A recursive construction of joint eigenfunctions for the commuting hyperbolic Calogero-Moser Hamiltonians
(Joint work with M. Hallnäs)

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Introduction

Reminders

- The hyperbolic \((A_{N-1})\) Calogero-Moser systems are integrable systems describing \(N\) particles on the line with hyperbolic pair interaction.

- The nonrelativistic quantum version is defined by the Hamiltonian

\[
H \equiv -\frac{\hbar^2}{2} \sum_{j=1}^{N} \partial^2 x_j + g(g - \hbar) \sum_{1 \leq j < l \leq N} V(x_j - x_l),
\]

with \(\hbar > 0\) (Planck’s constant), \(g \in \mathbb{R}\) (coupling constant), and pair potential

\[
V(x) \equiv \frac{\mu^2}{4 \sinh^2(\mu x/2)}, \quad \mu > 0.
\]

- The \(N = 2\) Schrödinger equation can be solved via the conical function, a specialization of the Gauss hypergeometric function.
Associated **integrable system** \((N\text{ commuting PDOs})\):

\[
H_1 \equiv -i\hbar \sum_{j=1}^{N} \partial_{x_j}, \quad H_2 \equiv -\hbar^2 \sum_{1 \leq j_1 < j_2 \leq N} \partial_{x_{j_1}} \partial_{x_{j_2}} - g(g - \hbar) \sum_{1 \leq j < l \leq N} V(x_j - x_l),
\]

\[
H_k \equiv (-i\hbar)^k \sum_{1 \leq j_1 < \ldots < j_k \leq N} \partial_{x_{j_1}} \ldots \partial_{x_{j_k}} + \text{l.o., } \quad k = 3, \ldots, N,
\]

where l.o. = lower order in partials. Thus, the defining Hamiltonian is given by

\[
H = H_1^2/2 - H_2.
\]

Integrable versions exist for Lie algebras \(B_N, \ldots, E_8, F_4, G_2\) (Olshanetsky/Perelomov, Oshima) and \(BC_N\) (Inozemtsev, Oshima).

\(N > 2\) eigenfunctions: Harish-Chandra, Heckman/Opdam, Felder/Varchenko, Chalykh,...
Introduction

Goal

- By proceeding recursively in $N$, construct joint eigenfunctions $\Psi_N(x, p)$ of the Hamiltonians $H_k$:

$$H_k(x)\Psi_N(x, p) = S_k(p)\Psi_N(x, p), \quad k = 1, \ldots, N,$$

where

$$S_k(p) := \sum_{1 \leq j_1 < \cdots < j_k \leq N} p_{j_1} \cdots p_{j_N}.$$

- It is crucial to switch from $\Psi_N$ to $F_N$ given by

$$\Psi_N(g; (x_1, \ldots, x_N), (p_1, \ldots, p_N)) =: W_N(g/\hbar; \mu x/2)^{1/2} \times F_N(g/\hbar; (\mu x_1/2, \ldots, \mu x_N/2), (2p_1/\hbar\mu, \ldots, 2p_N/\hbar\mu)),$$

with

$$W_N(\lambda; t) := \prod_{1 \leq j < k \leq N} [4 \sinh^2(t_j - t_k)]^\lambda.$$
Assuming $\text{Re} \lambda \geq 1$, $u \in \mathbb{R}^N$ and $|\text{Im} t_j| < \pi/2$, we obtain

$$F_N(\lambda; t, u) = \int_{\mathbb{R}^N(N-1)/2} \prod_{n=1}^{N-1} \frac{\prod_{1 \leq j < k \leq n}[4 \sinh^2(t_{nj} - t_{nk})]^{\lambda}}{n! \prod_{j=1}^{n+1} \prod_{k=1}^{n} [2 \cosh(t_{n+1,j} - t_{nk})]^{\lambda}} \times \exp \left( i \sum_{n=1}^{N} u_n \left( \sum_{j=1}^{n} t_{nj} - \sum_{j=1}^{n-1} t_{n-1,j} \right) \right) \prod_{n=1}^{N-1} \prod_{j=1}^{n} dt_{nj},$$

where

$$t_{Nj} := t_j, \quad j = 1, \ldots, N.$$

This integral can also be written

$$\exp \left( i u_N \sum_{j=1}^{N} t_j \right) \prod_{n=1}^{N-1} \int_{t_{nn} < \cdots < t_{n1}} \exp \left( i(u_n - u_{n+1})(t_{n1} + \cdots + t_{nn}) \right) \times \frac{\prod_{1 \leq j < k \leq n}[2 \sinh(t_{nj} - t_{nk})]^{2\lambda}}{\prod_{j=1}^{n+1} \prod_{k=1}^{n} [2 \cosh(t_{n+1,j} - t_{nk})]^{\lambda}} \prod_{j=1}^{n} dt_{nj}. $$
We proceed with an explicit description of the eigenvalue equations for $F_N$.

