Exact Solution of Critical Dense Polymers with Robin Boundary Conditions

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History

- Critical dense polymers: de Gennes 1979, Duplantier 1986, Saleur 1986/1992, ...
- Generic loop models: Nienhuis 1987, Blöte-Nienhuis 1989, ...
- Logarithmic Minimal Models \( \mathcal{L}M(p, p') \): Pearce-Rasmussen-Zuber 2006, ...
- Critical dense polymers \( \mathcal{L}M(1, 2) \) solved exactly as a Yang-Baxter integrable loop model on the square lattice: Pearce-Rasmussen 2007, ...
- The commuting transfer matrices of \( \mathcal{L}M(1, 2) \) satisfy inversion identities:

\[
D(u)D(u + \lambda) = f(u)I 
\Rightarrow \text{integrable on finite lattices}
\]

Critical dense polymers is exactly solvable on arbitrary size finite lattices for all topologies and all Yang-Baxter integrable boundary conditions!

- As a CFT, dense polymers is a protoypical logarithmic CFT described by symplectic fermions. It admits \((r, s)\) boundary conditions with conformal weights given by the Kac formula

\[
\Delta_{r,s} = \frac{(2r - s)^2 - 1}{8}, \quad r, s = 1, 2, 3, \ldots
\]

- The Kac representations do not exhaust all of the possible representations. There should exist boundary conditions yielding conformal weights with half-integer Kac labels: Saleur 1987, Duplantier 1986, Delfino 2013, ...
Critical Dense Polymers

- **Logarithmic Minimal Models:** Yang-Baxter integrable loop models on the square lattice. Face operators defined in diagrammatic planar Temperley-Lieb algebra *(Jones 1999)*

\[ X(u) = u = \sin(\lambda - u) + \sin u \]

1 ≤ p < p' coprime integers, \( \lambda = \frac{(p' - p)\pi}{p'} \) = crossing parameter

\( u = \) spectral parameter, \( \beta = 2 \cos \lambda = \) nonlocal loop fugacity

- **Critical Dense Polymers:** \((p, p') = (1, 2), \quad \lambda = \frac{\pi}{2}\)

\[ Z = \sum_{\text{loop configs}} \sin^{N_1}(\lambda - u) \sin^{N_2} u, \]

\( \beta = 0 \Rightarrow \) no closed loops \( \Rightarrow \) space filling dense polymer

\[ \Delta_{\text{path}}^{\text{SLE}} = 2 - 2\Delta_{1,1} = 2 \]

- There are no local degrees of freedom only **nonlocal degrees of freedom** in the form of extended polymer segments!
Commuting Double-Row Transfer Matrices on a Strip

Double row transfer matrix (nontrivial boundary on right)

\[
D(u) = \begin{bmatrix}
\frac{\pi}{2}u & \cdots & \cdots & \frac{\pi}{2}u \\
\quad & \ddots & \ddots & \\
\quad & \quad & & \\
\frac{\pi}{2}u & \cdots & \cdots & \frac{\pi}{2}u \\
\end{bmatrix}
\]

Yang-Baxter Equation

\[
(u - v) u = u (u - v)
\]

Commuting Family

\[
YBE + BYBE \implies D(u)D(v) = D(v)D(u) \implies \text{integrable}
\]

Crossing Symmetry

\[
D(u) = D(\lambda - u)
\]

- Multiplication is vertical concatenation of diagrams.
- The transfer matrix maps link states to link states \(\implies\) Matrix Reps \(\implies\) Spectra.
The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary diagrams

For $r, s = 1, 2, 3, \ldots$, the $(r, s)$ BYBE solution is built as the fusion product of $(r, 1)$ and $(1, s)$ integrable seams acting on the vacuum $(1, 1)$ triangle.

The column inhomogeneities are $\xi_k = \xi + k\lambda$ where $\xi$ is a boundary field.

The $s$-type seam introduces $d = s - 1$ defects into the bulk.

The $r$-type seams are realized with either $\rho = 2r - 1$ or $\rho = 2r$ with $\xi = \frac{1}{2}\lambda$.

