Exact Solution of Critical Dense Polymers with Robin Boundary Conditions

RAQIS, Dijon, 5 Sept 2014

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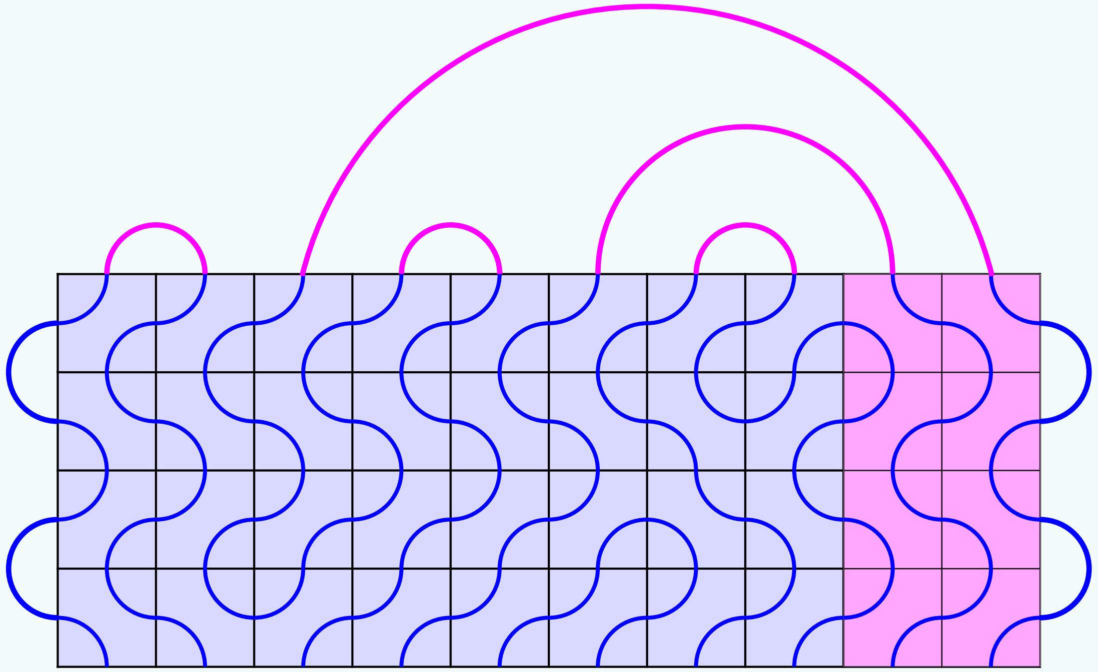


Collaborators: Jørgen Rasmussen and Ilya Tipunin.

- PAP, J.Rasmussen, J.-B.Zuber, *Logarithmic minimal models*, J.Stat.Mech. P11017 (2006).
- PAP, J.Rasmussen, *Solvable critical dense polymers*, J.Stat.Mech. P02015 (2007).
- PAP, J.Rasmussen, S.P.Villani, *Solvable critical dense polymers on the cylinder*, J.Stat.Mech. P02010 (2010).
- PAP, JR, S.P.Villani, *Infinitely extended Kac table of solvable dense polymers*, J.Phys. A46 (2013) 175202.
- PAP, JR, I.Y.Tipunin, *Critical dense polymers with Robin boundary conditions, half-integer Kac labels and* $\mathbb{Z}_4$ *fermions*, arXiv:1405.0550.

History

- Critical dense polymers: de Gennes 1979, Duplantier 1986, Saleur 1986/1992, ...
- Generic loop models: Nienhuis 1987, Blöte-Nienhuis 1989, ...
- Logarithmic Minimal Models $\mathcal{LM}(p,p')$: Pearce-Rasmussen-Zuber 2006, ...
- Critical dense polymers $\mathcal{LM}(1,2)$ solved exactly as a Yang-Baxter integrable loop model on the square lattice: Pearce-Rasmussen 2007, ...
- The commuting transfer matrices of $\mathcal{LM}(1,2)$ satisfy *inversion identities*:

$$\boldsymbol{D}(u)\boldsymbol{D}(u+\lambda) = f(u)\boldsymbol{I} \quad \Rightarrow \quad \text{integrable on finite lattices}$$

Critical dense polymers is exactly solvable on arbitrary size *finite* lattices for all topologies and all Yang-Baxter integrable boundary conditions!

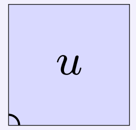
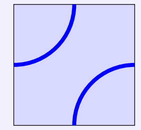
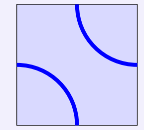
- As a CFT, dense polymers is a protoypical logarithmic CFT described by symplectic fermions. It admits (r,s) boundary conditions with conformal weights given by the Kac formula

$$\Delta_{r,s} = \frac{(2r-s)^2-1}{8}, \qquad r,s = 1,2,3,\ldots$$

- The Kac representations do not exhaust all of the possible representations. There should exist boundary conditions yielding conformal weights with half-integer Kac labels: Saleur 1987, Duplantier 1986, Delfino 2013, ...

