Open spin chains

Vincent PASQUIER

IPhT Saclay

Collaboration with Alexandre Lazarescu

Open XXZ with nonconserving boundaries

Many people made important contribution to the subject:

Open XXZ with nonconserving boundaries

Many people made important contribution to the subject:

From integrability point of view:

- Fabian Essler, Ian De Gier
- Ohlger Frahm
- Nicolai Kitanine
- Raphael Nepomechie
- Yupeng Wang
- Christian Korff
- Gulio Niccoli
- Eric Ragoucy, N. Crampé, D. Simpn
- ...

Non equilibrium

From nonequilibrium physics

- Bernard Derrida
- Joel Lebowitz
- Martin Evans
- Kirone Mallick
- Sylvain Prolhac
- $\cdots$

Non equilibrium

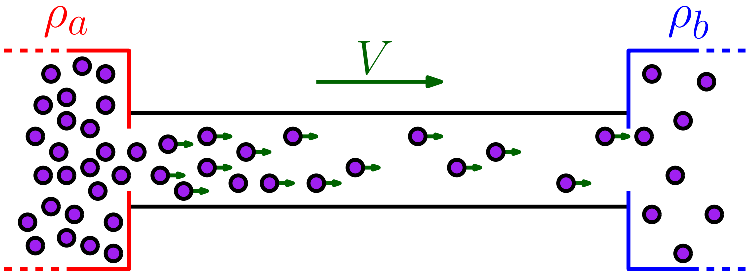
From nonequilibrium physics

- Bernard Derrida
- Joel Lebowitz
- Martin Evans
- Kirone Mallick
- Sylvain Prolhac
- $\cdots$

This Talk based on:
Alexandre thesis: arXiv 1311.7370
A.Lazarescu and V.P:arXiv 1403.6963

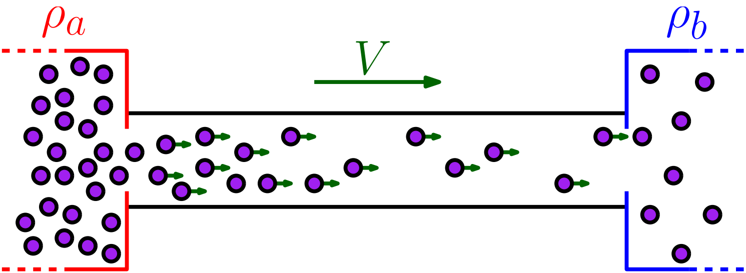
Introduction

Particles propagating and interacting between reservoirs.



Introduction

Particles propagating and interacting between reservoirs.

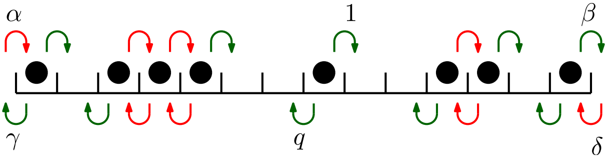


The field or the unbalance between reservoirs $\Rightarrow$ macroscopic current particles (entropy production).

Plan

- Introduction
- I – Open ASEP and quantum conductors
- II – Bethe Ansatz for XXX
- III – Comments about inhomogeneous Bethe Ansatz
- Conclusion

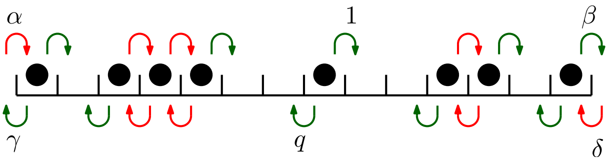
I - Asymetric exclusion model



Asymmetric Simple Exclusion Process, ASEP :

- One dimensional lattice L
- enter left with rate α and right with rate δ
- leave right with rate β and left with rate γ
- jump with rate $p = 1$ to the right et $q < 1$ to the left (if target is unoccupied)

I - Asymetric exclusion model

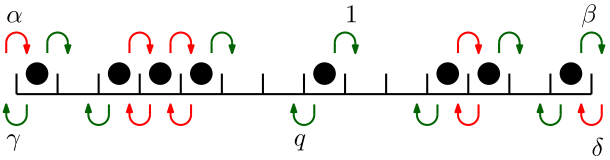


Asymmetric Simple Exclusion Process, ASEP :

- One dimensional lattice L
- enter left with rate α and right with rate δ
- leave right with rate β and left with rate γ
- jump with rate $p = 1$ to the right et $q < 1$ to the left (if target is unoccupied)

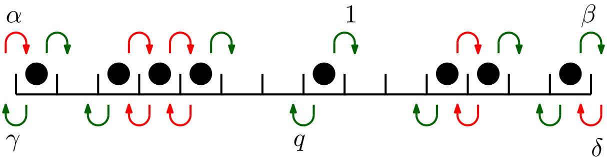
totally asymmetric (TASEP): $q = \gamma = \delta = 0$

I - Motivation



why ?

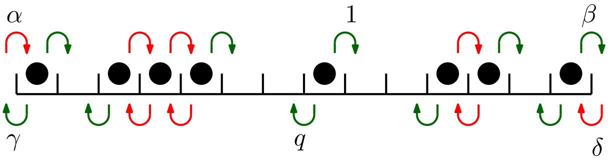
I - Motivation



why ?

- Simplicity

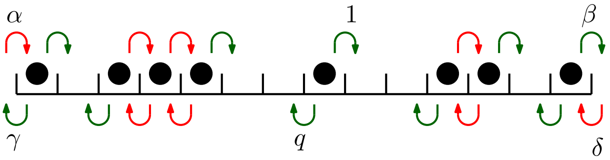
I - Motivation



why ?

- Simplicity
- Related to other topics (quantum XXZ, directed polymers, traffic, random matrices, orthogonal polynomials ...)

