Algebraic Bethe ansatz for 19-vertex models with upper triangular K-matrices

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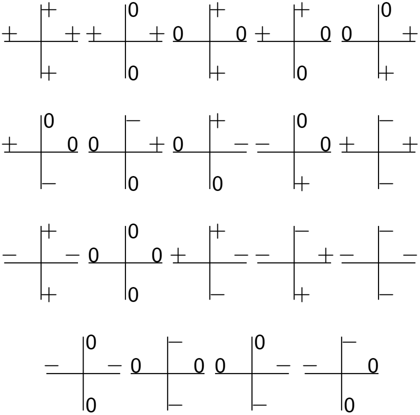
joint work with A. Lima-Santos, arXiv:1406.5757

Outline

The models

- Three-state vertex models constrained by the ice rule (19 non-null Boltzmann weights):



- Allows one to study spin-1 Hamiltonians.

R-matrix

- Solution of the Yang-Baxter equation: Zamolodchikov -Fateev (ZF) and Izergin-Korepin (IK) models

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$$R(u) = \left(\begin{array}{ccc|ccc|ccc} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f(u) & 0 & d(u) & 0 & h(u) & 0 & 0 \\ \hline 0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{d}(u) & 0 & e(u) & 0 & d(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0 \\ \hline 0 & 0 & \tilde{h}(u) & 0 & \tilde{d}(u) & 0 & f(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c(u) & 0 & b(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u) \end{array}\right)$$

The non-null entries are trigonometric functions.

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a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & f(u) & 0 & d(u) & 0 & h(u) & 0 & 0\\
\hline
0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \tilde{d}(u) & 0 & e(u) & 0 & d(u) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0\\
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0 & 0 & \tilde{h}(u) & 0 & \tilde{d}(u) & 0 & f(u) & 0 & 0\\
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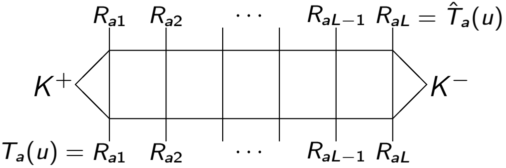
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u)
\end{array}\right)$$

The non-null entries are trigonometric functions.

- Yang-Baxter allow one to construct:
$t(u)=\mathrm{Tr}_a\left[R_{a1}(u)\dots R_{aL}(u)\right]$ such that $[t(u),t(v)]=0$.

Boundaries

- Sklyanin 88



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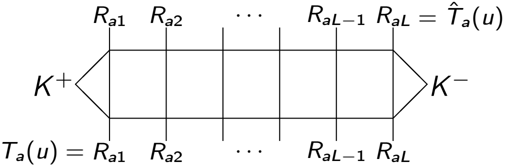

- Double-row monodromy: $U_a(u) = T_a(u)K^-(u)\hat{T}_a(u)$

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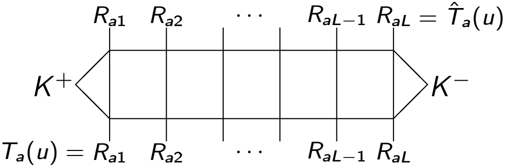

- Double-row monodromy: $U_a(u) = T_a(u)K^-(u)\hat{T}_a(u)$
- Double-row transfer matrix: $t(u) = \mathrm{Tr}_a[K^+(u)T_a(u)K^-(u)\hat{T}_a(u)]$

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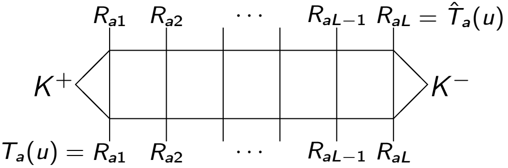

- Double-row monodromy: $U_a(u) = T_a(u)K^-(u)\hat{T}_a(u)$
- Double-row transfer matrix: $t(u) = \mathrm{Tr}_a[K^+(u)T_a(u)K^-(u)\hat{T}_a(u)]$
- Reflection equations have to take into account symmetries of the $R-$matrices (Mezincescu-Nepomechie 91),

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v)$$

$$\begin{aligned} & R_{12}(v-u)K_1^{+t_1}(u)M_1^{-1}R_{21}(-u-v-2\rho)M_1K_2^{+t_2}(v) \\ = \quad & K_2^{+t_2}(v)M_1R_{12}(-u-v-2\rho)M_1^{-1}K_1^{+t_1}(u)R_{21}(v-u) \end{aligned}$$

- We have $[t(u), t(v)] = 0$.