Critical Dense Polymers

- **Logarithmic Minimal Models:** Yang-Baxter integrable loop models on the square lattice. Face operators defined in diagrammatic planar Temperley-Lieb algebra (Jones 1999)

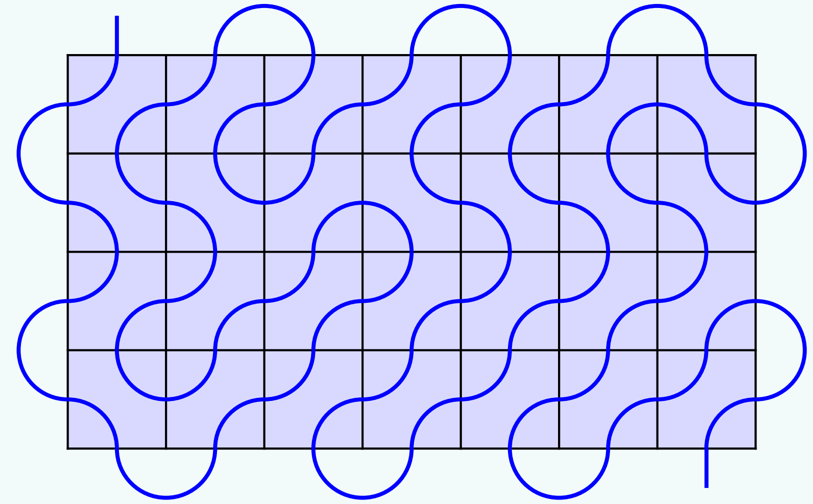
$X(u) =$  $= \sin(\lambda - u)$  $+ \sin u$ 

$1 \le p < p'$ coprime integers, $\qquad \lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

$u =$ spectral parameter, $\qquad \beta = 2\cos\lambda =$ nonlocal loop fugacity

- **Critical Dense Polymers:** $(p, p') = (1, 2)$, $\quad \lambda = \frac{\pi}{2}$

$$Z = \sum_{\text{loop configs}} \sin^{N_1}(\lambda - u)\, \sin^{N_2} u,$$

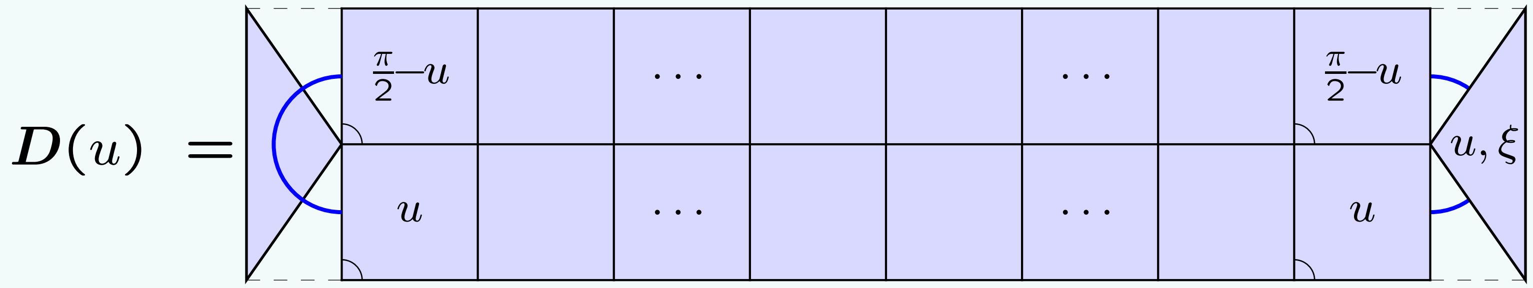


$\beta = 0 \;\Rightarrow$ no closed loops $\Rightarrow$ space filling dense polymer $\Rightarrow \quad d_{\text{path}}^{\text{SLE}} = 2 - 2\Delta_{1,1} = 2$

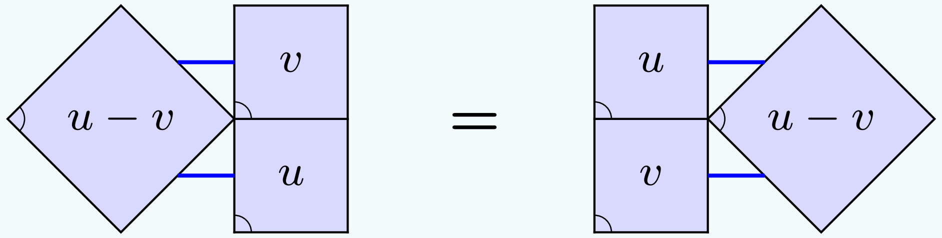
- There are no local degrees of freedom only nonlocal degrees of freedom in the form of extended polymer segments!

