3 - u_1)\big) \\
&\qquad - (u_3 - u_1)u - (p_3^2 - p_1^2)\ln(u_3 - u_1) \\
&+ A_{1,2,3}(u_1 - u_2)u_3 + (p_1^2 - p_2^2)\ln\big(A_{1,2,3}(u_1 - u_2)\big) \\
&\qquad - (u_1 - u_2)u - (p_1^2 - p_2^2)\ln(u_1 - u_2) = 0
\end{aligned}$$

Closure relation for other cases

The closure property was proven for many other lattice equations, but it often requires a specific form of the Lagrangian (taking into account that there is the freedom to add total-"derivative" terms).

- All equations in the ABS (Adler-Bobenko-Suris) list of scalar affine-linear equations;
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In many cases these Lagrangians contain the function $F(u) = u \ln u$ or the dilogarithm function

$$\mathrm{Li}_2(z) = -\int_0^z z^{-1} \ln(1-z)\, dz$$

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$$\begin{aligned}\mathrm{Li}_2\left(\frac{x}{1-y}\frac{y}{1-x}\right) &= \mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) \\ &\quad - \ln(1-x)\ln(1-y)\end{aligned}$$

In cases of elliptic lattice systems requires an elliptic analogue of the dilogarithm: $F(u) \sim \int^u \ln(\sigma(x))\, dx$.

An example of such a case is the Q4 equation, due to V. Adler:

$$p_i(u\,u_i + u_j u_{i,j}) - p_j(u\,u_j + u_i u_{i,j}) - p_{ij}(u\,u_{i,j} + u_i u_j) + p_i p_j p_{ij}(1 + u\,u_i u_j u_{i,j}) = 0$$

where (as before) $u_i := T_i u$, $u_{i,j} = T_i T_j u$. The parameters are in terms of Jacobi elliptic functions: $p_i = \sqrt{k}\,\mathrm{sn}(\alpha_i; k)$, $p_j = \sqrt{k}\,\mathrm{sn}(\alpha_j; k)$, $p_{ij} = \sqrt{k}\,\mathrm{sn}(\alpha_i - \alpha_j; k)$.

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Surface-dependent actions

The closure relation suggests the introduction of surface-dependent actions.

Discrete case:
action functional on a discrete surface σ:

$$S[u(\mathbf{n});\sigma] = \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathscr{L}_{ij}(\mathbf{n})$$

where $\mathscr{L}_{ij}(\mathbf{n})$ has the interpretation of a discrete Lagrangian 2-form:
These are oriented expressions of the form:

$$\mathscr{L}_{ij}(\mathbf{n}) = \mathscr{L}(u(\mathbf{n}), u(\mathbf{n}+\mathbf{e}_i), u(\mathbf{n}+\mathbf{e}_j); p_i, p_j)$$

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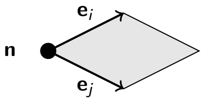

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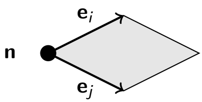

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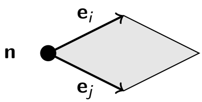

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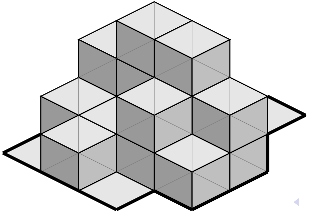

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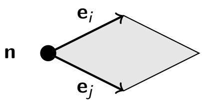
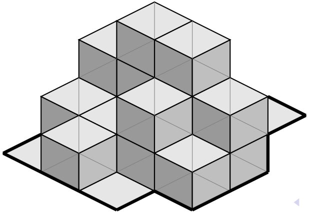
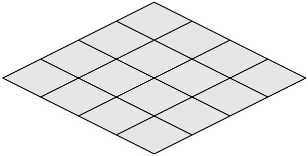


σ a discrete surface in the multidimensional lattice consisting of (a connected configuration of) elementary plaquettes $\sigma_{ij}(\mathbf{n})$



Surface independence

Independence of the action S under local deformations of the surface is equivalent to the closure relation holding.



$$\begin{aligned} S' \quad &= \quad S - \mathscr{L}(u, u_i, u_j; \alpha_i, \alpha_j) + \mathscr{L}(u_k, u_{i,k}, u_{j,k}; \alpha_i, \alpha_j) + \mathscr{L}(u_i, u_{i,j}, u_{i,k}; \alpha_j, \alpha_k) \\ &\quad + \mathscr{L}(u_j, u_{j,k}, u_{i,j}; \alpha_k, \alpha_i) - \mathscr{L}(u, u_j, u_k; \alpha_j, \alpha_k) - \mathscr{L}(u, u_k, u_i; \alpha_k, \alpha_i) \end{aligned}$$

taking into account the orientation of the plaquettes.