Diagrammatic fusion is applied on the bottom edge so that the $r$-arches can not close among themselves and similarly for the $s$-arches.
Kac Table of Critical Dense Polymers

- **Central charge:** \((p, p') = (1, 2)\)
  \[ c = -2 \]

- **Infinitely extended Kac table:**
  \[ \Delta_{r,s} = \frac{(2r - s)^2 - 1}{8}, \quad r, s = 1, 2, 3, \ldots \]

- **Kac representation characters:**
  \[ \chi_{r,s}(q) = q^{-c/24} \frac{q^{\Delta_{r,s}}(1 - q^{rs})}{\prod_{n=1}^{\infty}(1 - q^n)} \]

- **Modular nome:**
  \[ q = \exp(2\pi i \tau) = \text{modular nome} \]
  \[ \tau = i \delta \sin 2u = \text{geometric factor} \]
  \[ \delta = M/N = \text{aspect ratio} \]

- **Pink boxes are associated to critical exponents of off-critical Generalized Order Parameters**
  \[ \beta_{r,s} = \Delta_{r,s} \text{ accessible by CTMs (PAP-Seaton 2012).} \]
1-Bdy Temperley-Lieb Algebra

- The one-boundary TL algebra (NicholsEtAl05) is a planar diagrammatic algebra related to the blob algebra (Martin-Saleur94) and generated by two bulk tiles and two boundary triangles

- Fixing the direction of transfer gives the linear loop representation of the 1-bdy TL algebra

$$TL_N(\beta; \beta_1, \beta_2) := \langle I, e_j, f_N; \quad j = 1, \ldots, N - 1 \rangle$$

where $\beta_1, \beta_2$ are fugacities of loops that terminate on the boundary. The generators

- Multiplication is by vertical concatenation of diagrams

- We set $\beta_1 = \beta_2 = \tilde{\beta} = 1$. 
Twist/Robin Boundary Conditions

- The \((r, s)\) boundary conditions are \textit{Neumann} boundary conditions since loop segments are reflected at the boundary.

- Boundary triangles of \textit{Dirichlet} type \cite{JacobsenSaleur08} allow loop segments to terminate on the boundary. An elementary solution of the BYBE is given by the \textit{twist} boundary condition

\[
\begin{align*}
\left< u, \xi \right> &= \beta s_1(\xi - u) [s_1(\xi + u) - s_0(\xi + u)] + s_0(2u), \\
\end{align*}
\]

where \(\beta = 1\) is the fugacity of loops that terminate on the boundary. Such linear combinations of Neumann and Dirichlet boundary conditions are called \textit{Robin} boundary conditions.

- The twist boundary condition is known to be conjugate to the conformal weight

\[
\Delta_{0, \frac{1}{2}} = -\frac{3}{32}, \quad r = s = 1
\]

- Dressing the \textit{twist} boundary condition with \(r\)- and \(s\)-type seams, gives general integrable Robin boundary conditions
Robin Link States

- A link state on $N$ bulk and $w$ boundary nodes is a planar diagram of non-crossing arc segments. The transfer matrix $D(u)$ acts on Robin link states such that:

  (i) $d = s - 1 \geq 0$ defects (vertical segments) attach individual nodes to a point above at infinity
  (ii) $b \geq 0$ boundary links link individual nodes to the right boundary
  (iii) the remaining $N + w - d - b$ nodes are connected pairwise by half-arcs
  (iv) every boundary node is either a defect or is linked to a bulk node

\[
\begin{align*}
w &= \rho - 1 \equiv \text{width of } r\text{-type seam}, \\
N + w - d - b &\equiv 0 \pmod{2}
\end{align*}
\]

- The full vector space of Robin link states decomposes into $d$-fixed sectors

\[
\dim \mathcal{V}^{(N,w)} = \sum_{d=0}^{N+w} \dim \mathcal{V}_d^{(N,w)} = 2^N, \quad \dim \mathcal{V}_d^{(N,w)} = \binom{N}{\lfloor \frac{N-d}{2} \rfloor} + (-1)^{N-d-w} \lfloor \frac{w}{2} \rfloor
\]

\[
\begin{align*}
\mathcal{V}^{(3,1)}_1 &= \text{span} \{ \includegraphics[width=1cm]{31.png} \} \\
\mathcal{V}^{(3,2)}_1 &= \text{span} \{ \includegraphics[width=1cm]{32.png} , \includegraphics[width=1cm]{33.png} , \includegraphics[width=1cm]{34.png} \} \\
\mathcal{V}^{(4,2)}_0 &= \text{span} \{ \includegraphics[width=1cm]{42.png} , \includegraphics[width=1cm]{43.png} , \includegraphics[width=1cm]{44.png} , \includegraphics[width=1cm]{45.png} \}
\end{align*}
\]

The defects can be closed to an $s$-type seam on the right by passing under the boundary links.
Inversion Identities