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- Very recently, the spectrum of the XXX-s (Nepomechie 13 $(s, 1/2)$, Cao-Cui-Yang-Shi-Wang 14 (s, s)) and IK model (Hao-Cao-Li-Yang-Shi-Wang 14) with non-diagonal boundaries have been obtained (ODBA: talks of Y. Wang and W.L. Yang).

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- Other methods, for instance the functional-YB (Galleas 08), q-Onsager (Baseilhac-Koizumi 07), separation of variables (Frahm-Seel-Wirth 08, Frahm-Grelick-Seel-Wirth 11, Niccoli 12, Faldella-Kitanine-Niccoli 13), are still not available to 19-vertex models.

K-matrices

- 19-vertex models in the ABA framework:

K^+	K^-	6V	ZF	IK
diagonal	diagonal	Sklyanin 88	Kurak-Lima 04	Fan 97 Li-Shi-Yue 03 Kurak-Lima 04
diagonal	upper	Cao-Lin-Shi-Wang 03 Doikou 07 (XXZ-s)	Melo-Ribeiro-Martins (XXX-s) 05	$\subset$ PL 14
upper	upper	Belliard-Crampé-Ragoucy 13 (XXX) PL 13 (XXZ) António-Manojlović-Salom (XXX) 14	PL 14	PL 14
full	diagonal	Belliard-Crampé 13 (XXX)	-	-
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- In the 6V-XXX case: the full-diagonal and lower-upper cases solve the general case, up to similarity transformations. To other models, this fact remains to be investigated.

Bethe ansatz

- Representation for $U_a(u)$:

$$T_a(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{31}(u) & T_{33}(u) \end{pmatrix}, \hat{T}_a(u) = \begin{pmatrix} \hat{T}_{11}(u) & \hat{T}_{12}(u) & \hat{T}_{13}(u) \\ \hat{T}_{21}(u) & \hat{T}_{22}(u) & \hat{T}_{23}(u) \\ \hat{T}_{31}(u) & \hat{T}_{31}(u) & \hat{T}_{33}(u) \end{pmatrix}$$

$$U_a(u) = \begin{pmatrix} \mathcal{A}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{A}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{A}_3(u) \end{pmatrix}, \quad K^{\pm}(u) = \begin{pmatrix} k^{\pm}_{11}(u) & k^{\pm}_{12}(u) & k^{\pm}_{13}(u) \\ 0 & k^{\pm}_{22}(u) & k^{\pm}_{23}(u) \\ 0 & 0 & k^{\pm}_{33}(u) \end{pmatrix}$$

- For convenience, one introduce

$$\begin{aligned} \mathcal{D}_1(u) &= \mathcal{A}_1(u), \quad \mathcal{D}_2(u) = \mathcal{A}_2(u) - f_1(u)\mathcal{A}_1(u) \\ \mathcal{D}_3(u) &= \mathcal{A}_3(u) - f_2(u)\mathcal{A}_1 - f_3(u)\mathcal{D}_2(u) \end{aligned}$$

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- Transfer matrix:

$$t(u) = \underbrace{\omega_1(u)\mathcal{D}_1(u) + \omega_2(u)\mathcal{D}_2(u) + \omega_3(u)\mathcal{D}_3(u)}_{t_d(u):\text{ 'diagonal' term}} + \underbrace{k_{12}^{+}(u)\mathcal{C}_1(u) + k_{13}^{+}(u)\mathcal{C}_2(u) + k_{23}^{+}(u)\mathcal{C}_3(u)}_{t_u(u):\text{ anihilation term}}$$

Bethe ansatz

- Reference state:

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{(1)} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{(2)} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{(L)} ,$$

- Since:

$$\mathcal{D}_j(u)\Psi_0 = \Delta_j(u)\Psi_0 \quad \text{for} \quad j = 1, 2, 3$$

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- Ψ_0 is an eigenvector of $t(u)$:

$$t(u)\Psi_0 = \Lambda_0(u)\Psi_0$$

Bethe ansatz

- Eigenvectors of t_d (Fan 97, Li-Shi-Yue 03, Kurak-Lima 04):

$$\Psi_1(u_1) = \mathcal{B}_1(u_1)\Psi_0,$$

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- Tarasov-like recurrence relation:

$$\Psi_n(u_1, \dots, u_n) = \mathcal{B}_1(u_1)\Psi_{n-1}(u_2, \dots, u_n)$$

$$- \quad \mathcal{B}_2(u_1)\sum_{i=2}^{n}\Gamma_i^{(n)}(u_1, \dots, u_n)\Psi_{n-2}(u_2, \dots, \hat{u}_i, \dots, u_n)$$