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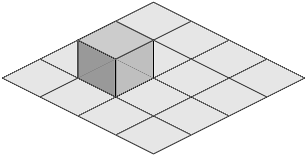

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Basic configurations for 2D lattice EL system

The following configurations constitute the elementary surface actions from which the system of 2D discrete EL equations for 2-forms are derived.

Figure : Flat 2d space.

This represents the "planar" EL equations.
The following represent EL over curved surfaces:

Figure : Configurations when embedded in 3d.

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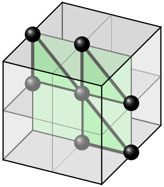

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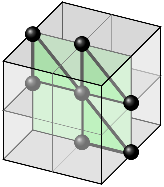

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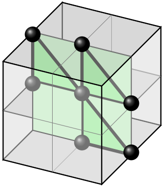

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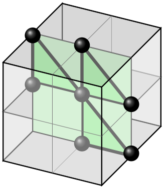

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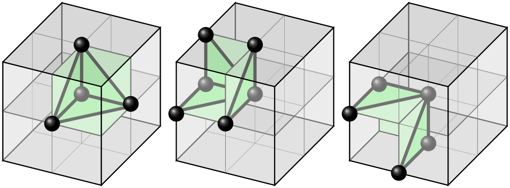

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The above configurations correspond to en exhaustive list of *elementary actions* given by the sum of 3-point Lagrangians $\mathscr{L}(u, T_i(u), T_j(u); \alpha_i, \alpha_j)$ in the multidimensional regular lattice, around a given vertex. Computing the EL equations for these actions we get the fundamental system of equations:

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Interplay Discrete $\longleftrightarrow$ Continuous

The lattice systems we consider here admit a role reversal:

lattice *parameters* p_i $\leftrightarrow$ lattice *variables* n_i

For all quadrilateral P∆Es we have *fully consistent system of equations* comprising three types of equations, all compatible discrete as well as continuous.

$$\boxed{\text{P}\Delta\text{E}} \quad \leftrightarrow \quad \boxed{\text{D}\Delta\text{E}} \quad \leftrightarrow \quad \boxed{\text{PDE}}$$

These D(∆)Es can be simultaneously imposed on the same dependent variables:

$$u = u(n_1, n_2, n_3, \ldots; p_1, p_2, p_3, \ldots)$$

and possess the property of multidimensional consistency.
The consistency can be expressed as the condition that on solutions of these equations the operators $(T_i u)(\ldots, n_i, \ldots) = u(\ldots, n_i + 1, \ldots)$, mutually commute in all lattice directions and with the differential operators $\partial_{p_i}, \partial_{p_j}, \ldots$, i.e. among these equations hold on the solutions

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Linear Case

$\boxed{\text{P}\Delta\text{E}}$ Fully discrete Lagrangian:

$$\mathscr{L}_{ij} = w(T_i w - T_j w) - \frac{1}{2}\left(\frac{p_i + p_j}{p_i - p_j}\right)((T_i - T_j)w)^2 \ ,$$

Linear quadrilateral lattice equation:

$$(p_i + p_j)(T_i - T_j)w - (p_i - p_j)(\mathrm{id} - T_i T_j)w = 0 \ .$$

The lattice Lagrangian $\mathscr{L}_{ij}$ obeys the closure relation:

$$\Delta_i \mathscr{L}_{jk} + \Delta_j \mathscr{L}_{ki} + \Delta_k \mathscr{L}_{ij} = 0$$

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$\boxed{\text{D}\Delta\text{E}}$ Linear differential-difference equation:

$$2p_i \frac{\partial w}{\partial p_i} = n_i (T_i^{-1} - T_i) w \ ,$$

Semi-discrete Lagrangian:

$$\mathscr{L}_i = n_i w \frac{\partial}{\partial p_i} T_i w - p_i \left(\frac{\partial w}{\partial p_i}\right)^2 \ .$$

Question: Is there a semi-discrete analogue of the closure relation?

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$$\Delta_i \mathscr{L}_{jk} + \Delta_j \mathscr{L}_{ki} + \Delta_k \mathscr{L}_{ij} = 0$$

on solutions of the lattice equation.