- The starting point consists of the Lax matrix

$$\mathcal{L}(t, u)_{jk} \equiv \delta_{jk} u_j + (1 - \delta_{jk}) \frac{i \lambda}{\sinh(t_j - t_k)},$$

and the diagonal matrix

$$\mathcal{E}(t) \equiv \text{diag} (w_1(t), \ldots, w_N(t)), $$

with

$$w_j(t) \equiv -i \lambda \sum_{k \neq j} \coth(t_j - t_k).$$
We let \( \hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t) \) denote the normal-ordered PDOs obtained from the symmetric functions

\[
\Sigma_k(\mathcal{L}(t, u) + \mathcal{E}(t)) \equiv \sum_{I \subset \{1, \ldots, N\}, |I|=k} \det(\mathcal{L}(t, u) + \mathcal{E}(t))_I
\]

by performing the substitutions

\[
u_j \rightarrow -i\partial_t, \quad j = 1, \ldots, N.
\]

The Hamiltonians \( \mathcal{H}_k(\lambda; t) \equiv (2/\hbar\mu)^k \mathcal{H}_k(\lambda\hbar; 2t/\mu) \) are then given by (S. R., 1994)

\[
\mathcal{H}_k(\lambda; t) = W(t)^{1/2} : \hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t) : W(t)^{-1/2}.
\]

It follows that \( F_N(t, u) \) should satisfy the eigenvalue equations

\[
: \hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t) : F_N(t, u) = S_k(u)F_N(t, u), \quad k = 1, \ldots, N.
\]
Another key tool is given by so-called kernel functions.

- Given a pair of operators $H_1(v)$ and $H_2(w)$, a kernel function is a function $\Psi(v, w)$ satisfying

$$H_1(v)\Psi(v, w) = H_2(w)\Psi(v, w).$$

Here, $v$ and $w$ may vary over spaces of different dimension.

- There exist elementary kernel functions that connect the PDOs $\hat{\Sigma}_k(L + E)(t)$ to a sum of PDOs in variables $s_1, \ldots, s_{N-\ell}$. (Langmann for $k=2$, Hallnäs/S. R. for $k>2$.)

- For $\ell = 1$ this connection can be used to set up a recursive scheme yielding the above explicit integral representations of the joint eigenfunctions $F_N$.

- For $\lambda = 1/2$ recursive $H$-eigenfunctions were previously found by Gerasimov/Kharchev/Lebedev, and for $\lambda = -1, -2, \ldots$ by Felder/Veselov. (Relation unclear to date.)
$N = 2$ case

From $N = 1$ to $N = 2$

- For $N = 1$ we set
  \[
  F_1(t, u) \equiv \exp(itu),
  \]
  which obviously satisfies
  \[
  -i\partial_t F_1(t, u) = uF_1(t, u).
  \]

- Now consider
  \[
  F_2(\lambda; t, u) \equiv e^{iu(t_1 + t_2)} \int_{\mathbb{R}} ds K_2^\#(\lambda; t, s) F_1(s, u_1 - u_2)
  \]
  with kernel function
  \[
  K_2^\#(\lambda; t, s) \equiv \prod_{j=1}^{2} [2 \cosh(t_j - s)]^{-\lambda}.
  \]
If \( \text{Re} \lambda > 0 \) and \( u \in \mathbb{R}^2 \), then the integrand decays exponentially as \( |s| \to \infty \). It has singularities only at

\[
s = t_j \pm \frac{i\pi}{2} (2n + 1), \quad j = 1, 2, \quad n \in \mathbb{N}.
\]

Hence \( F_2(\lambda; t, u) \) is well defined as long as

\[
\text{Re} \lambda > 0, \quad u \in \mathbb{R}^2,
\]

and \( t \in \mathbb{C}^2 \) satisfies

\[
|\text{Im} t_j| < \pi/2, \quad j = 1, 2.
\]
\( \mathcal{N} = 2 \) case

**Holomorphy**

- \( F_2(\lambda; t, u) \) has analytic continuation in \((\lambda, t)\) to

\[
\{ \lambda \in \mathbb{C} \mid \text{Re } \lambda > 0 \} \times \{ t \in \mathbb{C}^2 \mid |\text{Im} (t_1 - t_2)| < \pi \}.
\]