Commuting Double-Row Transfer Matrices on a Strip

Double row transfer matrix (nontrivial boundary on right)



Yang-Baxter Equation



Commuting Family

$$\text{YBE} + \text{BYBE} \quad \Rightarrow \quad \boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u) \quad \Rightarrow \quad \text{integrable}$$

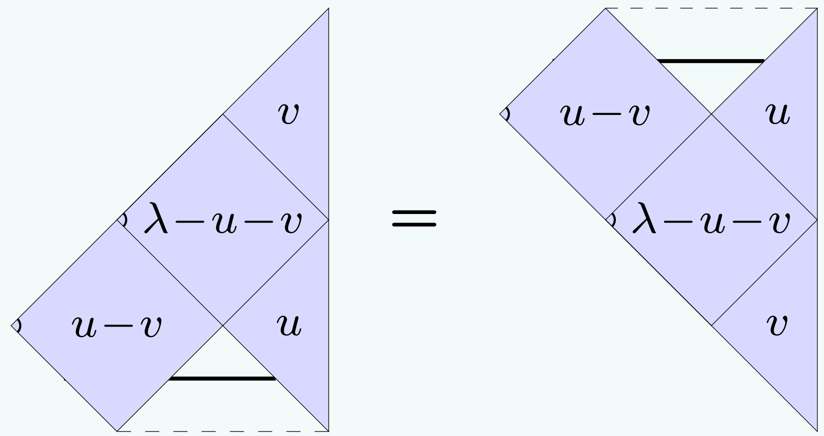
Crossing Symmetry

$$\boldsymbol{D}(u) = \boldsymbol{D}(\lambda - u)$$

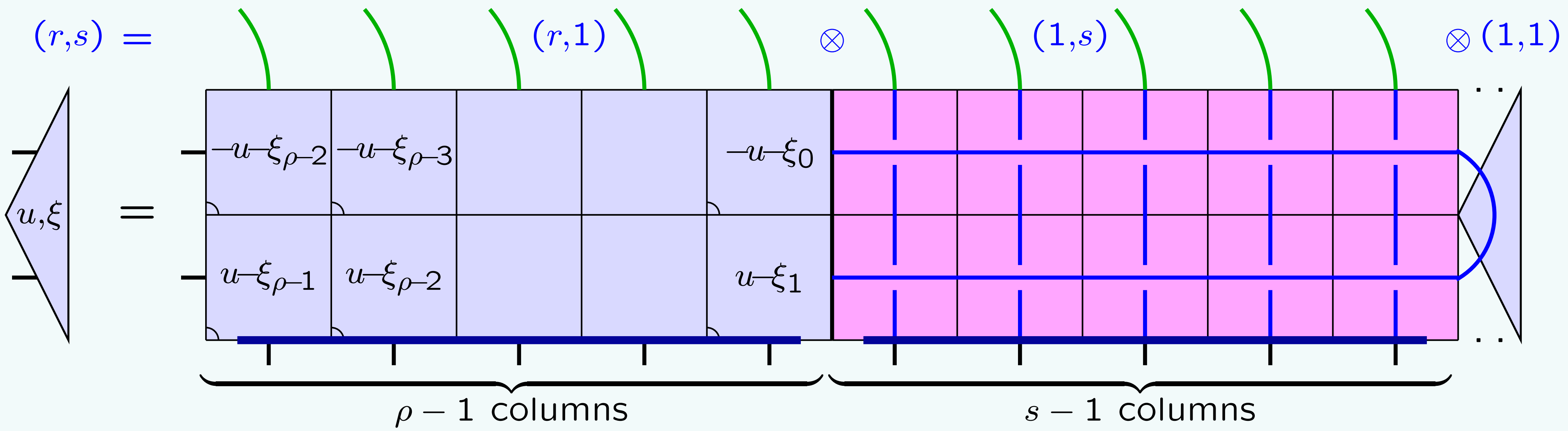
- Multiplication is vertical concatenation of diagrams.
- The transfer matrix maps link states to link states $\Rightarrow$ Matrix Reps $\Rightarrow$ Spectra.

Boundary Yang-Baxter Equation

- The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary diagrams



- For $r, s = 1, 2, 3, \ldots$, the (r, s) BYBE solution is built as the fusion product of $(r, 1)$ and $(1, s)$ integrable seams acting on the vacuum $(1, 1)$ triangle



- The column inhomogeneities are $\xi_k = \xi + k\lambda$ where ξ is a boundary field.
- The s-type seam introduces $d = s - 1$ defects into the bulk.
- The r-type seams are realized with either $\rho = 2r - 1$ or $\rho = 2r$ with $\xi = \frac{1}{2}\lambda$.
- Diagrammatic fusion is applied on the bottom edge so that the r-arches can not close among themselves and similarly for the s-arches.