$\boxed{\text{D}\Delta\text{E}}$ Linear differential-difference equation:

$$2p_i \frac{\partial w}{\partial p_i} = n_i (T_i^{-1} - T_i) w \ ,$$

Semi-discrete Lagrangian:

$$\mathscr{L}_i = n_i w \frac{\partial}{\partial p_i} T_i w - p_i \left(\frac{\partial w}{\partial p_i}\right)^2 \ .$$

Question: Is there a semi-discrete analogue of the closure relation?

$\boxed{\text{PDE}}$ Fully continuous equation;

$$\partial_{p_i}\partial_{p_j}(p_i^2 - p_j^2)\partial_{p_i}\partial_{p_j} w = 4(n_j\partial_{p_i} - n_i\partial_{p_j})\frac{1}{p_i^2 - p_j^2}(n_j p_i^2 \partial_{p_i} - n_i p_j^2 \partial_{p_j})w \ ,$$

has a Lagrangian:

$$\mathscr{L}_{ij} = \frac{1}{n_j n_i}\left\{\frac{1}{2}(p_i^2 - p_j^2)(\partial_{p_i}\partial_{p_j} w)^2 + (n_j^2(\partial_{p_i} w)^2 - n_i^2(\partial_{p_j} w)^2) + \frac{p_i^2 + p_j^2}{p_i^2 - p_j^2}(n_j\partial_{p_i} w - n_i\partial_{p_j} w)^2\right\}$$

The continuous Lagrangian $\mathscr{L}_{ij}$ obeys the closure relation:

$$\partial_{p_i}\mathscr{L}_{jk} + \partial_{p_j}\mathscr{L}_{ki} + \partial_{p_k}\mathscr{L}_{ij} = 0$$

on solutions of the continuous equation.
Since the PDE is of higher order, this requires additional relations, e.g.,

$$n_k(p_i^2 - p_j^2)\partial_{p_i}\partial_{p_j} w + n_i(p_j^2 - p_k^2)\partial_{p_j}\partial_{p_k} w + n_j(p_k^2 - p_i^2)\partial_{p_i}\partial_{p_k} w = 0$$

Nonlinear case

P∆E Fully discrete Lagrangian:

$$\mathscr{L}_{ij} = u(T_i u - T_j u) + (p_i^2 - p_j^2)\ln\left(T_i u - T_j u\right) \;,$$

Linear quadrilateral lattice equation(H1):

$$(u - T_i T_j u)(T_i u T_j u) + p_i^2 - p_j^2 = 0 \;.$$

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D∆E The D∆E analogue of H1 is given by the relations:

$$\frac{\partial u}{\partial p_i} = \frac{2n_i p_i}{T_i u - T_i^{-1} u}$$

It can be shown that this equation is compatible with H1.
A Lagrangian for the D∆E is given by

$$\mathscr{L}_i = u \frac{\partial}{\partial p_i} T_i u + 2n_i p_i \log\left(\frac{\partial}{\partial p_i} T_i u\right)$$

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PDE The following coupled system of PDEs follows from the DΔE and the PΔE:

$$\frac{\partial^2 u}{\partial p_i\,\partial p_j} = \frac{1}{p_i^2 - p_j^2}\left[2n_i p_i \frac{\partial u}{\partial p_j} - 2n_j p_j \frac{\partial u}{\partial p_i} - 2v_{ij}\frac{\partial u}{\partial p_i}\frac{\partial u}{\partial p_j}\right] ,$$

$$\frac{\partial^2 v_{ij}}{\partial p_i\,\partial p_j} = \frac{\partial}{\partial p_i}\left[\frac{(\partial u/\partial p_j)v_{ij}^2 + 2n_j p_j v_{ij}}{p_i^2 - p_j^2}\right] + \frac{\partial}{\partial p_j}\left[\frac{(\partial u/\partial p_i)v_{ij}^2 - 2n_i p_i v_{ij}}{p_i^2 - p_j^2}\right] ,$$

where $v_{ij} = (T_i - T_j)u$, and which derives as EL equations from the following Lagrangian:

$$\mathscr{L}_{ij} = v_{ij}\left(\frac{\partial^2 u}{\partial p_i\,\partial p_j} - \frac{2n_i p_i(\partial u/\partial p_j) - 2n_j p_j(\partial u/\partial p_i)}{p_i^2 - p_j^2}\right) + \frac{1}{p_i^2 - p_j^2} v_{ij}^2 \frac{\partial u}{\partial p_i}\,\frac{\partial u}{\partial p_j} .$$