- **Kac \((r, s)\) Boundary Conditions:** \((\text{PRVillani2013})\) \(\xi = \frac{1}{2}\lambda = \frac{\pi}{4}\)

\[
d(u)d(u + \frac{\pi}{2}) = \begin{cases} 
(\cos^{2N}u + \sin^{2N}u)^2 I & 
\text{if } \rho = 2r \\
(\cos^{2N}u - \sin^{2N}u)^2 I & 
\text{if } \rho = 2r - 1 
\end{cases}
\]

\[
d(u) = \begin{cases} 
\frac{2^{\rho-1}D(u)}{\sin 2u \cos^{\rho-2}2u}, & \rho = 2r \\
\frac{2^{\rho-1}D(u)}{\sin 2u \cos^{\rho-1}2u}, & \rho = 2r - 1 
\end{cases}
\]

- **Robin Boundary Conditions:** Acting on \(\mathcal{V}^{(N,w)}\) with \(w \geq 0\), \(\xi = -\frac{1}{2}\lambda = -\frac{\pi}{4}\)

\[
d(u)d(u + \frac{\pi}{2}) = \frac{\cos^{4N+2}u - \sin^{4N+2}u}{\cos^2u - \sin^2u} I, \quad d(u) = \frac{2^w(\cos u - \sin u)}{\sin 2u \cos^w2u[\cos u - (-1)^w \sin u]} D(u)
\]

- The inversion identities determine all of the eigenvalues of \(d(u)\) subject to the initial and crossing relations

\(d(0) = d(\frac{\pi}{2}) = I, \quad d(u) = d(\frac{\pi}{2} - u)\)

- For Robin boundary conditions, \(w = \rho - 1\) and \(r\) are related in the \((r, s)\) sector by

\[
s = d + 1, \quad r = \begin{cases} 
(-1)^d + w \left\lfloor \frac{w}{2} \right\rfloor, & \text{if } N \text{ even} \\
-(1)^d + w \left\lfloor \frac{w}{2} \right\rfloor, & \text{if } N \text{ odd} 
\end{cases}, \quad r \in \mathbb{Z}
\]

- Note that \(s \geq 1\) but \(r\) can be zero or negative!
Inversion Identity Solution

- The $2^N$ solutions to the inversion relation are given explicitly by
  
  $$d_n(u) = \prod_{j=1}^{N} \left(1 + \epsilon_j^{(n)} \sin t_j \sin 2u\right), \quad n = 0, 1, 2, \ldots$$

  $$\epsilon_j^{(n)} = \pm 1, \quad t_j = \frac{(j - \frac{1}{2})\pi}{2N + 1}, \quad j = 1, \ldots, N$$

- The partition function for an $M \times N$ strip is
  
  $$Z_{M,N} = \text{Tr } d(u)^M = \sum_n d(u)^M = \sum_n e^{-ME_n(u)}$$

  $$E_n(u) = -\ln d(u) = 2N f_{\text{bulk}}(u) + f_{\text{bdy}}(u) + \frac{2\pi \sin 2u}{N} \left(-\frac{c}{24} + \Delta + k\right) + \cdots$$

- The Euler-Maclaurin formula gives the bulk and boundary free energies
  
  $$f_{\text{bulk}}(u) = -\frac{1}{\pi} \int_0^{\pi/2} \ln(1 + \sin t \sin 2u) dt, \quad f_{\text{bdy}}(u) = \frac{1}{2} \ln(1 + \sin 2u)$$

- It also gives the central charge $c = -2$ and finite excitations
  
  $$\Delta = -\frac{3}{32} + \sum_{j \in \mathcal{E}_n} \frac{1}{2}(j - \frac{1}{2}), \quad \mathcal{E}_n = \{j : \epsilon_j^{(n)} = -1\} \subset \{1, 2, \ldots, N\}$$

  $$\Delta_{r,s-\frac{1}{2}} = \frac{1}{32}(L^2 - 4) = \frac{3}{32}, \frac{5}{32}, \frac{21}{32}, \frac{45}{32}, \frac{77}{32}, \ldots$$

  $$L = 2s - 1 - 4r, \quad r \in \mathbb{Z}, s \in \mathbb{N}$$
Finitized Partition Function

- Taking the Hamiltonian limit, the associated quantum Hamiltonian acting on \( \mathcal{V}_d^{(N,w)} \) is

\[
\mathcal{H} = - \sum_{j=1}^{N} e_j + \sum_{k=1}^{w} (-1)^{\lfloor \frac{k+1}{2} \rfloor + (k-1)w} e_{N} e_{N+1} \cdots e_{N+k}, \quad e_{N+w} \equiv f_{N+w}
\]