Kac Table of Critical Dense Polymers

- **Central charge:** $(p, p') = (1, 2)$

$$c \;=\; -2$$

- **Infinitely extended Kac table:**

$$\Delta_{r,s} \;=\; \frac{(2r-s)^2-1}{8}, \qquad r,s = 1,2,3,\ldots$$

- **Kac representation characters:**

$$\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- **Modular nome:**

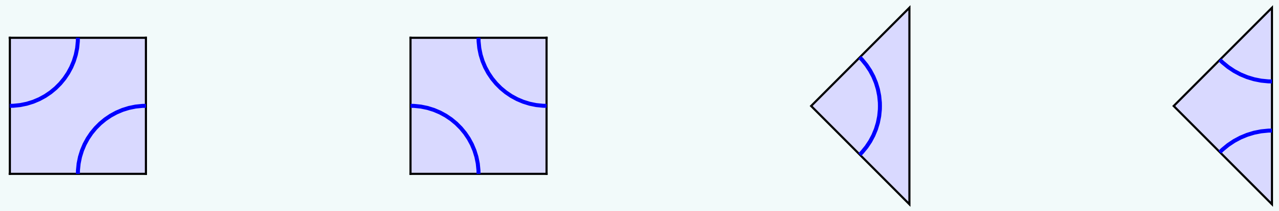
$$\begin{aligned} q &= \exp(2\pi i\tau) = \text{modular nome}\\ \tau &= i\,\delta\,\sin 2u \;=\; \text{geometric factor}\\ \delta &= M/N = \text{aspect ratio} \end{aligned}$$

- Pink boxes are associated to critical exponents of off-critical Generalized Order Parameters $\beta_{r,s} = \Delta_{r,s}$ accessible by CTMs (PAP-Seaton 2012).

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

1-Bdy Temperley-Lieb Algebra

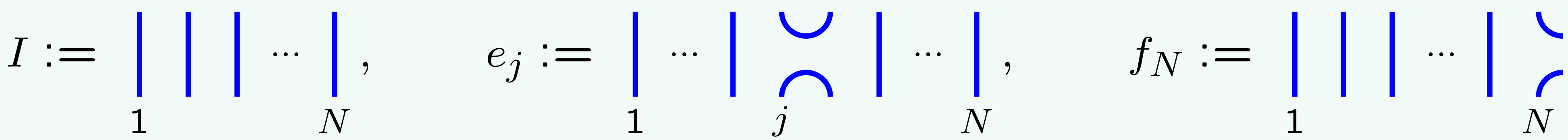
- The one-boundary TL algebra (NicholsEtAl05) is a planar diagrammatic algebra related to the blob algebra (Martin-Saleur94) and generated by two bulk tiles and two boundary triangles



- Fixing the direction of transfer gives the linear loop representation of the 1-bdy TL algebra

$$TL_N(\beta;\beta_1,\beta_2) := \big\langle I,\, e_j,\, f_N;\;\; j=1,\ldots,N-1 \big\rangle$$

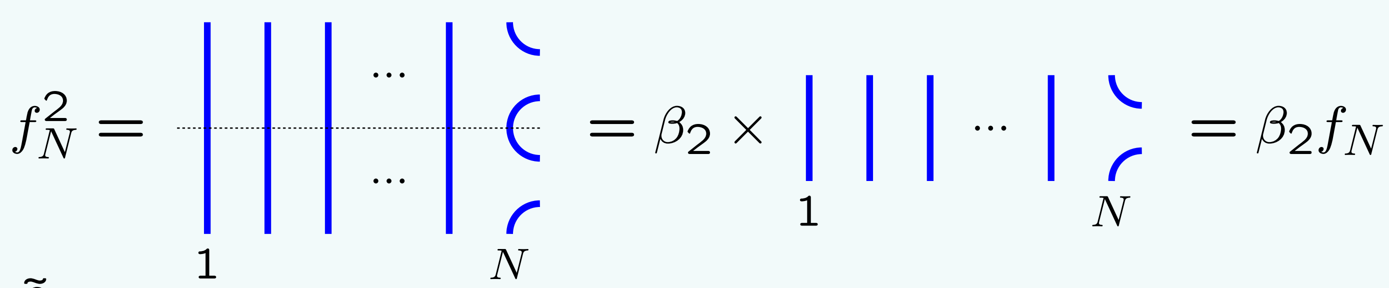
where β_1, β_2 are fugacities of loops that terminate on the boundary. The generators



satisfy

$$\begin{array}{rl} [e_i, e_j] = 0, & |i-j| > 1 \\ e_i e_j e_i = e_i, & |i-j| = 1 \\ e_j^2 = \beta e_j, & j = 1,\ldots,N-1 \\ [e_j, f_N] = 0, & j = 1,\ldots,N-2 \\ e_{N-1} f_N e_{N-1} = \beta_1 e_{N-1}, & f_N^2 = \beta_2 f_N \end{array}$$

- Multiplication is by vertical concatenation of diagrams

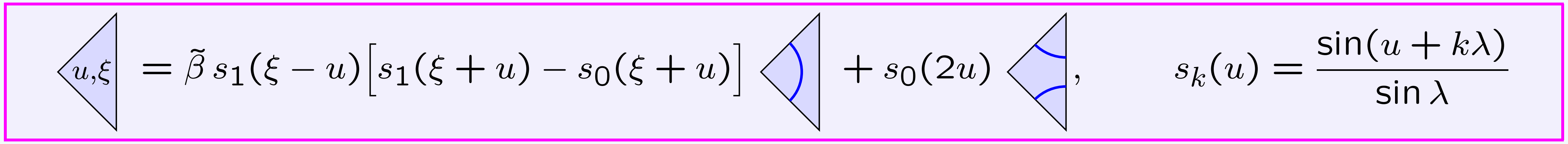


- We set $\beta_1 = \beta_2 = \tilde{\beta} = 1$.