The latter Lagrangian obeys the *closure relation*

$$\frac{\mathscr{L}_{ij}}{\partial p_k} + \frac{\mathscr{L}_{jk}}{\partial p_i} + \frac{\mathscr{L}_{ki}}{\partial p_j} = 0 ,$$

(albeit with a condition on the "form field" v_{ij}).
Alternatively, there is a Lagrangian for u alone yielding the higher-order PDE in u:

$$\mathscr{L}_{ij} = \frac{1}{4}(p_i^2 - p_j^2)\frac{(\partial_{p_i}\partial_{p_j} u)^2}{(\partial_{p_i} u)(\partial_{p_j} u)} + \frac{1}{p_i^2 - p_j^2}\left(n_i^2 p_i^2 \frac{\partial_{p_j} u}{\partial_{p_i} u} + n_j^2 p_j^2 \frac{\partial_{p_i} u}{\partial_{p_j} u}\right) .$$

Remark: The corresponding "hierarchy generating PDE" [FWN, A Hone and N Joshi, (2000)] (it encodes the entire KdV hierarchy), was shown to be closely related to the Ernst equation of General Relativity.

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Variational formalism for continuous Lagrangian 2-forms

In the continuous case, the closure relation suggests again that the $\mathscr{L}_{ij}$ should be considered as components of a *Lagrangian 2-form*, which is closed *on the solutions of the equations of the motion!*

The action S, defined on an arbitrary surface σ (embedded in an arbitrary number of dimensions) takes the form:

$$S[u(\mathbf{p});\sigma] = \int_\sigma \sum_{i<j} \mathscr{L}_{i,j} dp_i \wedge dp_j = \iint_\Omega \sum_{i<j} \left\{ \mathscr{L}_{i,j} \frac{\partial(p_i,p_j)}{\partial(s,t)} \right\} \mathrm{d}s\,\mathrm{d}t \;,$$

where $\mathscr{L}_{ij} = -\mathscr{L}_{ji}$, and where in the latter form we assume that the surface σ can be (smoothly) parametrized by functions $\mathbf{p}(s,t) = (p_i(s,t))$, Ω being an open set in the space of s and t-parameters. Here the Lagrangian typically takes the form:

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The variational equations, under which we demand S to be stationary, follow from two types of variations:

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Variational principle for Lagrange 1-forms

To propose a system of Lagrangians associated with higher-time variables $\mathbf{t} = (t_1, t_2, \dots)$ we consider a *Lagrangian 1-form*:

$$\mathbf{L} = \sum_k \mathscr{L}_k(\mathbf{x}(\mathbf{t}), \mathbf{x}_{t_1}(\mathbf{t}), \mathbf{x}_{t_2}(\mathbf{t}), \dots)\, dt_k$$

with components $\mathscr{L}_k$. The action becomes a functional of the type

$$S[\mathbf{x}(\mathbf{t}); \Gamma\,] = \int_\Gamma \mathbf{L}(\mathbf{x}(\mathbf{t}), \mathbf{x}_\mathbf{t}) = \int_{s_0}^{s_1} \sum_k \left(\mathscr{L}_k(\mathbf{x}(\mathbf{t}(s)), \mathbf{x}_{t_1}(\mathbf{t}(s)), \mathbf{x}_{t_2}(\mathbf{t}(s)), \dots) \frac{dt_k}{ds} \right) ds$$

where the functions $(\mathbf{t}(s))$, $s_0 \leq s \leq s_1$ form a parametrization of the curve Γ.
The variational calculus in terms of 1-forms involves two types of variations:

- variations $\mathbf{t}(s) \to \mathbf{t}(s) + \delta\mathbf{t}(s)$ of the curve, i.e. $\Gamma \to \Gamma'$
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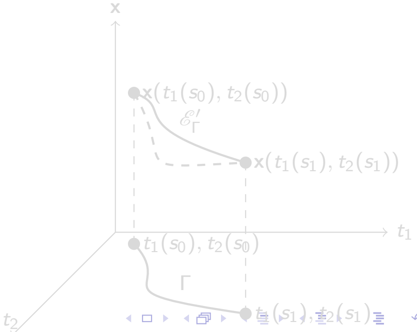

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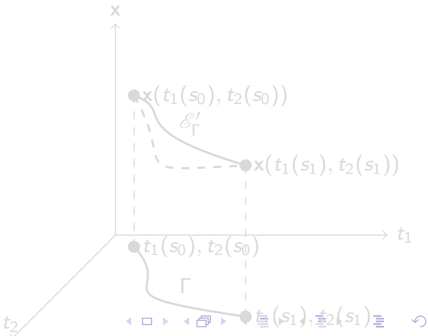