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- We have:

$$t_d(u)\Psi_n(u_1,\ldots,u_n) = \Lambda_n(u,u_1,\ldots,u_n)\Psi_n(u_1,\ldots,u_n)$$
$$+ \quad \mathcal{B}_1(u)\sum_{j=1}^{n}\mathcal{F}_j^{(n)}(u,u_1,\ldots,u_n)\Psi_{n-1}(u_1,\ldots,\hat{u}_j,\ldots,u_n)$$
$$+ \quad \mathcal{B}_3(u)\sum_{j=1}^{n}\mathcal{G}_j^{(n)}(u,u_1,\ldots,u_n)\Psi_{n-1}(u_1,\ldots,\hat{u}_j,\ldots,u_n)$$

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- To compensate the action of the $t_u(u)$, we consider a superposition of the "diagonal" Bethe states:

$$\Phi_n = \sum_{k=0}^{n} g^{(k)} \Psi_k$$

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- We implemented this ansatz for the trigonometric six vertex model, presenting a constructive way to fix the $g-$coefficients: we impose the vanishing of extra unwanted terms in the ABA analysis.
- In this work, we show that this is also possible for $19-$vertex models.

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$$t_u(u)\Psi_n(u_1,\dots,u_n) = \sum_{j=1}^{n} \mathcal{T}_j^{(n)}(u,u_1,\dots,u_n)\Psi_{n-1}(u_1,\dots,\hat{u}_j,\dots,u_n)$$

$$+ \quad \sum_{j<k}^{n} \mathcal{U}_{jk}^{(n)}(u,u_1,\dots,u_n)\Psi_{n-2}(u_1,\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,u_n)$$

$$+ \quad \mathcal{B}_1(u)\sum_{j<k}^{n} \mathcal{V}_{jk}^{(n)}(u,u_1,\dots,u_n)\Psi_{n-2}(u_1,\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,u_n)$$

$$+ \quad \mathcal{B}_3(u)\sum_{j<k}^{n} \mathcal{W}_{jk}^{(n)}(u,u_1,\dots,u_n)\Psi_{n-2}(u_1,\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,u_n)$$

$$+ \quad \mathcal{B}_1(u)\sum_{j<k<\ell}^{n} \mathcal{X}_{jk\ell}^{(n)}(u,u_1,\dots,u_n)\Psi_{n-3}(u_1,\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,\hat{u}_\ell,\dots,u_n)$$

$$+ \quad \mathcal{B}_3(u)\sum_{j<k<\ell}^{n} \mathcal{Y}_{jk\ell}^{(n)}(u,u_1,\dots,u_n)\Psi_{n-3}(u_1,\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,\hat{u}_\ell,\dots,u_n)$$

$$+ \quad \mathcal{B}_2(u)\sum^{n} \mathcal{Z}_{jk\ell}^{(n)}(u,u_1,\dots,u_n)\Psi_{n-3}(u_1,\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,\hat{u}_\ell,\dots,u_n)$$

Bethe ansatz - 1st excited state

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- Acting with $t(u)$ on $\Phi_1(u)$,

$$t(u)\Phi_1(u_1) = t_d(u)\Psi_1(u_1) + g(u_1)t_d(u)\Psi_0 + t_u(u)\Psi_1(u_1)$$

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- Off-shell structure

$$\begin{aligned} t(u)\Phi_1(u_1) &= \Lambda_1(u,u_1)\Phi_1(u_1) \\ &+ \mathcal{F}_1^{(1)}(u,u_1)\mathcal{B}_1(u)\Psi_0 + \mathcal{G}_1^{(1)}(u,u_1)\mathcal{B}_3(u)\Psi_0 \\ &+ \left\{ g(u_1)\left[\Lambda_0(u) - \Lambda_1(u,u_1)\right] + \mathcal{T}_1^{(1)}(u,u_1) \right\} \Psi_0 \end{aligned}$$

Bethe ansatz - 1st excited state

- Usual unwanted terms: Bethe equation

$$\mathcal{F}_1^{(1)}(u,u_1)=\mathcal{G}_1^{(1)}(u,u_1)=0 \Leftrightarrow \frac{\Delta_1(u_1)}{\Delta_2(u_1)}=-\Theta(u_1)$$

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- Extra unwanted term: fix g

$$g(u_1)\left[\Lambda_1(u, u_1) - \Lambda_0(u)\right] = \mathcal{T}_1^{(1)}(u, u_1)$$