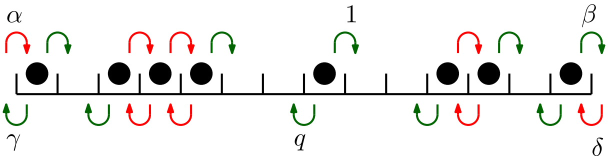
I - Motivation



why ?

- Simplicity
- Related to other topics (quantum XXZ, directed polymers, traffic, random matrices, orthogonal polynomials ...)
- Integrable $\Rightarrow$ susceptible of analytical results

I - Motivation

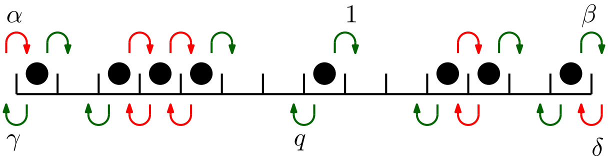


why ?

- Simplicity
- Related to other topics (quantum XXZ, directed polymers, traffic, random matrices, orthogonal polynomials ...)
- Integrable $\Rightarrow$ susceptible of analytical results

Interesting quantity :

I - Motivation

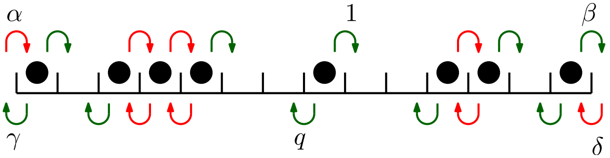


why ?

- Simplicity
- Related to other topics (quantum XXZ, directed polymers, traffic, random matrices, orthogonal polynomials ...)
- Integrable $\Rightarrow$ susceptible of analytical results

Interesting quantity : The macroscopic current.

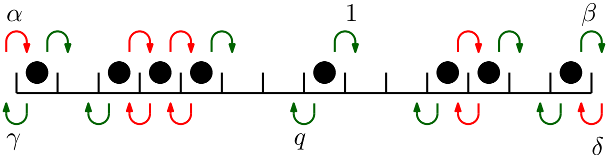
I - Asymetric exclusion model



Asymmetric Simple Exclusion Process, ASEP :

- One dimensional lattice L
- enter left with rate α and right with rate δ
- leave right with rate β and left with rate γ
- jump with rate $p = 1$ to the right et $q < 1$ to the left (if target is unoccupied)

I - Asymetric exclusion model

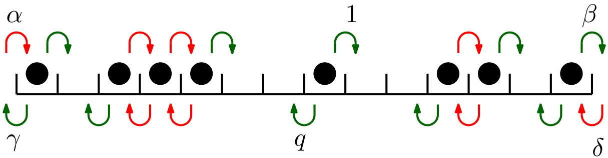


Asymmetric Simple Exclusion Process, ASEP :

- One dimensional lattice L
- enter left with rate α and right with rate δ
- leave right with rate β and left with rate γ
- jump with rate $p = 1$ to the right et $q < 1$ to the left (if target is unoccupied)

totally asymmetric (TASEP): $q = \gamma = \delta = 0$

I - Master equation



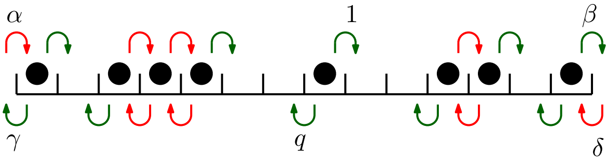
The vector $|P_t\rangle$ components are the probabilities to observe a configuration $\mathcal{C}$ at time t. It obeys the master equation:

$$\frac{d}{dt}|P_t\rangle = M|P_t\rangle$$

with M is a sum of local matrices M_i (one for each link i) STOCHASTIC (in the basis $\{0,1\}$ and $\{00,01,10,11\}$)

$$M_0 = \begin{bmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{bmatrix} , \; M_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \; M_L = \begin{bmatrix} -\delta & \beta \\ \delta & -\beta \end{bmatrix}$$

I - Master equation



The vector $|P_t\rangle$ components are the probabilities to observe a configuration $\mathcal{C}$ at time t. It obeys the master equation:

$$|P_t\rangle \to |P^\star\rangle$$

with M is a sum of local matrices M_i (one for each link i) STOCHASTIC (in the basis $\{0,1\}$ and $\{00,01,10,11\}$)

$$M_0 = \begin{bmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{bmatrix} , \; M_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \; M_L = \begin{bmatrix} -\delta & \beta \\ \delta & -\beta \end{bmatrix}$$

I - Quantum Hamiltonian

A different problem

The vector $|\Psi_t\rangle$ components are the amplitudes to observe a configuration $\mathcal{C}$ at time t. It obeys the master equation:

$$\frac{d}{dt}|\Psi_t\rangle = H|\Psi_t\rangle$$

with H is a sum of local matrices H_i (one for each link i) HERMITIAN (in the basis $\{0,1\}$ and $\{00,01,10,11\}$)

$$h_0 = \begin{bmatrix} -\alpha & \gamma \\ \bar{\gamma} & -\alpha \end{bmatrix} \;,\; H_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\Delta & 1 & 0 \\ 0 & 1 & -\Delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;,\; h_L = \begin{bmatrix} -\delta & \beta \\ -\bar{\beta} & -\delta \end{bmatrix}$$

I - Quantum Hamiltonian

A different problem

The vector $|\Psi_t\rangle$ components are the amplitudes to observe a configuration $\mathcal{C}$ at time t. It obeys the master equation:

$$|\Psi_t\rangle \to |P^\star\rangle$$

with H is a sum of local matrices H_i (one for each link i) HERMITIAN (in the basis $\{0,1\}$ and $\{00,01,10,11\}$)

$$h_0 = \begin{bmatrix} -\alpha & \gamma \\ \bar{\gamma} & -\alpha \end{bmatrix} \; , \; H_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\Delta & 1 & 0 \\ 0 & 1 & -\Delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; , \; h_L = \begin{bmatrix} -\delta & \beta \\ -\bar{\beta} & -\delta \end{bmatrix}$$

The Boundary matrices = Boundary Magnetic fields.