Twist/Robin Boundary Conditions

- The (r, s) boundary conditions are *Neumann* boundary conditions since loop segments are reflected at the boundary.

- Boundary triangles of *Dirichlet* type (Jacobsen-Saleur08) allow loop segments to terminate on the boundary. An elementary solution of the BYBE is given by the *twist* boundary condition

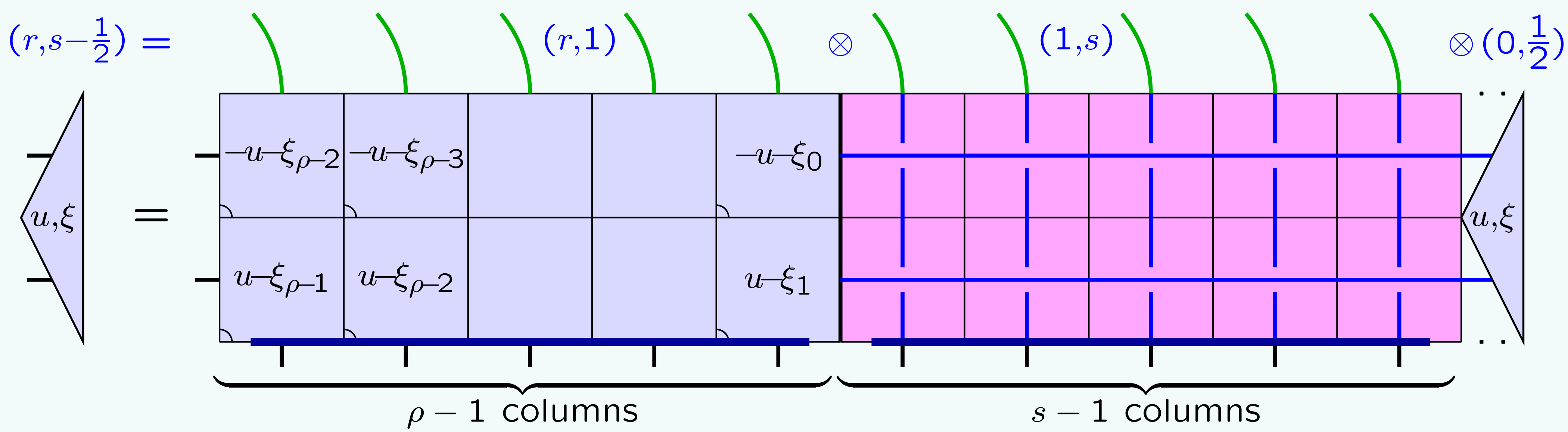


where $\tilde{\beta} = 1$ is the fugacity of loops that terminate on the boundary. Such linear combinations of Neumann and Dirichlet boundary conditions are called *Robin* boundary conditions.

- The twist boundary condition is known to be conjugate to the conformal weight

$$\Delta_{0,\frac{1}{2}} = -\frac{3}{32}, \qquad r = s = 1$$

- Dressing the twist boundary condition with r- and s-type seams, gives general integrable Robin boundary conditions



Robin Link States

- A link state on N bulk and w boundary nodes is a planar diagram of non-crossing arc segments.

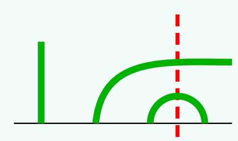
The transfer matrix $\boldsymbol{D}(u)$ acts on *Robin link states* such that:

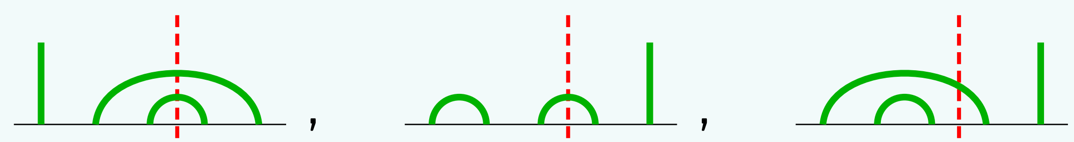
(i) $d = s-1 \geq 0$ *defects* (vertical segments) attach individual nodes to a point above at infinity
(ii) $b \geq 0$ *boundary links* link individual nodes to the right boundary
(iii) the remaining $N+w-d-b$ nodes are connected pairwise by half-arcs
(iv) every boundary node is either a defect or is linked to a bulk node

$$w = \rho - 1 = \text{width of } r\text{-type seam}, \qquad N+w-d-b \equiv 0 \mod 2$$

- The full vector space of Robin link states decomposes into d-fixed sectors

$$\dim \mathcal{V}^{(N,w)} = \sum_{d=0}^{N+w} \dim \mathcal{V}_d^{(N,w)} = 2^N, \qquad \dim \mathcal{V}_d^{(N,w)} = \binom{N}{\left\lfloor \frac{N-d}{2} \right\rfloor + (-1)^{N-d-w} \left\lceil \frac{w}{2} \right\rceil}$$

$\mathcal{V}_1^{(3,1)} = \text{span}\Big\{$  $\Big\}$

$\mathcal{V}_1^{(3,2)} = \text{span}\Big\{$  $\Big\}$

$\mathcal{V}_0^{(4,2)} = \text{span}\Big\{$  $\Big\}$

The defects can be closed to an s-type seam on the right by passing under the boundary links.

