Twisted Yangians for symmetric pairs of types B, C, D

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1 Abstract

Twisted Yangians are a class of quantized enveloping algebras associated with symmetric pairs of Lie algebras that can be presented as coideal subalgebras of the (non-twisted) Yangians or as quotients of a reflection algebra by additional symmetry relations. Twisted Yangians of types AI and AII, corresponding to the symmetric pairs $(\mathfrak{gl}_N, \mathfrak{so}_N)$ and $(\mathfrak{gl}_N, \mathfrak{sp}_N)$ and the twisted reflection equation were introduced by G. Olshanskii in [Ol] and [MNO] and have been studied extensively over the past twenty years. Those of type AIII were introduced in [MoRa] where they were called reflection algebras since they can be defined using the non-twisted reflection equation.

In this poster we present new twisted Yangians for the classical Lie algebras of types B, C and D: they are in bijection with the symmetric pairs of types BDI, CI, CII and DIII. (This notation refers to Cartan's classification of symmetric spaces.) We also introduce twisted Yangians of type BCD0 corresponding to the symmetric pair $(\mathfrak{g}_N[x], \mathfrak{g}_N[x^2])$.

2 Yangians $X(\mathfrak{g}_N)$ and $Y(\mathfrak{g}_N)$ for $\mathfrak{g}_N = \mathfrak{so}_N, \mathfrak{sp}_N$

The extended Yangian $X(\mathfrak{g}_N)$ was introduced in [AACFR] and was studied furthermore in [AMR]. It admits as quotients the (non-twisted) orthogonal and symplectic Yangians $Y(\mathfrak{g}_N)$.

Let $n \in \mathbb{N}$ and set $N = 2n$ or $N = 2n+1$. Let $i(j,k,\ldots) \in \{\mp n, \ldots, \mp 1\}$ if $N = 2n$ and $i(j,k,\ldots) \in \{\mp n, \ldots, \mp 1, 0\}$ if $N = 2n+1$. Set $\theta_{ij} = 1$ in the orthogonal case and $\theta_{ij} = \mathrm{sign}(i)\cdot\mathrm{sign}(j)$ in the symplectic case. Let $F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i}$ where E_{ij} is the elementary matrix of $\mathfrak{gl}_N$. Then $\mathfrak{so}_N, \mathfrak{sp}_N = \mathrm{span}_{\mathbb{C}}\{F_{ij}\}_{-n\le i,j\le n}$ satisfying

$$F_{ij} + \theta_{ij} F_{-j,-i} = 0, \qquad [F_{ij}, F_{kl}] = \delta_{jk} F_{il} - \delta_{il} F_{kj} + \theta_{ij}\delta_{j,-l} F_{k,-i} - \theta_{ij}\delta_{i,-k} F_{-j,l}.$$

Let P denote the permutation operator and Q denote the transposed projector on $\mathbb{C}^N \otimes \mathbb{C}^N$, namely

$$P = \sum_{-n\le i,j\le n} E_{ij} \otimes E_{ji}, \qquad Q = \sum_{-n\le i,j\le n} \theta_{ij} E_{ij} \otimes E_{-i,-j}.$$

Set $(E_{ij})^t = \theta_{ij} E_{-j,-i}$ and let I denote the identity matrix. Then $P^2 = I$, $Q = P^{t_1} = P^{t_2}$ and also $PQ = QP = \pm Q$ and $Q^2 = NQ$. Here the upper sign corresponds to the orthogonal case and the lower sign to the symplectic case. Set $\kappa = N/2 \mp 1$. The R matrix $R(u)$ that we will need is defined by [AACFR]

$$R(u) = I - \frac{P}{u} + \frac{Q}{u-\kappa} \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N).$$

It is a solution of the quantum Yang-Baxter equation on $(\mathbb{C}^N)^{\otimes 3}$ with spectral parameter,

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u).$$

Definition 2.1. *[AACFR, AMR] The extended Yangian $X(\mathfrak{g}_N)$ is the associative $\mathbb{C}$-algebra with generators $t_{ij}^{(r)}$ for $-n \le i,j \le n$ and $r \in \mathbb{Z}_{\ge 0}$, which satisfy the following relations:*

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v), \tag{1}$$

where the T-matrix $T(u) \in \mathrm{End}\,\mathbb{C}^N \otimes X(\mathfrak{g}_N)[[u^{-1}]]$ is defined by

$$T(u) = \sum_{-n\le i,j\le n}^{n} E_{ij} \otimes t_{ij}(u), \qquad t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} \in X(\mathfrak{g}_N)[[u^{-1}]], \qquad t_{ij}^{(0)} = \delta_{ij}.$$