I - Quantum Hamiltonian

A different problem

The vector $|\Psi_t\rangle$ components are the amplitudes to observe a configuration $\mathcal{C}$ at time t. It obeys the master equation:

$$|\Psi_t\rangle \to |P^\star\rangle$$

with H is a sum of local matrices H_i (one for each link i) HERMITIAN (in the basis $\{0, 1\}$ and $\{00, 01, 10, 11\}$)

$$h_0 = \begin{bmatrix} -\alpha & \gamma \\ \bar{\gamma} & -\alpha \end{bmatrix} , \; H_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\Delta & 1 & 0 \\ 0 & 1 & -\Delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \; h_L = \begin{bmatrix} -\delta & \beta \\ -\bar{\beta} & -\delta \end{bmatrix}$$

The Boundary matrices = Boundary Magnetic fields.

Non parallel fields induce a current= helix classical sin configuration

I - Matrix Ansatz

[B. Derrida, M. R. Evans, V. Hakim, V.P., **J. Phys. A**, 1993]

Define matrices D et E and states $\langle\langle W\|$ et $\|V\rangle\rangle$ obeying

$$DE - qED = (1-q)(D+E)$$
$$\langle\langle W\|(\alpha E - \gamma D) = (1-q)\langle\langle W\|$$
$$(\beta D - \delta E)\|V\rangle\rangle = (1-q)\|V\rangle\rangle$$

I - Matrix Ansatz

[B. Derrida, M. R. Evans, V. Hakim, V.P., **J. Phys. A**, 1993]

Define matrices D et E and states $\langle\langle W\|$ et $\|V\rangle\rangle$ obeying

$$\begin{aligned} DE - qED &= (1-q)(D+E) \\ \langle\langle W\|(\alpha E - \gamma D) &= (1-q)\langle\langle W\| \\ (\beta D - \delta E)\|V\rangle\rangle &= (1-q)\|V\rangle\rangle \end{aligned}$$

Then, for $\mathcal{C} = \{n_i\}$, with $n_i = 0$ (hole) or 1 (particle),

$$P^\star(\mathcal{C}) = \frac{1}{Z_L}\langle\langle W\| \prod_{i=1}^{L} \left(n_i D + (1-n_i)E\right) \|V\rangle\rangle$$

where $Z_L = \langle\langle W\|(D+E)^L\|V\rangle\rangle$.

I - Matrix Ansatz

[B. Derrida, M. R. Evans, V. Hakim, V.P., **J. Phys. A**, 1993]

Define matrices D et E and states $\langle\langle W\|$ et $\|V\rangle\rangle$ obeying

$$DE - qED = (1-q)(D+E)$$
$$\langle\langle W\|(\alpha E - \gamma D) = (1-q)\langle\langle W\|$$
$$(\beta D - \delta E)\|V\rangle\rangle = (1-q)\|V\rangle\rangle$$

Then, for $\mathcal{C} = \{n_i\}$, with $n_i = 0$ (hole) or 1 (particle),

$$P^\star(\mathcal{C}) = \frac{1}{Z_L}\langle\langle W\| \prod_{i=1}^{L} \left(n_i D + (1-n_i)E\right) \|V\rangle\rangle$$

where $Z_L = \langle\langle W\|(D+E)^L\|V\rangle\rangle$.

for example, $P^\star(110101) = \frac{1}{Z_6}\langle\langle W\|DDEDED\|V\rangle\rangle$.

I - Matrix Ansatz

[B. Derrida, M. R. Evans, V. Hakim, V.P., **J. Phys. A**, 1993]

Define matrices D et E and states $\langle\langle W \|$ et $\| V \rangle\rangle$ obeying

$$\begin{aligned} DE - qED &= (1-q)(D+E) \\ \langle\langle W \| (\alpha E - \gamma D) &= (1-q) \langle\langle W \| \\ (\beta D - \delta E) \| V \rangle\rangle &= (1-q) \| V \rangle\rangle \end{aligned}$$

Then, for $\mathcal{C} = \{n_i\}$, with $n_i = 0$ (hole) or 1 (particle),

$$P^\star(\mathcal{C}) = \frac{1}{Z_L} \langle\langle W \| \prod_{i=1}^{L} \left(n_i D + (1-n_i) E \right) \| V \rangle\rangle$$

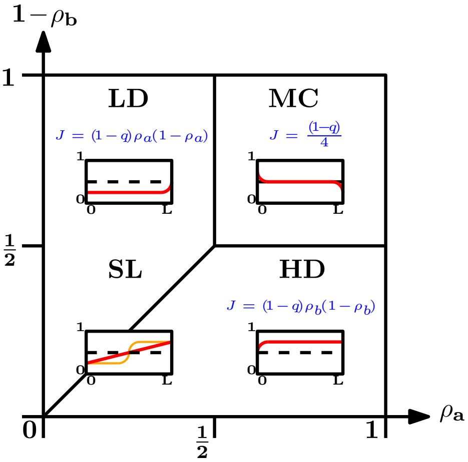
where $Z_L = \langle\langle W \| (D+E)^L \| V \rangle\rangle$.

for example, $P^\star(110101) = \frac{1}{Z_6} \langle\langle W \| DDEDED \| V \rangle\rangle$.