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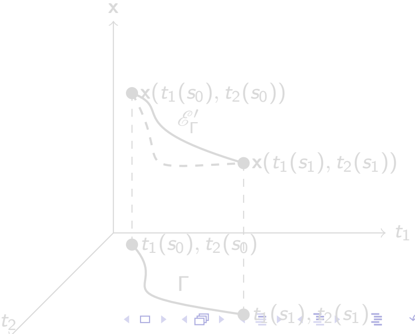
with components $\mathscr{L}_k$. The action becomes a functional of the type

$$S[\mathbf{x}(\mathbf{t}); \Gamma\,] = \int_\Gamma \mathbf{L}(\mathbf{x}(\mathbf{t}), \mathbf{x}_\mathbf{t}) = \int_{s_0}^{s_1} \sum_k \left(\mathscr{L}_k(\mathbf{x}(\mathbf{t}(s)), \mathbf{x}_{t_1}(\mathbf{t}(s)), \mathbf{x}_{t_2}(\mathbf{t}(s)), \dots) \frac{dt_k}{ds} \right) ds$$

where the functions $(\mathbf{t}(s))$, $s_0 \leq s \leq s_1$ form a parametrization of the curve Γ.

The variational calculus in terms of 1-forms involves two types of variations:

- variations $\mathbf{t}(s) \to \mathbf{t}(s) + \delta\mathbf{t}(s)$ of the curve, i.e. $\Gamma \to \Gamma'$
- variations $\mathbf{x}(\mathbf{t}) \to \mathbf{x} + \delta\mathbf{x}(\mathbf{t})$ of the dependent variables on a fixed curve Γ;



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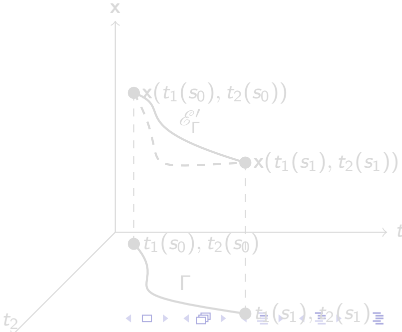

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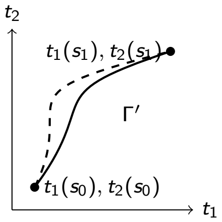

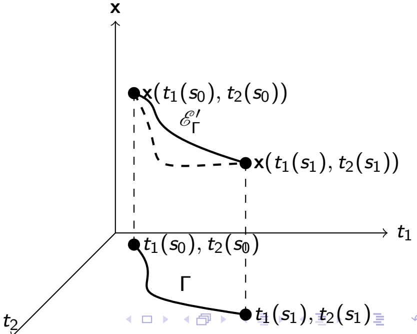

Generalized Euler-Lagrange equations

The corresponding set of EL equations (in the 2-time case) comprises the following relations:

- variations w.r.t. the *independent* variables leads to a *closure relation* in the form

$$\frac{\partial \mathscr{L}_2}{\partial t_1} = \frac{\partial \mathscr{L}_1}{\partial t_2} \ ,$$

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Remark: This scheme applies to the cases of the Calogero-Moser, the (finite) Toda chain and the Ruijsenaars-Schneider systems, and leads to the construction of the system of Lagrangians for the higher-order (commuting flows). These are mixed Lagrangians in all the time-derivatives.

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Possible implications for physics

1. Integrable systems seem to correspond to topological field theories. In fact, integrable hierarchies of matrix PDEs in 1+1 dimensions can be shown to have a Lagrangians formalism in terms of the Wess-Zumino-Witten-Novikov (WZNW) action with:[3]:

$$\mathscr{L} = \mathrm{tr}\left(\partial_\ell g \cdot \partial_{\ell'} g^{-1}\right) + \frac{\ell' + \ell}{\ell' - \ell} \int_0^1 \mathrm{d}t\, \mathrm{tr}\left([\partial_\ell g \cdot g^{-1}, \partial_{\ell'} g \cdot g^{-1}] \frac{dg}{dt} \cdot g^{-1} \right) .$$

(where the ℓ and ℓ' are variables like the lattice parameters p_i, i.e. so-called *Miwa variables*). In 2+1-dimensional hierarchies of integrable systems there are connections with Chern-Simons theory over loop algebras.