The Hopf algebra structure on $X(\mathfrak{g}_N)$ is given by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=-n}^{n} t_{ik}(u) \otimes t_{kj}(u), \qquad S : T(u) \mapsto T^{-1}(u), \qquad \epsilon : T(u) \mapsto I.$$

It follows from the properties of Q and (1) that

$$T^t(u+\kappa)\, T(u) = T(u)\, T^t(u+\kappa) = z(u)\cdot I,$$

where $z(u) = 1 + \sum_{i\ge 1} z_i u^{-i}$ is called the quantum contraction of the matrix $T(u)$; its coefficients z_i generate the centre $ZX(\mathfrak{g}_N)$ of $X(\mathfrak{g}_N)$. This leads to the following tensor product decomposition of $X(\mathfrak{g}_N)$ ([AMR], Theorem 3.1):

$$X(\mathfrak{g}_N) = ZX(\mathfrak{g}_N) \otimes Y(\mathfrak{g}_N). \tag{2}$$

The Yangian $Y(\mathfrak{g}_N)$ is thus isomorphic to the quotient of $X(\mathfrak{g}_N)$ by the ideal generated by the elements z_i, that is, $Y(\mathfrak{g}_N) \cong X(\mathfrak{g}_N)/(z(u)-1)$. Moreover, z_i's are algebraically independent over $Y(\mathfrak{g})$.

3 Twisted Yangians as subalgebras of Yangians

The symmetric pairs we are interested in are of the form $(\mathfrak{g}_N, \mathfrak{g}_N^\rho)$ where ρ is an involution of $\mathfrak{g}_N$. The fixed-point subalgebra is $\mathfrak{g}_N^\rho = \mathrm{span}_{\mathbb{C}}\{X \in \mathfrak{g}_N \,|\, X = \mathcal{G}X\mathcal{G}^{-1}\} = \mathrm{span}_{\mathbb{C}}\{X + \mathcal{G}X\mathcal{G}^{-1} \,|\, X \in \mathfrak{g}_N\}$ where:

- BCD0: $\mathcal{G} = I$, ρ is trivial and $\mathfrak{g}_N^\rho = \mathfrak{g}_N$.
- CI : N is even, $\mathfrak{g}_N = \mathfrak{sp}_N$, $\mathcal{G} = \sum_{i=1}^{\frac{N}{2}} (E_{ii} - E_{-i,-i})$ and $\mathfrak{g}_N^\rho \cong \mathfrak{gl}_{\frac{N}{2}}$.
- DIII : N is even, $\mathfrak{g}_N = \mathfrak{so}_N$, $\mathcal{G} = \sum_{i=1}^{\frac{N}{2}} (E_{ii} - E_{-i,-i})$ and $\mathfrak{g}_N^\rho \cong \mathfrak{gl}_{\frac{N}{2}}$.
- CII : N, p and q are even and > 0, $N = p+q$, $\mathfrak{g}_N = \mathfrak{sp}_N$,

$$\mathcal{G} = -\sum_{i=1}^{\frac{q}{2}} (E_{ii} + E_{-i,-i}) + \sum_{i=\frac{q}{2}+1}^{\frac{N}{2}} (E_{ii} + E_{-i,-i})$$

  and $\mathfrak{g}_N^\rho = \mathfrak{sp}_p \oplus \mathfrak{sp}_q$. More precisely, the subalgebra of $\mathfrak{g}_N^\rho$ spanned by F_{ij} with $-\frac{q}{2} \le i,j \le \frac{q}{2}$ is isomorphic to $\mathfrak{sp}_q$ and the subalgebra of $\mathfrak{g}_N^\rho$ spanned by F_{ij} with $|i|,|j| > \frac{q}{2}$ is isomorphic to $\mathfrak{sp}_p$.
- BDI : $\mathfrak{g}_N = \mathfrak{so}_N$, $\mathfrak{g}_N^\rho = \mathfrak{so}_p \oplus \mathfrak{so}_q$ where $p > q > 0$ if N is odd, and $p \ge q > 0$ if N is even. (If $q = 1$, then $\mathfrak{so}_q$ is the zero Lie algebra.) When N is even, p and q have the same parity; when N is odd, $p - q$ is also odd. Now $\mathcal{G}$ is given by

$$\mathcal{G} = \sum_{i=1}^{\frac{p-q}{2}} (E_{ii} + E_{-i,-i}) + \sum_{i=\frac{p-q}{2}+1}^{\frac{N}{2}} (E_{-i,i} + E_{i,-i}), \qquad \mathcal{G} = \sum_{i=-\frac{p-q-1}{2}}^{\frac{p-q-1}{2}} E_{ii} + \sum_{i=\frac{p-q+1}{2}}^{\frac{N-1}{2}} (E_{-i,i} + E_{i,-i}),$$

  for even and odd cases, respectively.