One deduces the mean current $J = (1-q) \frac{Z_{L-1}}{Z_L}$

I - Phase Diagram

As a function of $\rho_a(\alpha, \gamma, q)$ and $\rho_b(\beta, \delta, q)$:



In each case : $J = (1-q)\rho_c(1-\rho_c)$.

II - Large Deviation

History $\mathcal{C}(t)$ with a current $Q_t[\mathcal{C}]$.

II - Large Deviation

History $\mathcal{C}(t)$ with a current $Q_t[\mathcal{C}]$.

Large deviations for J_t

Probability $Q_t = tj$: for $t \to \infty$,

$$\mathrm{P}\big(Q_t = tj\big) \sim \mathrm{e}^{-tg(j)}$$

$g(j)$ is the large deviation function.

II - Large Deviation

History $\mathcal{C}(t)$ with a current $Q_t[\mathcal{C}]$.

Large deviations for J_t

Probability $Q_t = tj$: for $t \to \infty$,

$$\mathrm{P}\big(Q_t = tj\big) \sim \mathrm{e}^{-tg(j)}$$

$g(j)$ is the large deviation function.

Generating function : $E(\mu)$ such that $t \to \infty$,

$$\mathrm{e}^{tE(\mu)} = \left\langle \mathrm{e}^{\mu Q_t} \right\rangle$$

II - Large Deviation

History $\mathcal{C}(t)$ with a current $Q_t[\mathcal{C}]$.

Large deviations for J_t

Probability $Q_t = tj$: for $t \to \infty$,

$$\mathrm{P}\big(Q_t = tj\big) \sim \mathrm{e}^{-tg(j)}$$

$g(j)$ is the large deviation function.

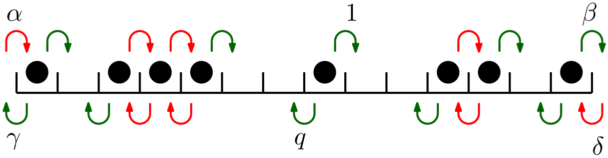
Generating function : $E(\mu)$ such that $t \to \infty$,

$$\mathrm{e}^{tE(\mu)} = \left\langle \mathrm{e}^{\mu Q_t} \right\rangle$$

Gärtner-Ellis

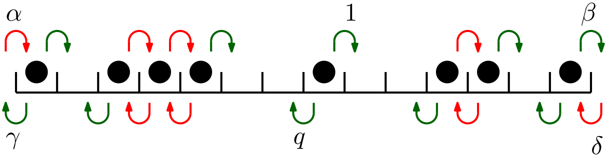
Theorem:

$$E(\mu) = \mu j^\star - g(j^\star) \quad , \quad \frac{\mathrm{d}}{\mathrm{d}j} g(j^\star) = \mu$$



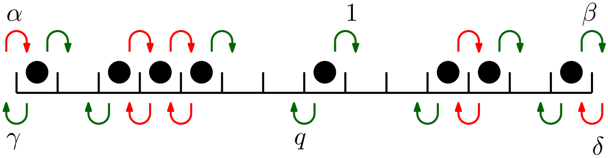
α
1
β
γ
q
δ

II - Measure the current



$$M_0 = \begin{bmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{bmatrix}$$

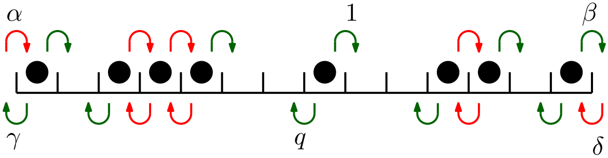
II - Measure the current



$$M_0(\mu) = \begin{bmatrix} -\alpha & \gamma\, e^{-\mu} \\ \alpha\, e^{\mu} & -\gamma \end{bmatrix}$$

Add fugacities to the off diagonal terms enables to count the current of particles.

II - Measure the current

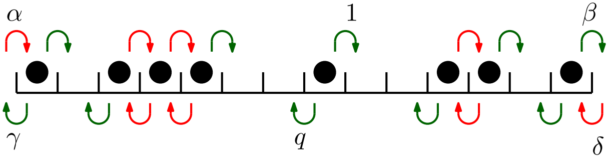


$$M_0(\mu) = \begin{bmatrix} -\alpha & \gamma \ e^{-\mu} \\ \alpha \ e^{\mu} & -\gamma \end{bmatrix}$$

Add fugacities to the off diagonal terms enables to count the current of particles.

$$M_\mu = M_0(\mu) + \sum_{i=1}^{L-1} M_i + M_L$$

II - Measure the current



$$M_0(\mu) = \begin{bmatrix} -\alpha & \gamma\ e^{-\mu} \\ \alpha\ e^{\mu} & -\gamma \end{bmatrix}$$

Add fugacities to the off diagonal terms enables to count the current of particles.

$$M_\mu = M_0(\mu) + \sum_{i=1}^{L-1} M_i + M_L$$

$E(\mu)$ largest eigenvalue M_μ

Corresponding eigenvectors : $|P_\mu\rangle$ and $\langle \tilde{P}_\mu|$

IV - Strategy- closed chain XXX

- Obtain a Two Parameters $T(u, v)$ family of transfer matrices commuting with the Hamiltonian

IV - Strategy- closed chain XXX

- Obtain a Two Parameters $T(u, v)$ family of transfer matrices commuting with the Hamiltonian

- Factorize it as $T(u, v) = P(u)Q(v)$

IV - Strategy- closed chain XXX

- Obtain a Two Parameters $T(u,v)$ family of transfer matrices commuting with the Hamiltonian

- Factorize it as $T(u,v) = P(u)Q(v)$

- For specific values of $u+v=1-k$ observe it decomposes as:

$$T(u,1-k-u) = t^{(k)}(u) + N_k T(u+k,1-u)$$

IV - Strategy- closed chain XXX

- Obtain a Two Parameters $T(u,v)$ family of transfer matrices commuting with the Hamiltonian