Inversion Identities

- **Kac (r,s) Boundary Conditions:** (PRVillani2013) $\xi=\frac{1}{2}\lambda=\frac{\pi}{4}$

$$\boldsymbol{d}(u)\boldsymbol{d}(u+\tfrac{\pi}{2}) = \begin{cases}\Big(\cos^{2N}u+\sin^{2N}u\Big)^2\boldsymbol{I}\\ \Big(\dfrac{\cos^{2N}u-\sin^{2N}u}{\cos^2 u-\sin^2 u}\Big)^2\boldsymbol{I}\end{cases} \qquad \boldsymbol{d}(u)=\begin{cases}\dfrac{2^{\rho-1}\boldsymbol{D}(u)}{\sin 2u\cos^{\rho-2}2u}, & \rho=2r\\ \dfrac{2^{\rho-1}\boldsymbol{D}(u)}{\sin 2u\cos^{\rho-1}2u}, & \rho=2r-1\end{cases}$$

- **Robin Boundary Conditions:** Acting on $\mathcal{V}^{(N,w)}$ with $w\ge 0$, $\xi=-\frac{1}{2}\lambda=-\frac{\pi}{4}$

$$\boldsymbol{d}(u)\boldsymbol{d}(u+\tfrac{\pi}{2}) = \frac{\cos^{4N+2}u-\sin^{4N+2}u}{\cos^2 u-\sin^2 u}\boldsymbol{I}, \qquad \boldsymbol{d}(u)=\frac{2^w(\cos u-\sin u)}{\sin 2u\,\cos^w 2u[\cos u-(-1)^w\sin u]}\boldsymbol{D}(u)$$

- The inversion identities determine all of the eigenvalues of $\boldsymbol{d}(u)$ subject to the initial and crossing relations

$$\boldsymbol{d}(0)=\boldsymbol{d}(\tfrac{\pi}{2})=\boldsymbol{I}, \qquad \boldsymbol{d}(u)=\boldsymbol{d}(\tfrac{\pi}{2}-u)$$

- For Robin boundary conditions, $w=\rho-1$ and r are related in the (r,s) sector by

$$s=d+1, \qquad r=\begin{cases}(-1)^{d+w}\lceil\frac{w}{2}\rceil, & N \text{ even}\\ -(-1)^{d+w}\lceil\frac{w}{2}\rceil, & N \text{ odd}\end{cases} \qquad r\in\mathbb{Z}$$

- Note that $s\ge 1$ but r can be zero or negative!

Inversion Identity Solution

- The 2^N solutions to the inversion relation are given explicitly by

$$d_n(u)=\prod_{j=1}^{N}\big(1+\epsilon_j^{(n)}\sin t_j\sin 2u\big),\qquad n=0,1,2,\ldots$$

$$\epsilon_j^{(n)}=\pm 1,\qquad t_j=\frac{(j-\frac{1}{2})\pi}{2N+1},\qquad j=1,\ldots,N$$

- The partition function for an $M\times N$ strip is

$$Z_{M,N}\ =\ \mathrm{Tr}\,\boldsymbol{d}(u)^M\ =\ \sum_n d(u)^M\ =\ \sum_n e^{-ME_n(u)}$$

$$E_n(u)\ =\ -\ln d(u)\ =\ 2Nf_{bulk}(u)+f_{bdy}(u)+\frac{2\pi\sin 2u}{N}\Big(-\frac{c}{24}+\Delta+k\Big)+\cdots$$

- The Euler-Maclaurin formula gives the bulk and boundary free energies

$$f_{bulk}(u)\ ==-\frac{1}{\pi}\int_0^{\pi/2}\ln(1+\sin t\,\sin 2u)dt,\qquad f_{bdy}(u)\ =\ \tfrac{1}{2}\ln(1+\sin 2u)$$

- It also gives the central charge $c=-2$ and finite excitations

$$\Delta=-\tfrac{3}{32}+\sum_{j\in\mathcal{E}_n}\tfrac{1}{2}(j-\tfrac{1}{2}),\qquad \mathcal{E}_n=\{j\,:\,\epsilon_j^{(n)}=-1\}\subset\{1,2,\ldots,N\}$$

$$\Delta_{r,s-\frac{1}{2}}=\frac{1}{32}(L^2-4)=-\frac{3}{32},\frac{5}{32},\frac{21}{32},\frac{45}{32},\frac{77}{32},\ldots\qquad L=2s-1-4r,\qquad r\in\mathbb{Z},s\in\mathbb{N}$$