- Follows via contour shifts:
  
  - \( \lambda \in \mathbb{C} \)
  
  \[
  \begin{align*}
  &\bullet t_2 + i\pi/2 \\
  &\bullet t_1 + i\pi/2 \\
  &\bullet \mathbb{R} + i\eta \\
  &\bullet t_2 - i\pi/2 \\
  &\bullet t_1 - i\pi/2
  \end{align*}
  \]

  where we can choose \( \eta = \text{Im} (t_1 + t_2)/2 \).

- Can allow \( u \in \mathbb{C}^2 \) such that \( |\text{Im} (u_1 - u_2)| < 2\text{Re } \lambda \).
We claim that $F_2(\lambda; t, u)$ is a joint eigenfunction of the PDOs

\[
\hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) := -i(\partial_{t_1} + \partial_{t_2}),
\]

\[
\hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) := -\partial_{t_1}\partial_{t_2} + \lambda \coth(t_1 - t_2)(\partial_{t_1} - \partial_{t_2}) + \lambda^2.
\]

▶ Key point: we have kernel identities

\[
\hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_2^\#(t, s) = 0,
\]

and

\[
\hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_2^\#(t, s) =: \hat{\Sigma}_1^{(1)}(\mathcal{L} + \mathcal{E})(-s) : \mathcal{K}_2^\#(t, s) = i\partial_s\mathcal{K}_2^\#(t, s).
\]

▶ By analyticity, need only consider $t \in \mathbb{R}$ and $\lambda > 0$ (say).
To establish the eigenfunction property for \( \hat{\Sigma}_k^{(2)}(\mathcal{L} + \mathcal{E})(t) : \), \( k = 1, 2 \), we take 7 steps. First let \( k = 1 \).

1. Recall that
   \[
   F_2(\lambda; t, u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds K_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).
   \]

2. Act with \( \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : \), and shift through plane wave:
   \[
   e^{iu_2(t_1+t_2)} \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E} + u_2\mathbf{1}_2)(t) : \int_{\mathbb{R}} ds K_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).
   \]

3. Use determinant expansion to get
   \[
   \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E} + u_2\mathbf{1}_2)(t) := \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : + 2u_2.
   \]

4. Act with PDO under integral sign and invoke second kernel identity:
   \[
   e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds F_1(s, u_1 - u_2) : \hat{\Sigma}_1^{(1)}(\mathcal{L} + \mathcal{E})(-s) : K_2^\#(\lambda; t, s)
   \]
   \[
   + 2u_2 e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds K_2^\#(\lambda; t, s) F_1(s, u_1 - u_2).
   \]
5. Use: \( \hat{\Sigma}_1^{(1)}(L + \mathcal{E})(-s) := i\partial_s \), and integrate by parts:

\[
e^{iu(t_1+t_2)} \int_{\mathbb{R}} ds K^\#_2(\lambda; t, s) : \hat{\Sigma}_1^{(1)}(L + \mathcal{E})(s) : F_1(s, u_1 - u_2)
\]
\[
+ 2u_2 e^{iu(t_1+t_2)} \int_{\mathbb{R}} ds K^\#_2(\lambda; t, s) F_1(s, u_1 - u_2).
\]

6. Invoke eigenfunction property for \( F_1 \) to obtain

\[
(u_1 - u_2 + 2u_2) e^{iu(t_1+t_2)} \int_{\mathbb{R}} ds K^\#_2(\lambda; t, s) F_1(s, u_1 - u_2).
\]

7. Conclude that

\[
: \hat{\Sigma}_1^{(2)}(L + \mathcal{E})(t) : F_2(t, u) = S_1^{(2)}(u) F_2(t, u),
\]

with \( S_1^{(2)}(a_1, a_2) \equiv a_1 + a_2. \)
Now take $k = 2$. Then steps 1 and 2 are clear, and step 3 yields

\[ : \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E} + u_2 \mathbf{1}_2)(t) : = : \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : + u_2 : \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : + u_2^2. \]

In step 4 we use both kernel identities. Then \( : \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : \) drops out and we can take step 5 just as for \( k = 1 \). As a result, we obtain as the counterpart of step 6

\[ (u_2(u_1 - u_2) + u_2^2) e^{iu_2(t_1 + t_2)} \int_{\mathbb{R}} ds K_2^{\#}(\lambda; t, s) F_1(s, u_1 - u_2). \]