- Factorize it as $T(u,v) = P(u)Q(v)$

- For specific values of $u+v=1-k$ observe it decomposes as:

$$T(u,1-k-u) = t^{(k)}(u) + N_k T(u+k,1-u)$$

- combine with factorization:

$$\boxed{P(u)Q(1-k-u) = t^{(k)}(u) + N_k P(u+k)Q(1-u)}$$

IV - Strategy- closed chain XXX

- Obtain a Two Parameters $T(u,v)$ family of transfer matrices commuting with the Hamiltonian

- Factorize it as $T(u,v) = P(u)Q(v)$

- For specific values of $u+v = 1-k$ observe it decomposes as:

$$T(u, 1-k-u) = t^{(k)}(u) + N_k T(u+k, 1-u)$$

- combine with factorization:

$$\boxed{P(u)Q(1-k-u) = t^{(k)}(u) + N_k P(u+k)Q(1-u)}$$

- Use the above equation with $k = 1, 2$ to obtain the spectrum.

Bethe Ansatz- closed chain XXX

- First two relations:

$$Q(v)P(-v) = h(v) + N_1 Q(1+v)P(1-v)$$
$$Q(v)P(-v-1) = t^{(2)}(v) + N_2 Q(2+v)P(1-v)$$

First relation is the the Wronsky identity.

Bethe Ansatz- closed chain XXX

- First two relations:

$$Q(v)P(-v) = h(v) + N_1 Q(1+v)P(1-v)$$
$$Q(v)P(-v-1) = t^{(2)}(v) + N_2 Q(2+v)P(1-v)$$

First relation is the the Wronsky identity.

- which combine into

$$t^{(2)}(v)Q(v+1) = h(v+1)Q(v) + N_1 h(v)Q(v+2)$$

Transfer Matrix- closed chain XXX

- Use Oscillatory internal space:

$$S^+ = x^2 \frac{d}{dx} + 2\lambda x$$
$$S^- = -\frac{d}{dx}$$
$$S^3 = x\frac{d}{dx} + \lambda$$

Transfer Matrix- closed chain XXX

- Use Oscillatory internal space:

$$S^+ = x^2 \frac{d}{dx} + 2\lambda x$$
$$S^- = -\frac{d}{dx}$$
$$S^3 = x\frac{d}{dx} + \lambda$$

- construct Lax matrix:

$$L(z) = \begin{pmatrix} z + S^3 & S^- \\ S^+ & z - S^3 \end{pmatrix},$$

Transfer Matrix- closed chain XXX

- Use Oscillatory internal space:

$$S^+ = x^2 \frac{d}{dx} + 2\lambda x$$
$$S^- = -\frac{d}{dx}$$
$$S^3 = x\frac{d}{dx} + \lambda$$

- construct Lax matrix:

$$L(z) = \begin{pmatrix} z + S^3 & S^- \\ S^+ & z - S^3 \end{pmatrix},$$

- construct Transfer Matrix:

$$T(u,v) = tr(L_1(z,\lambda)\cdots L_N(z,\lambda))$$

with:

$$\lambda = \frac{u+v}{2},\ z = \frac{u-v}{2}$$

Commutes with the Hamiltonian for all u, v.

Transfer Matrix Open chain XXX

- We require H to commute with the product of two transfer matrices:

$$\Lambda(v_1, v_2) = \Lambda(v_1)\Lambda'(v_2) = W^L.T_{v_1} W^R * W'^L.T_{v_2} W'^R$$

Transfer Matrix Open chain XXX

- We require H to commute with the product of two transfer matrices:

$$\Lambda(v_1, v_2) = \Lambda(v_1)\Lambda'(v_2) = W^L.T_{v_1} W^R * W'^L.T_{v_2} W'^R$$

- Boundary vectors are coherent states

$$(2h_R.S_{v_1} + 1)W = 0$$
$$(2h_R.S_{v_2} - 1)W' = 0$$

- Boundary Hamiltonian with spinors:

$$h = \frac{1}{\mu(r_1 - r_2)} \begin{pmatrix} \frac{r_1+r_2}{2} & -r_1 r_2 \\ 1 & -\frac{r_1+r_2}{2} \end{pmatrix}.$$

$$W = (1 - r_1 x)^{-\mu - v_1}(1 - r_2 x)^{\mu - v_1}$$
$$W' = (1 - r_1 x')^{\mu - v_2}(1 - r_2 x')^{-\mu - v_2}.$$

K Matrix XXX

- Can be rewritten using K-Matrix

$$\Lambda(v_1, v_2) = tr(K^L_{v_2,v_1} T(v_1, v_2) K^R_{v_1,v_2} \tilde{T}^T(v_2, v_1)),$$

$$\boxed{K = \int_0^\infty dt t^{v_1+v_2-1} G_{v_1,v_2}(t, \mu, x, r_1, r_2) G_{v_1,v_2}(t, -\mu, y, r_1, r_2)}$$

K Matrix XXX

- Can be rewritten using K-Matrix

$$\Lambda(v_1, v_2) = tr(K^L_{v_2,v_1} T(v_1, v_2) K^R_{v_1,v_2} \tilde{T}^T(v_2, v_1)),$$

$$\boxed{K = \int_0^\infty dt t^{v_1+v_2-1} G_{v_1,v_2}(t, \mu, x, r_1, r_2) G_{v_1,v_2}(t, -\mu, y, r_1, r_2)}$$

with:

$$G_{v_1,v_2}(t, \mu, r, x, y) = (1 - xr)^{-v_1-v_2}(1 + t\tfrac{1-yr}{1-xr})^{\mu-v_2}$$