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- Extra unwanted term: fix g

$$g(u_1)\left[\Lambda_1(u,u_1) - \Lambda_0(u)\right] = \mathcal{T}_1^{(1)}(u,u_1)$$

- Use Bethe equation:

$$g(u_1) = \beta_+ \left[\frac{\sinh(2u_1+\eta)}{\sinh(u_1+\frac{\eta}{2}-\xi_+)}\right]\Delta_2(u_1)$$

in the case of the ZF model and

$$g(u_1) = \beta_+ \left[\frac{\sinh(2u_1+\eta)}{\sinh(u_1+\frac{3\eta}{4}-\epsilon\frac{i\pi}{4})}\right]\Delta_2(u_1)$$

for the IK model

Bethe ansatz - 2nd excited state

$$\begin{aligned}\Phi_2(u_1, u_2) &= \Psi_2(u_1, u_2) \\ &+ g_2^{(1)}(u_1, u_2)\Psi_1(u_1) + g_1^{(1)}(u_1, u_2)\Psi_1(u_2) \\ &+ g_{12}^{(0)}(u_1, u_2)\Psi_0\end{aligned}$$

Bethe - 2nd excited state

$$
\begin{aligned}
& t(u)\Phi_2(u_1,u_2) = \Lambda_2(u,u_1,u_2)\Phi_2(u_1,u_2) \\
+ \quad & \mathcal{F}_1^{(2)}(u,u_1,u_2)\mathcal{B}_1(u)\Psi_1(u_2) + \mathcal{F}_2^{(2)}(u,u_1,u_2)\mathcal{B}_1(u)\Psi_1(u_1) \\
+ \quad & \mathcal{G}_1^{(2)}(u,u_1,u_2)\mathcal{B}_3(u)\Psi_1(u_2) + \mathcal{G}_2^{(2)}(u,u_1,u_2)\mathcal{B}_3(u)\Psi_1(u_1) \\
+ \quad & \mathcal{H}_{12}^{(2)}(u,u_1,u_2)\mathcal{B}_2(u)\Psi_0 \\
+ \quad & \left\{ g_2^{(1)}(u_1,u_2)\mathcal{F}_1^{(1)}(u,u_1) + g_1^{(1)}(u_1,u_2)\mathcal{F}_1^{(1)}(u,u_2) + \mathcal{V}_{12}^{(2)}(u,u_1,u_2) \right\} \mathcal{B}_1(u)\Psi_0 \\
+ \quad & \left\{ g_2^{(1)}(u_1,u_2)\mathcal{G}_1^{(1)}(u,u_1) + g_1^{(1)}(u_1,u_2)\mathcal{G}_1^{(1)}(u,u_2) + \mathcal{W}_{12}^{(2)}(u,u_1,u_2) \right\} \mathcal{B}_3(u)\Psi_0 \\
+ \quad & \left\{ g_1^{(1)}(u_1,u_2)\left[\Lambda_1(u,u_2) - \Lambda_2(u,u_1,u_2)\right] + \mathcal{T}_1^{(2)}(u,u_1,u_2) \right\} \Psi_1(u_2) \\
+ \quad & \left\{ g_2^{(1)}(u_1,u_2)\left[\Lambda_1(u,u_1) - \Lambda_2(u,u_1,u_2)\right] + \mathcal{T}_2^{(2)}(u,u_1,u_2) \right\} \Psi_1(u_1) \\
+ \quad & \left\{ g_{12}^{(0)}(u_1,u_2)\left[\Lambda_0(u) - \Lambda_2(u,u_1,u_2)\right] \right. \\
+ \quad & \left. g_1^{(1)}(u_1,u_2)\mathcal{T}_1^{(1)}(u,u_2) + g_2^{(1)}(u_1,u_2)\mathcal{T}_1^{(1)}(u,u_1) + \mathcal{U}_{12}^{(2)}(u,u_1,u_2) \right\} \Psi_0
\end{aligned}
$$

Bethe - 2nd excited state

- Bethe equations:

$$\frac{\Delta_1(u_j)}{\Delta_2(u_j)} = -\Theta(u_j) \prod_{k=1,k\neq j}^{2} \frac{a_{21}(u_j, u_k)}{a_{11}(u_j, u_k)}.$$

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$$\frac{\Delta_1(u_j)}{\Delta_2(u_j)} = -\Theta(u_j) \prod_{k=1,k\neq j}^{2} \frac{a_{21}(u_j,u_k)}{a_{11}(u_j,u_k)}.$$