2. An unexpected relation between KdV theory and the Ernst equations of General Relativity (describing gravitational waves) emerge from the *generating PDEs*, which are the PDEs in terms of the lattice parameters p_i. In the case of the Boussinesq system those same equations are related to the Einstein-Maxwell-Weyl equations (for gravitational waves in the presence of Maxwell and neutrino fields). [Tongas, Tsoubelis & Xenitidis, 2001; Tongas & FN, 2005]

3. Within the Lagrangian multi-form scheme, "integrable Lagrangians" arise as critical points, reminiscent of *renormalization theory*. In fact, at those critical points the system loses "sensitivity for dimensionality" and allows for the co-existence of a discrete and continuous spaces of independent variables on which they are defined (insensitivity w.r.t. the lattice scale).

[3] F.W. Nijhoff, Integrable Hierarchies, Lagrangian Structures and Non-commuting Flows, Eds. M.J. Ablowitz, B. Fuchssteiner and M. Kruskal, in: Topics in Soliton Theory and Exactly Solvable Nonlinear Equations, pp. 150–181, Signapore, World Scientific Publ. Co. , 1987.

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$$\mathscr{L} = \mathrm{tr}\left(\partial_\ell g \cdot \partial_{\ell'} g^{-1}\right) + \frac{\ell'+\ell}{\ell'-\ell} \int_0^1 \mathrm{d}t\, \mathrm{tr}\left([\partial_\ell g \cdot g^{-1}, \partial_{\ell'} g \cdot g^{-1}] \frac{dg}{dt} \cdot g^{-1} \right) .$$

   (where the ℓ and ℓ' are variables like the lattice parameters p_i, i.e. so-called *Miwa variables*). In 2+1-dimensional hierarchies of integrable systems there are connections with Chern-Simons theory over loop algebras.

2. An unexpected relation between KdV theory and the Ernst equations of General Relativity (describing gravitational waves) emerge from the *generating PDEs*, which are the PDEs in terms of the lattice parameters p_i. In the case of the Boussinesq system those same equations are related to the Einstein-Maxwell-Weyl equations (for gravitational waves in the presence of Maxwell and neutrino fields). [Tongas, Tsoubelis & Xenitidis, 2001; Tongas & FN, 2005]

3. Within the Lagrangian multi-form scheme, "integrable Lagrangians" arise as critical points, reminiscent of *renormalization theory*. In fact, at those critical points the system loses "sensitivity for dimensionality" and allows for the co-existence of a discrete and continuous spaces of independent variables on which they are defined (insensitivity w.r.t. the lattice scale).

[3] F.W. Nijhoff, Integrable Hierarchies, Lagrangian Structures and Non-commuting Flows, Eds. M.J. Ablowitz, B. Fuchssteiner and M. Kruskal, in: Topics in Soliton Theory and Exactly Solvable Nonlinear Equations, pp. 150–181, Signapore, World Scientific Publ. Co. , 1987.

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Possible implications for quantum theory

Paul Dirac, in his seminal paper[4] of 1933, stated:
"The two formulations [namely that of Hamilton and of Lagrange] *are, of course, closely related but there are reasons for believing that the Lagrangian one is more fundamental."*
Dirac's paper contains already the key ideas underlying the path integral, later introduced by Feynman.

Based on the *Lagrangian 1-form structure* that was shown to hold for several many-body systems (e.g. the CM and RS system) one could make a (tentative) proposal for a quantum Lagrange multi-form theory in the form of a path integral. Thus, one could postulate (à la Feynman) a 1-form quantum propagator:

$$K(\mathbf{x}_b, \mathbf{t}_b, s_b; \mathbf{x}_a, \mathbf{t}_a, s_a) = \int_{\mathbf{t}(s_a)=\mathbf{t}_a}^{\mathbf{t}(s_b)=\mathbf{t}_b} [\mathscr{D}\mathbf{t}(s)] \int_{\mathbf{x}(\mathbf{t}_a)=\mathbf{x}_a}^{\mathbf{x}(\mathbf{t}_b)=\mathbf{x}_b} [\mathscr{D}_\Gamma \mathbf{x}(\mathbf{t})] \exp\left(\frac{i}{\hbar} S[\mathbf{x}(\mathbf{t});\Gamma]\right) .$$

Here:

- $[\mathscr{D}_\Gamma \mathbf{x}(\mathbf{t})]$ is some path integral measure along a curve Γ in the space of dependent variables $\mathbf{x}(\mathbf{t})$;
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