K Matrix XXX

- Can be rewritten using K-Matrix

$$\Lambda(v_1, v_2) = tr(K^L_{v_2,v_1} T(v_1, v_2) K^R_{v_1,v_2} \tilde{T}^T(v_2, v_1)),$$

$$\boxed{K = \int_0^\infty dt t^{v_1+v_2-1} G_{v_1,v_2}(t, \mu, x, r_1, r_2) G_{v_1,v_2}(t, -\mu, y, r_1, r_2)}$$

with:

$$G_{v_1,v_2}(t, \mu, r, x, y) = (1 - xr)^{-v_1-v_2}(1 + t\frac{1-yr}{1-xr})^{\mu-v_2}$$

On which you can observe the symmetry:

$$K(x, y, \mu) = K(y, x, -\mu)$$

Bethe Equations for open chain XXX

- TQ equation:

$$\lambda_2(u)q(u) = b(-u)q(u-1) + b(u)q(u+1)$$

Bethe Equations for open chain XXX

- TQ equation:

$$\lambda_2(u)q(u) = b(-u)q(u-1) + b(u)q(u+1)$$

$$b(u) = \frac{1}{4u(u+\frac{1}{2})}(u+\tfrac{1}{2}-\mu_L)(u+\tfrac{1}{2}-\mu_R)\prod_{i=1}^{N}(u-\tfrac{1}{2}-\alpha_i)(u-\tfrac{1}{2}+\alpha_i)$$

Bethe Equations for open chain XXX

- TQ equation:

$$\lambda_2(u)q(u) = b(-u)q(u-1) + b(u)q(u+1)$$

$$b(u) = \frac{1}{4u(u+\frac{1}{2})}(u+\tfrac{1}{2}-\mu_L)(u+\tfrac{1}{2}-\mu_R)\prod_{i=1}^{N}(u-\tfrac{1}{2}-\alpha_i)(u-\tfrac{1}{2}+\alpha_i)$$

- Incompatible with polynomial at infinity if boundary fields are not parallel

Bethe Equations for open chain XXX

- TQ equation:

$$\lambda_2(u)q(u) = b(-u)q(u-1) + b(u)q(u+1)$$

$$b(u) = \frac{1}{4u(u+\frac{1}{2})}(u+\tfrac{1}{2}-\mu_L)(u+\tfrac{1}{2}-\mu_R)\prod_{i=1}^{N}(u-\tfrac{1}{2}-\alpha_i)(u-\tfrac{1}{2}+\alpha_i)$$

- Incompatible with polynomial at infinity if boundary fields are not parallel
- Wang et al. propose to modify with inhomogeneous term., but we were able to solve it in some cases without? at the price of nonsymmetric, nonpolynomial solutions.

II - Cumulants : ASEP XXZ [Gorissen, Lazarescu, Mallick, Vanderzande; P.R.L., 2012]

$$E(\mu) = -(1-q)\oint_S \frac{dz}{\imath 2\pi(1+z)^2} W_B(z) \qquad = -(1-q)\sum_{k=1}^{\infty} D_k \frac{B^k}{k}$$

$$\mu = -\oint_S \frac{dz}{\imath 2\pi z} W_B(z) \qquad = -\sum_{k=1}^{\infty} C_k \frac{B^k}{k}$$

II - Cumulants : ASEP XXZ [Gorissen, Lazarescu, Mallick, Vanderzande; P.R.L., 2012]

$$E(\mu) = -(1-q)\oint_S \frac{dz}{\imath 2\pi (1+z)^2} W_B(z) \qquad = -(1-q)\sum_{k=1}^{\infty} D_k \frac{B^k}{k}$$

$$\mu = -\oint_S \frac{dz}{\imath 2\pi z} W_B(z) \qquad = -\sum_{k=1}^{\infty} C_k \frac{B^k}{k}$$

avec

$$W_B(z) = -\frac{1}{2}\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

II - Cumulants : ASEP XXZ [Gorissen, Lazarescu, Mallick, Vanderzande; P.R.L., 2012]

$$E(\mu) = -(1-q)\oint_S \frac{dz}{\imath 2\pi (1+z)^2} W_B(z) \qquad = -(1-q)\sum_{k=1}^{\infty} D_k \frac{B^k}{k}$$

$$\mu = -\oint_S \frac{dz}{\imath 2\pi z} W_B(z) \qquad = -\sum_{k=1}^{\infty} C_k \frac{B^k}{k}$$

avec

$$W_B(z) = -\frac{1}{2}\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

avec

$$F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(az)_\infty(az^{-1})_\infty(\tilde{a}z)_\infty(\tilde{a}z^{-1})_\infty(bz)_\infty(bz^{-1})_\infty(\tilde{b}z)_\infty(\tilde{b}z^{-1})_\infty}$$

$$X[f](z) = \oint_S \frac{d\tilde{z}}{\imath 2\pi \tilde{z}} f(\tilde{z})K(z,\tilde{z})$$

II - Cumulants : ASEP XXZ [Gorisssen, Lazarescu, Mallick, Vanderzande; P.R.L., 2012]

$$E(\mu) = -(1-q)\oint_S \frac{dz}{\imath 2\pi (1+z)^2} W_B(z) \qquad = -(1-q)\sum_{k=1}^{\infty} D_k \frac{B^k}{k}$$

$$\mu = -\oint_S \frac{dz}{\imath 2\pi z} W_B(z) \qquad = -\sum_{k=1}^{\infty} C_k \frac{B^k}{k}$$

avec

$$W_B(z) = -\frac{1}{2}\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

avec

$$F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(az)_\infty(az^{-1})_\infty(\tilde{a}z)_\infty(\tilde{a}z^{-1})_\infty(bz)_\infty(bz^{-1})_\infty(\tilde{b}z)_\infty(\tilde{b}z^{-1})_\infty}$$

$$X[f](z) = \oint_S \frac{d\tilde{z}}{\imath 2\pi \tilde{z}} f(\tilde{z})K(z,\tilde{z})$$

avec

$$(x)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$$

$$K(z,\tilde{z}) = 2\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\left((z/\tilde{z})^k + (z/\tilde{z})^{-k}\right)$$