- Coefficients:

$$\begin{aligned} g_1^{(1)}(u_1,u_2) &= g(u_1)a_{21}(u_1,u_2), \\ g_2^{(1)}(u_1,u_2) &= g(u_2)a_{21}(u_2,u_1)\Omega(u_2,u_1), \\ g_{12}^{(0)}(u_1,u_2) &= g(u_1)g(u_2)s(u_1,u_2) \end{aligned}$$

Bethe - 2nd excited state

$$s(u_1, u_2) = \frac{\sinh(u_1 + u_2)\sinh(u_1 - u_2 - \eta)\sinh\left(u_1 + u_2 + \frac{3\eta}{2}\right)}{\sinh\left(u_1 - u_2 - \frac{\eta}{2}\right)\sinh\left(u_1 + u_2 + \frac{\eta}{2}\right)^2}$$

for the ZF model and,

$$\begin{aligned} s(u_1, u_2) \quad &= \quad \frac{\cosh\left(u_1 + u_2 + \frac{\eta}{2}\right)\sinh(u_1 + u_2 + 2\eta)}{\sinh(u_1 + u_2)\cosh\left(u_1 - u_2 + \frac{\eta}{2}\right)\cosh\left(u_1 + u_2 + \frac{3\eta}{2}\right)^2} \\ &\times \quad \left[\cosh\left(u_1 + \frac{\eta}{4} - i\epsilon\frac{\pi}{4}\right)\cosh\left(u_1 + \frac{3\eta}{4} + i\epsilon\frac{\pi}{4}\right)\right. \\ &\quad + \left.\cosh\left(u_2 - \frac{\eta}{4} + i\epsilon\frac{\pi}{4}\right)\cosh\left(u_2 + \frac{5\eta}{4} - i\epsilon\frac{\pi}{4}\right)\right] \end{aligned}$$

for the IK solution.

Bethe - general excited state

- Eigenstates:

$$\Phi_n(u_1,\dots,u_n) = \Psi_n(u_1,\dots,u_n)$$
$$+ \sum_{k=0}^{n-1} \sum_{\ell_1<\dots<\ell_{n-k}=1}^{n} g^{(k)}_{\ell_1,\dots,\ell_{n-k}}(u_1,\dots,u_n)\Psi_k(u_1,\dots,\hat{u}_{\ell_1},\dots,\hat{u}_{\ell_{n-k}},\dots,u_n)$$

- $g-$functions:

$$g^{(k)}_{\ell_1,\dots,\ell_{n-k}}(u_1,\dots,u_n) = \prod_{m\in\bar{\ell}} g(u_m) \prod_{m'\in\bar{\ell},m'<m} s(u_{m'},u_m) \prod_{m''=1,m''\notin\bar{\ell}}^{n} a_{21}(u_m,u_{m''})\tilde{\Omega}_{m}$$

with $\bar{\ell} = \{\ell_1,\dots,\ell_{n-k}\}$ and

$$\tilde{\Omega}_{m,m''} = \begin{cases} \Omega(u_m,u_{m''}), & \text{if } m > m'' \\ 1, & \text{otherwise.} \end{cases}$$

- Eigenvalues:

$$\Lambda_n(u,u_1,\dots,u_n) = \sum_{\alpha=1}^{3} \omega_\alpha(u)\Delta_\alpha(u) \prod_{j=1}^{n} a_{\alpha 1}(u,u_j)$$

- Bethe equations:

$$\frac{\Delta_1(u_j)}{\Delta_2(u_j)} = -\Theta(u_j) \prod_{k=1,k\neq j}^{n} \frac{a_{21}(u_j,u_k)}{a_{11}(u_j,u_k)}$$

for $j = 1,\dots,n$

Conclusion

- Upper-triangular boundaries: we still have a simple reference state and, thus, we can execute the ABA.
- Same spectrum and Bethe equations, eigenvectors are a superposition of auxiliary Bethe vectors.
- More general $K-$matrices, with constrained boundary parameters can, in principle, be studied with our results.
- Concerning 19-vertex and other higher rank models, the ABA is still a powerfull method to construct eigenvectors of vertex models with non-diagonal boundaries.

Open problems

- Upper-upper case:
  - Investigate simpler off-shell Bethe vectors;
  - Gaudin magnet: spectrum and eigenvectors, connection with KZ-equations;
  - Beyond the spectral level: scalar products (still an open problem even for the periodic case - SOV?);
  - Other 19- and higher hank vertex models.
- General case:
  - Gauge-similarity transformations for 19-vertex models. More general reflection matrices with constrained boundary parameters.
  - Investigate if the full-diagonal and lower-upper case can solve the general integrable boundary parameters case.