Finitized Partition Function

- Taking the Hamiltonian limit, the associated quantum Hamiltonian acting on $\mathcal{V}_d^{(N,w)}$ is

$$\mathcal{H} = -\sum_{j=1}^{N} e_j + \sum_{k=1}^{w} (-1)^{\lfloor\frac{k+1}{2}\rfloor+(k-1)w} e_N e_{N+1}\dots e_{N+k}, \qquad e_{N+w}\equiv f_{N+w}$$

- Summing over all sectors, the finitized partition function for $\boldsymbol{d}(u)$ or $\mathcal{H}$ is

$$Z^{(N)}(q,z) = q^{-\frac{c}{24}-\frac{3}{32}} \sum_{d=0}^{N+w} q^{\frac{1}{8}[(d+\frac{1}{2}-2(-1)^{N-d-w}\lceil\frac{w}{2}\rceil)^2-\frac{1}{4}]} z^{(-1)^{N+w}\lceil\frac{w}{2}\rceil-(-1)^d\lceil\frac{d}{2}\rceil} \begin{bmatrix} N \\ \lfloor\frac{N-d}{2}\rfloor+(-1)^{N-d-w}\lceil\frac{w}{2}\rceil \end{bmatrix}_q$$

$$= \sum_{\sigma=-\lfloor\frac{N+1}{2}\rfloor}^{\lfloor\frac{N}{2}\rfloor} z^{-\sigma}\chi_\sigma^{(N)}(q) \;=\; q^{-\frac{c}{24}-\frac{3}{32}} \prod_{k=1}^{\lfloor\frac{N}{2}\rfloor}\left(1+q^{k-\frac{1}{4}}z^{-1}\right) \prod_{k=1}^{\lfloor\frac{N+1}{2}\rfloor}\left(1+q^{k-\frac{3}{4}}z\right)$$

is independent of w. Here $Z^{(N)}(1,1)=2^N$ gives the correct counting of states and the quantum number σ, coming from physical combinatorics, is

$$\sigma = \begin{cases} \frac{d}{2}-r = -\frac{1}{2}(2r-s+1), & d \text{ even } (s \text{ odd}) \\ r-\frac{d+1}{2} = \frac{1}{2}(2r-s), & d \text{ odd } (s \text{ even}) \end{cases}$$

This is the known partition function of the $\mathbb{Z}_4$ sector of symplectic fermions (Kausch 2000).

- The character of the sector labelled by σ

$$\lim_{N\to\infty}\chi_\sigma^{(N)}(q) = q^{-\frac{c}{24}-\frac{3}{32}+\frac{1}{2}\sigma(\sigma+\frac{1}{2})} \lim_{N\to\infty} \begin{bmatrix} N \\ \lfloor\frac{N}{2}\rfloor-\sigma \end{bmatrix}_q = \frac{q^{-\frac{c}{24}-\frac{3}{32}+\frac{1}{2}\sigma(\sigma+\frac{1}{2})}}{\prod_{n=1}^{\infty}(1-q^n)}$$

is associated with an irreducible (highest weight) Virasoro Verma module.

Half-Integer Kac Table

- For $r \in \mathbb{Z}, s \in \mathbb{N}$, Robin boundary conditions gives rise to a half-integer Kac table

$$\Delta_{r,s-\frac{1}{2}} = \frac{(4r-2s+1)^2-4}{32}$$

- We extend to $s \leq 0$ by identifying

$$(r,s) \equiv (-r, 1-s), \quad r \in \mathbb{Z}, \quad s \leq 0$$

The physical boundary conditions correspond to $s \geq 1$.

- The extended Kac table encodes the $s\ell(2)$ fusion rules with the fundamentals $(r,s) = (2,1),(1,2)$.