- Summing over all sectors, the finitized partition function for \( d(u) \) or \( \mathcal{H} \) is

\[
Z^{(N)}(q, z) = q^{-\frac{c}{24} - \frac{3}{32} N + w} \sum_{d=0}^{\lfloor N/2 \rfloor} q^{\frac{1}{8} [(d+\frac{1}{2}) - 2(-1)^{N-d-w}]^2 - \frac{1}{4}} z^{(-1)^{N+w} \lfloor \frac{w}{2} \rfloor - (-1)^{d+\frac{1}{2}} \lfloor \frac{N-d}{2} \rfloor} q^{N \lfloor \frac{N-d}{2} \rfloor + (1) N^\frac{1}{2}} \prod_{k=1}^{\lfloor \frac{N}{2} \rfloor} (1 + q^{k-\frac{1}{2}} z^{-1}) \prod_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} (1 + q^{k-\frac{3}{2}} z)
\]

is independent of \( w \). Here \( Z^{(N)}(1, 1) = 2^N \) gives the correct counting of states and the quantum number \( \sigma \), coming from physical combinatorics, is

\[
\sigma = \begin{cases} 
\frac{d}{2} - r = -\frac{1}{2} (2r - s + 1), & d \text{ even (} s \text{ odd)} \\
\frac{r - d+1}{2} = \frac{1}{2} (2r - s), & d \text{ odd (} s \text{ even)}
\end{cases}
\]

This is the known partition function of the \( \mathbb{Z}_4 \) sector of symplectic fermions (Kausch 2000).

- The character of the sector labelled by \( \sigma \)

\[
\lim_{N \to \infty} \chi^{(N)}_{\sigma}(q) = q^{-\frac{c}{24} - \frac{3}{32} + \frac{1}{2} \sigma(\sigma+\frac{1}{2})} \prod_{n=1}^{\infty} (1 - q^n)
\]

is associated with an irreducible (highest weight) Virasoro Verma module.
For \( r \in \mathbb{Z}, s \in \mathbb{N} \), Robin boundary conditions gives rise to a half-integer Kac table

\[
\Delta_{r, s - \frac{1}{2}} = \frac{(4r - 2s + 1)^2 - 4}{32}
\]

We extend to \( s \leq 0 \) by identifying

\[
(r, s) \equiv (-r, 1 - s), \quad r \in \mathbb{Z}, \quad s \leq 0
\]

The physical boundary conditions correspond to \( s \geq 1 \).

The extended Kac table encodes the \( \mathfrak{sl}(2) \) fusion rules with the fundamentals \( (r, s) = (2, 1), (1, 2) \).
Summary

- A Yang-Baxter integrable model of critical dense polymers is exactly solvable on arbitrary finite size lattices by solving transfer matrix functional equations in the form of inversion identities.
- On the strip, general integrable Robin boundary conditions are constructed as linear combinations of Neumann and Dirichlet boundary conditions.
- The commuting double row transfer matrices satisfy simple functional equations in the form of inversion identities.
- Solving the inversion identities using Euler-Maclaurin gives the bulk and boundary free energies as well as the central charge $c = -2$ and conformal weights

\[ \Delta_{r,s-\frac{1}{2}} = \frac{(4r - 2s + 1)^2 - 4}{32}, \quad r, s = 1, 2, 3, \ldots \]

- The Robin boundary conditions are thus conjugate to scaling fields with conformal weights $\Delta_{r,s-\frac{1}{2}}$ giving rise to an infinitely extended Virasoro Kac table with half-integer Kac labels.
- The partition function is the known partition function of the $\mathbb{Z}_4$ sector of symplectic fermions.
- It would be of interest to obtain numerically the conformal weights of the $\mathcal{L}M(p, p')$ models with Robin boundary conditions to see if half-integer Kac labels appear.
### Free Fermions Versus Symplectic Fermions

#### Ising Model
\( c = \frac{1}{2} \) rational CFT = free fermions

Crossing parameter: \( \lambda = \pi/4 \)

Scaling fields: \( \{I, \sigma, \epsilon\} = \{\varphi_{1,1}, \varphi_{1,2}, \varphi_{1,3}\} \)

\[ \alpha = 0, \ \nu = 1 \iff \Delta_\epsilon = \Delta_{1,3} = \frac{1 - \alpha}{2 - \alpha} = \frac{1}{2} \]

\[ \beta = \frac{1}{8} \iff \Delta_\sigma = \Delta_{1,2} = \frac{1}{2} \beta = \frac{1}{16} \]

#### Critical Dense Polymers
\( c = -2 \log \) CFT = symplectic fermions

Crossing parameter: \( \lambda = \pi/2 \)

Scaling fields: \( \{\varphi_{r,s}\}_{r,s \in \mathbb{N}} \)

\[ \alpha = 1, \ \nu = \frac{1}{2} \iff \Delta_t = \Delta_{1,3} = \frac{1 - \alpha}{2 - \alpha} = 0 \]

\[ \beta_{r,s} = \Delta_{r,s}, \quad (2r - s)^2 < 8 \]