Definition 3.1. *Let the matrix $\mathcal{G}$ be as described above. The extended twisted Yangian $X(\mathfrak{g}_N, \mathcal{G})^{tw}$ is the subalgebra of $X(\mathfrak{g}_N)$ generated by the coefficients of the entries of the S-matrix*

$$S(u) = T(u-\kappa/2)\, \mathcal{G}(u)\, T^t(-u+\kappa/2), \tag{3}$$

where

- *$\mathcal{G}(u) = \mathcal{G}$ for cases BCD0, CI, DIII and DI, CII when $p = q$;*
- *$\mathcal{G}(u) = (I - c\,u\,\mathcal{G})(1 - c\,u)^{-1}$ with $c = \frac{4}{p-q}$ for cases BDI, CII when $p > q$.*

In the algebra $X(\mathfrak{g}_N, \mathcal{G})^{tw}$, the product $S(u)\,S(-u)$ is a scalar matrix

$$S(u)\,S(-u) = w(u)\cdot I,$$

where $w(u)$ is an even formal power series in u^{-1} with coefficients $w_2, w_4, \ldots$, central in $X(\mathfrak{g}_N, \mathcal{G})^{tw}$.

Definition 3.2. *The twisted Yangian $Y(\mathfrak{g}_N, \mathcal{G})^{tw}$ is the quotient of $X(\mathfrak{g}_N, \mathcal{G})^{tw}$ by the ideal generated by the coefficients of the unitarity relation, i.e.,*

$$Y(\mathfrak{g}_N, \mathcal{G})^{tw} = X(\mathfrak{g}_N, \mathcal{G})^{tw} / (S(u)\,S(-u) - I).$$

Let us denote by $ZX(\mathfrak{g}_N, \mathcal{G})^{tw}$ the commutative algebra generated by the coefficients of $w(u)$. This algebra is the centre of $X(\mathfrak{g}_N, \mathcal{G})^{tw}$. The two twisted Yangians are related via the following tensor product decomposition (which is the analogue of (2)):

$$X(\mathfrak{g}_N, \mathcal{G})^{tw} \cong ZX(\mathfrak{g}_N, \mathcal{G})^{tw} \otimes Y(\mathfrak{g}_N, \mathcal{G})^{tw}.$$

The new algebras, as Olshanskii's twisted Yangians, are coideal subalgebras of a larger Yangian.

Proposition 3.1. *The algebra $X(\mathfrak{g}_N, \mathcal{G})^{tw}$ is a left coideal subalgebra of $X(\mathfrak{g}_N)$:*

$$\Delta(X(\mathfrak{g}_N, \mathcal{G})^{tw}) \subset X(\mathfrak{g}_N) \otimes X(\mathfrak{g}_N, \mathcal{G})^{tw}.$$

namely, for any $s_{ij}(u) \in X(\mathfrak{g}_N, \mathcal{G})^{tw}[[u^{-1}]]$ we have

$$\Delta(s_{ij}(u)) = \sum_{a,b=-n}^{n} \theta_{jb}\, t_{ia}(u-\kappa/2)\, t_{-j,-b}(-u+\kappa/2) \otimes s_{ab}(u),$$

The same property holds for $Y(\mathfrak{g}_N, \mathcal{G})^{tw}$; it is a left coideal subalgebra of $Y(\mathfrak{g}_N)$.

4 Twisted Yangians as reflection algebras

Definition 4.1. *The extended reflection algebra $\mathcal{XB}(\mathcal{G})$ is the unital associative algebra generated by elements $\mathsf{s}_{ij}^{(r)}$ for $-n \le i,j \le n$, $r \in \mathbb{Z}_{\ge 0}$ satisfying the reflection equation*

$$R(u-v)\, \mathsf{S}_1(u)\, R(u+v)\, \mathsf{S}_2(v) = \mathsf{S}_2(v)\, R(u+v)\, \mathsf{S}_1(u)\, R(u-v), \tag{4}$$

where the S-matrix $\mathsf{S}(u)$ is defined by

$$\mathsf{S}(u) = \sum_{i,j=-n}^{n} E_{ij} \otimes \mathsf{s}_{ij}(u) \in \mathrm{End}(\mathbb{C}^N) \otimes \mathcal{XB}(\mathcal{G})[[u^{-1}]], \quad \mathsf{s}_{ij}(u) = \sum_{r=0}^{\infty} \mathsf{s}_{ij}^{(r)} u^{-r}, \quad \mathsf{s}_{ij}^{(0)} = g_{ij}.$$