II - Cumulants : ASEP XXZ [Gorissen, Lazarescu, Mallick, Vanderzande; P.R.L., 2012]

$$E(\mu) = -(1-q)\oint_S \frac{dz}{\imath 2\pi(1+z)^2} W_B(z) \qquad = -(1-q)\sum_{k=1}^{\infty} D_k \frac{B^k}{k}$$

$$\mu = -\oint_S \frac{dz}{\imath 2\pi z} W_B(z) \qquad = -\sum_{k=1}^{\infty} C_k \frac{B^k}{k}$$

avec

$$W_B(z) = -\frac{1}{2}\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

avec

$$F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(az)_\infty(az^{-1})_\infty(\tilde{a}z)_\infty(\tilde{a}z^{-1})_\infty(bz)_\infty(bz^{-1})_\infty(\tilde{b}z)_\infty(\tilde{b}z^{-1})_\infty}$$

$$X[f](z) = \oint_S \frac{d\tilde{z}}{\imath 2\pi\tilde{z}} f(\tilde{z})K(z,\tilde{z})$$

avec

$$(x)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$$

$$K(z,\tilde{z}) = 2\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\left((z/\tilde{z})^k + (z/\tilde{z})^{-k}\right)$$

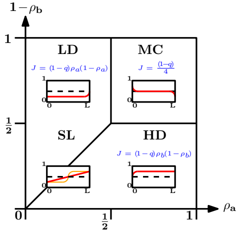
Similaire à [S. Prolhac, **J. Phys. A**, 2010] pour le ASEP périodique.

II - Cumulants : Large size

- low density :

$$E(\mu) = (1-q)\left(\frac{e^{\mu}}{e^{\mu}+a} - \frac{1}{1+a}\right)$$

de Gier Essler PRL 2011.



II - Cumulants : Large size

- low density :

$$E(\mu) = (1-q)\left(\frac{e^{\mu}}{e^{\mu}+a} - \frac{1}{1+a}\right)$$

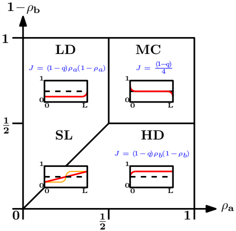
de Gier Essler PRL 2011.

- maximal current phase:

$$E(\mu) - \frac{(1-q)}{4}\mu = -(1-q)\frac{L^{-3/2}}{16\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(2k)!}{k!k^{(k+5/2)}}B^k$$

$$\mu = -\frac{L^{-1/2}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(2k)!}{k!k^{(k+3/2)}}B^k$$

$E_k \sim L^{(k-3)/2}$



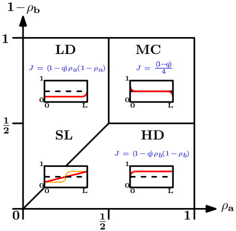
II - Cumulants : Large size

- low density :

$$E(\mu) = (1-q)\left(\frac{e^{\mu}}{e^{\mu}+a} - \frac{1}{1+a}\right)$$

de Gier Essler PRL 2011.

- maximal current phase:



$$\begin{aligned} E(\mu) - \frac{(1-q)}{4}\mu &= -(1-q)\frac{L^{-3/2}}{16\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+5/2)}} B^k \\ \mu &= \quad -\frac{L^{-1/2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(2k)!}{k!k^{(k+3/2)}} B^k \end{aligned}$$

$E_k \sim L^{(k-3)/2}$

- choc line :

$$\begin{aligned} E(\mu) - (1-q)\frac{a}{(1+a)^2}\mu &= -(1-q)\frac{2}{L^2}\frac{a}{a^2-1} \sum_{k=1}^{\infty} \frac{k^{2k-2}}{(2k)!} B^k \\ \mu &= \quad -\frac{2}{L}\frac{a+1}{a-1} \sum_{k=1}^{\infty} \frac{k^{2k-1}}{(2k)!} B^k \end{aligned}$$

$E_k \sim L^{k-2}$.

Conclusion

We have:

- Constructed a Q matrix commuting wth the ASEP Hamiltonian
- Used it to obtain functional equations equivalent to the Bethe Ansatz
- Used analiticity properties to obtain a particular eigenvalue yielding the large deviation function.

Conclusion

We have:

- Constructed a Q matrix commuting wth the ASEP Hamiltonian
- Used it to obtain functional equations equivalent to the Bethe Ansatz
- Used analiticity properties to obtain a particular eigenvalue yielding the large deviation function.

We would like to:

- Explore more ahead our results especially in the maximal current phase.
- Obtain the correct analiticity properties to solve XXZ chain and other models with open boundary.
- Simplify our derivations.

Thank you!

Bethe Equations

Bethe equations?

Bethe Equations

Bethe equations?

- Unknown are $t(u)$ and $q(u) = x^u \prod_{i=1}^{k}(u - u_i)$ polynomials of degree N and k.

Bethe Equations

Bethe equations?

- Unknown are $t(u)$ and $q(u) = x^u \prod_{i=1}^{k}(u - u_i)$ polynomials of degree N and k.
- known $b(u) = \prod_{i=1}^{N}(u - \alpha_i)$

Bethe Equations

Bethe equations?

- Unknown are $t(u)$ and $q(u) = x^u \prod_{i=1}^{k}(u - u_i)$ polynomials of degree N and k.
- known $b(u) = \prod_{i=1}^{N}(u - \alpha_i)$
- Baxter equation:

$$t\hat{q} = b_+\hat{q}_- + b\hat{q}_+,$$

Bethe Equations

Bethe equations?