s	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
7	$\cdots$	$\frac{1085}{32}$	$\frac{837}{32}$	$\frac{621}{32}$	$\frac{437}{32}$	$\frac{285}{32}$	$\frac{165}{32}$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\cdots$
6	$\cdots$	$\frac{957}{32}$	$\frac{725}{32}$	$\frac{525}{32}$	$\frac{357}{32}$	$\frac{221}{32}$	$\frac{117}{32}$	$\frac{45}{32}$	$\frac{5}{32}$	$-\frac{3}{32}$	$\frac{21}{32}$	$\frac{77}{32}$	$\cdots$
5	$\cdots$	$\frac{837}{32}$	$\frac{621}{32}$	$\frac{437}{32}$	$\frac{285}{32}$	$\frac{165}{32}$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\frac{117}{32}$	$\cdots$
4	$\cdots$	$\frac{725}{32}$	$\frac{525}{32}$	$\frac{357}{32}$	$\frac{221}{32}$	$\frac{117}{32}$	$\frac{45}{32}$	$\frac{5}{32}$	$-\frac{3}{32}$	$\frac{21}{32}$	$\frac{77}{32}$	$\frac{165}{32}$	$\cdots$
3	$\cdots$	$\frac{621}{32}$	$\frac{437}{32}$	$\frac{285}{32}$	$\frac{165}{32}$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\frac{117}{32}$	$\frac{221}{32}$	$\cdots$
2	$\cdots$	$\frac{525}{32}$	$\frac{357}{32}$	$\frac{221}{32}$	$\frac{117}{32}$	$\frac{45}{32}$	$\frac{5}{32}$	$-\frac{3}{32}$	$\frac{21}{32}$	$\frac{77}{32}$	$\frac{165}{32}$	$\frac{285}{32}$	$\cdots$
1	$\cdots$	$\frac{437}{32}$	$\frac{285}{32}$	$\frac{165}{32}$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\frac{117}{32}$	$\frac{221}{32}$	$\frac{357}{32}$	$\cdots$
0	$\cdots$	$\frac{357}{32}$	$\frac{221}{32}$	$\frac{117}{32}$	$\frac{45}{32}$	$\frac{5}{32}$	$-\frac{3}{32}$	$\frac{21}{32}$	$\frac{77}{32}$	$\frac{165}{32}$	$\frac{285}{32}$	$\frac{437}{32}$	$\cdots$
-1	$\cdots$	$\frac{285}{32}$	$\frac{165}{32}$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\frac{117}{32}$	$\frac{221}{32}$	$\frac{357}{32}$	$\frac{525}{32}$	$\cdots$
-2	$\cdots$	$\frac{221}{32}$	$\frac{117}{32}$	$\frac{45}{32}$	$\frac{5}{32}$	$-\frac{3}{32}$	$\frac{21}{32}$	$\frac{77}{32}$	$\frac{165}{32}$	$\frac{285}{32}$	$\frac{437}{32}$	$\frac{621}{32}$	$\cdots$
-3	$\cdots$	$\frac{165}{32}$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\frac{117}{32}$	$\frac{221}{32}$	$\frac{357}{32}$	$\frac{525}{32}$	$\frac{725}{32}$	$\cdots$
-4	$\cdots$	$\frac{117}{32}$	$\frac{45}{32}$	$\frac{5}{32}$	$-\frac{3}{32}$	$\frac{21}{32}$	$\frac{77}{32}$	$\frac{165}{32}$	$\frac{285}{32}$	$\frac{437}{32}$	$\frac{621}{32}$	$\frac{837}{32}$	$\cdots$
-5	$\cdots$	$\frac{77}{32}$	$\frac{21}{32}$	$-\frac{3}{32}$	$\frac{5}{32}$	$\frac{45}{32}$	$\frac{117}{32}$	$\frac{221}{32}$	$\frac{357}{32}$	$\frac{525}{32}$	$\frac{725}{32}$	$\frac{957}{32}$	$\cdots$
	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
		-5	-4	-3	-2	-1	0	1	2	3	4	5	r

Summary

- A Yang-Baxter integrable model of critical dense polymers is exactly solvable on arbitrary *finite size* lattices by solving transfer matrix functional equations in the form of *inversion identities*.

- On the strip, general integrable Robin boundary conditions are constructed as linear combinations of Neumann and Dirichlet boundary conditions.

- The commuting double row transfer matrices satisfy simple functional equations in the form of inversion identities.

- Solving the inversion identities using Euler-Maclaurin gives the bulk and boundary free energies as well as the central charge $c = -2$ and conformal weights

$$\Delta_{r,s-\frac{1}{2}} = \frac{(4r-2s+1)^2-4}{32}, \qquad r,s = 1,2,3,\ldots$$

- The Robin boundary conditions are thus conjugate to scaling fields with conformal weights $\Delta_{r,s-\frac{1}{2}}$ giving rise to an infinitely extended Virasoro Kac table with half-integer Kac labels.

- The partition function is the known partition function of the $\mathbb{Z}_4$ sector of symplectic fermions.

- It would be of interest to obtain numerically the conformal weights of the $\mathcal{LM}(p,p')$ models with Robin boundary conditions to see if half-integer Kac labels appear.

Free Fermions Versus Symplectic Fermions

Ising Model

$c = \frac{1}{2}$ rational CFT = free fermions

Crossing parameter: $\lambda = \pi/4$

Scaling fields = $\{I, \sigma, \epsilon\} = \{\varphi_{1,1}, \varphi_{1,2}, \varphi_{1,3}\}$

s		
3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$
	1	2 r

$$\alpha = 0,\ \nu = 1 \quad \Leftrightarrow \quad \Delta_\varepsilon = \Delta_{1,3} = \frac{1-\alpha}{2-\alpha} = \frac{1}{2}$$

$$\beta = \tfrac{1}{8} \quad \Leftrightarrow \quad \Delta_\sigma = \Delta_{1,2} = \tfrac{1}{2}\beta = \frac{1}{16}$$

Critical Dense Polymers

$c = -2$ log CFT = symplectic fermions

Crossing parameter: $\lambda = \pi/2$

Scaling fields = $\{\varphi_{r,s}\}_{r,s\in\mathbb{N}}$

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

$$\alpha = 1,\ \nu = \tfrac{1}{2} \quad \Leftrightarrow \quad \Delta_t = \Delta_{1,3} = \frac{1-\alpha}{2-\alpha} = 0$$

$$\beta_{r,s} = \Delta_{r,s}, \qquad (2r-s)^2 < 8$$