There exists a formal power series $\mathsf{c}(u)$ in u^{-1} with coefficients c_i $(i = 1, 2, \ldots)$ central in $\mathcal{XB}(\mathcal{G})$, such that the following identity holds

$$Q\, \mathsf{S}_1(u)\, R(2u-\kappa)\, \mathsf{S}_2^{-1}(\kappa-u) = \mathsf{S}_2^{-1}(\kappa-u)\, R(2u-\kappa)\, \mathsf{S}_1(u)\, Q = p(u)\, \mathsf{c}(u)\, Q,$$

where

$$p(u) = (\pm)\, 1 \mp \frac{1}{2u-\kappa} + \frac{\mathrm{tr}(\mathcal{G}(u))}{2u-2\kappa}.$$

The minus sign in the special notation $(\pm)$ is for the cases CI and DIII only.

Definition 4.2. *The reflection algebra $\mathcal{B}(\mathcal{G})$ is isomorphic the quotient of $\mathcal{XB}(\mathcal{G})$ by the ideal generated by the coefficients of the series $\mathsf{c}(u)$:*

$$\mathcal{B}(\mathcal{G}) \cong \mathcal{XB}(\mathcal{G})/(\mathsf{c}(u)-1).$$

The odd coefficients of $\mathsf{c}(u)$ are algebraically independent over $\mathcal{XB}(\mathcal{G})$. Moreover, the constraint $\mathsf{c}(u) = 1$ is equivalent to the symmetry relation

$$\mathsf{S}^t(u) = (\pm)\, \mathsf{S}(\kappa-u) \pm \frac{\mathsf{S}(u) - \mathsf{S}(\kappa-u)}{2u-\kappa} + \frac{\mathrm{tr}(\mathcal{G}(u))\, \mathsf{S}(k-u) - \mathrm{tr}(\mathsf{S}(u))\cdot I}{2u-2\kappa}. \tag{5}$$

In the algebra $\mathcal{B}(\mathcal{G})$ the product $\mathsf{S}(u)\,\mathsf{S}(-u) = \mathsf{w}(u)\cdot I$ is a scalar matrix, where $\mathsf{w}(u)$ is an even formal power series in u^{-1} with coefficients w_i $(i = 2, 4, \ldots)$ central in $\mathcal{B}(\mathcal{G})$.

Definition 4.3. *The unitary reflection algebra $\mathcal{UB}(\mathcal{G})$ is the quotient of the reflection algebra $\mathcal{B}(\mathcal{G})$ by the ideal generated by the unitarity constrain*

$$\mathsf{S}(u)\,\mathsf{S}(-u) = I.$$

Let us comment on the choice of the reflection equation (4). Consider the twisted reflection equation

$$R(u-v)\, S_1'(u)\, R^t(-u-v)\, S_2'(v) = S_2'(v)\, R^t(-u-v)\, S_1'(u)\, R(u-v). \tag{6}$$

Observe that $R^t(u) = R(\kappa-u)$. Then it is possible to see that (6) is equivalent to (4) upon identification $\mathsf{S}'(u) = \mathsf{S}(u+\kappa/2)$. Moreover, the choice of (4) has motivated the form of the S-matrix $S(u)$ in (3). For the twisted reflection equation (6) the natural choice would be $S'(u) = T(u)\,\mathcal{G}(u+\kappa/2)\,T^t(-u)$, the unitarity relation would become $S'(u)\,S'(-\kappa-u) = I$.

The S-matrix $S(u)$ given by (3) automatically satisfies the reflection equation (4) and the symmetry relation (5). Moreover, the map

$$\phi \;:\; \mathcal{B}(\mathcal{G}) \to X(\mathfrak{g}_N, \mathcal{G})^{tw}, \quad \mathsf{S}(u) \mapsto S(u) = T(u-\kappa/2)\, \mathcal{G}(u)\, T^t(-u+\kappa/2),$$

is both injective and surjective.

Theorem 4.1. *The extended twisted Yangian $X(\mathfrak{g}_N, \mathcal{G})^{tw}$ is isomorphic via ϕ to $\mathcal{B}(\mathcal{G})$.*

Theorem 4.2. *The twisted Yangian $Y(\mathfrak{g}_N, \mathcal{G})^{tw}$ is isomorphic to $\mathcal{UB}(\mathcal{G})$.*

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