- Unknown are $t(u)$ and $q(u) = x^u \prod_{i=1}^{k}(u - u_i)$ polynomials of degree N and k.
- known $b(u) = \prod_{i=1}^{N}(u - \alpha_i)$
- Baxter equation:

$$t\hat{q} = b_+\hat{q}_- + b\hat{q}_+,$$

- Bethe equations:

$$\frac{q_-(u_i)}{q_+(u_i)} = -\frac{b(u_i)}{b_+(u_i)}$$

Pronko Stroganov other Side of the equator

- There is a second solution $r(u) = x^{-u} \prod_{i=1}^{N-k} (u - v_i)$ of degree N-k.

Pronko Stroganov other Side of the equator

- There is a second solution $r(u) = x^{-u} \prod_{i=1}^{N-k} (u - v_i)$ of degree N-k.

$$t\hat{r} = b_+ \hat{r}_- + b\hat{r}_+,$$

Pronko Stroganov other Side of the equator

- There is a second solution $r(u) = x^{-u} \prod_{i=1}^{N-k}(u - v_i)$ of degree N-k.

$$t\hat{r} = b_+\hat{r}_- + b\hat{r}_+,$$

- Satisfy Wronsky identity:

$$\frac{b}{\hat{q}\hat{q}_-} = \frac{\hat{r}}{\hat{q}} - \frac{\hat{r}_-}{\hat{q}_-}$$

Why should the right hand side be dividable by b?

Pronko Stroganov other Side of the equator

- There is a second solution $r(u) = x^{-u} \prod_{i=1}^{N-k}(u - v_i)$ of degree N-k.

$$t\hat{r} = b_+\hat{r}_- + b\hat{r}_+,$$

- Satisfy Wronsky identity:

$$\frac{b}{\hat{q}\hat{q}_-} = \frac{\hat{r}}{\hat{q}} - \frac{\hat{r}_-}{\hat{q}_-}$$

Why should the right hand side be dividable by b?

- Simple element decomposition of Baxter$/\hat{q}\hat{q}_-\hat{q}_+$ gives

Pronko-Stroganov identity:

$$\frac{b}{\hat{q}\hat{q}_-} = \frac{\hat{\alpha}}{\hat{q}} - \frac{\hat{\alpha}_-}{\hat{q}_-} + \hat{\beta}$$

Pronko Stroganov other Side of the equator

- There is a second solution $r(u) = x^{-u} \prod_{i=1}^{N-k}(u - v_i)$ of degree N-k.

$$t\hat{r} = b_+\hat{r}_- + b\hat{r}_+,$$

- Satisfy Wronsky identity:

$$\frac{b}{\hat{q}\hat{q}_-} = \frac{\hat{r}}{\hat{q}} - \frac{\hat{r}_-}{\hat{q}_-}$$

Why should the right hand side be dividable by b?

- Simple element decomposition of Baxter$/\hat{q}\hat{q}_-\hat{q}_+$ gives

Pronko-Stroganov identity:

$$\frac{b}{\hat{q}\hat{q}_-} = \frac{\hat{\alpha}}{\hat{q}} - \frac{\hat{\alpha}_-}{\hat{q}_-} + \hat{\beta}$$

- solve $\hat{\beta} = \hat{h} - \hat{h}_-$ to obtain:

$$\frac{\hat{r}}{\hat{q}} = \frac{\hat{\alpha}}{\hat{q}} + \hat{h}$$

Inhomogeneous Bethe Equations

Homogeneous B.A. can always be made inhomogeneous.

Inhomogeneous Bethe Equations

Homogeneous B.A. can always be made inhomogeneous.

- We can always recast B.A.:

$$t\hat{Q} = b_+\hat{Q}_- + b\hat{Q}_+ + (x + x^{-1} - y - y^{-1})bb_+y^u$$

with $Q(u) = y^u$ polynomial of degree N.

Inhomogeneous Bethe Equations

Homogeneous B.A. can always be made inhomogeneous.

- We can always recast B.A.:

$$t\hat{Q} = b_+\hat{Q}_- + b\hat{Q}_+ + (x + x^{-1} - y - y^{-1})bb_+y^u$$

with $Q(u) = y^u$ polynomial of degree N.

- apply variation of parameters to obtain:

$$\hat{c}_+ - \hat{c} = (y + y^{-1} - x - x^{-1})\hat{q}y^u$$

$$\frac{\hat{c}\hat{b}}{\hat{q}\hat{q}_-} = \frac{\hat{Q}}{\hat{q}} - \frac{\hat{Q}_-}{\hat{q}_-}$$

Inhomogeneous Bethe Equations

Homogeneous B.A. can always be made inhomogeneous.

- We can always recast B.A.:

$$t\hat{Q} = b_+ \hat{Q}_- + b\hat{Q}_+ + (x + x^{-1} - y - y^{-1}) b b_+ y^u$$

with $Q(u) = y^u$ polynomial of degree N.

- apply variation of parameters to obtain:

$$\hat{c}_+ - \hat{c} = (y + y^{-1} - x - x^{-1}) \hat{q} y^u$$

$$\frac{\hat{c}\hat{b}}{\hat{q}\hat{q}_-} = \frac{\hat{Q}}{\hat{q}} - \frac{\hat{Q}_-}{\hat{q}_-}$$

- How can you know $Q(u)$ is polynomial?.

Use Pronko-Stroganov identity!

$$\hat{Q} = \hat{\alpha}\hat{c} + \hat{q}\hat{h}$$

with $\hat{h}$ solution of:

$$\hat{h} - \hat{h}_- = \beta - (y + y^{-1} - x - x^{-1}) \hat{\alpha}_- / y$$