Hence step 7 gives

\[ : \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : F_2(t, u) = S_2^{(2)}(u) F_2(t, u), \]

with \( S_2^{(2)}(a_1, a_2) \equiv a_1 a_2 \). Thus we have proved the claim and obtained both eigenvalues.
$N = 2$ case

A bound

Let $u \in \mathbb{R}^2$. For $\Re \lambda > 0$ and $|\Im (t_1 - t_2)| < \pi$, we have the $F_2$-bound

$$|F_2(\lambda; t, u)| < C(\lambda, |\Im (t_1 - t_2)|)$$

$$\times \exp(-\Im (t_1 + t_2)(u_1 + u_2)/2) \frac{\Re (t_1 - t_2)}{\sinh(\Re \lambda \Re (t_1 - t_2))}.$$ 

We need this bound to get analytic control on the step $N = 2 \rightarrow N = 3$ in the recursive scheme we now turn to. The bound readily follows from the integral evaluation

$$\int_{\mathbb{R}} \prod_{\delta = +, -} ds \frac{1}{2 \cosh(s + \delta z/2)} = \frac{z}{2 \sinh z}.$$
The function

\[ \mathcal{K}_N^\#(\lambda; t, s) \equiv \prod_{j=1}^{N} \prod_{k=1}^{N-1} [2 \cosh(t_j - s_k)]^{-\lambda}, \quad N > 1, \]

satisfies the kernel identities

\[ : \hat{\Sigma}_N^{(N)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_N^\#(t, s) = 0, \]

and

\[ ( : \hat{\Sigma}_k^{(N)}(\mathcal{L} + \mathcal{E})(t) : - : \hat{\Sigma}_k^{(N-1)}(\mathcal{L} + \mathcal{E})(-s) : ) \mathcal{K}_N^\#(t, s) = 0, \quad k < N. \]

This connection between the \( N \) and \( N - 1 \) variable cases can be used to recursively construct the joint eigenfunctions \( F_N \) of the \( N \) PDOs \( : \hat{\Sigma}_k^{(N)}(\mathcal{L} + \mathcal{E})(t) :, \quad k = 1, \ldots, N. \)
Assume the function $F_{N-1}(t, u), t, u \in \mathbb{C}^{N-1}$, has been constructed.

Then $F_N(t, u), t, u \in \mathbb{C}^N$, is formally given by

$$F_N(t, u) \equiv \frac{e^{iu_N \sum_{j=1}^{N} t_j}}{(N-1)!} \int_{\mathbb{R}^{N-1}} ds W_{N-1}(s) K_N^\#(t, s) \times F_{N-1}(s, (u_1 - u_N, \ldots, u_{N-1} - u_N)).$$

Using explicit bounds we have shown $F_N(\lambda; t, u)$ is well defined for $\text{Re } \lambda \geq 1$ and $t \in \mathbb{C}^N$ such that $|\text{Im } t_j| < \pi/2$ (and $u \in \mathbb{R}^N$), and continues analytically to

$$\{ \lambda \in \mathbb{C} \mid \text{Re } \lambda > 1 \} \times \{ t \in \mathbb{C}^N \mid \max_{1 \leq j < k \leq N} |\text{Im } (t_j - t_k)| < \pi \}.$$
$A_{N-1}$ Heckman-Opdam hypergeometric function

- $F_{A_{N-1}}(\tilde{\lambda}, k; h)$ depends on three types of parameters:
  - coupling parameter $k \in \mathbb{C}$,
  - eigenvalue vector $\tilde{\lambda} \in \mathbb{C}^N$,
  - a quantity $h$, which can be viewed as a diagonal $N \times N$ matrix with $\det(h) = 1$.

- For $t \in \mathbb{C}^N$ such that $\sum t_j = 0$, let
  
  $$h(t) \equiv \text{diag} \left( e^{2t_1}, \ldots, e^{2t_N} \right).$$

- Comparing eigenvalue equations and normalisations, we deduce
  
  $$F_{A_{N-1}}(iu/2, \lambda; h(t)) = \frac{F_N(\lambda; t, u)}{F_N(\lambda; 0, u)},$$

  where $\sum t_j = \sum u_j = 0$. 
Given by $2N$ commuting analytic difference operators

$$A_{k,\delta} \equiv \sum_{|l|=k} \prod_{m \in l, n \notin l} s_{\delta}(x_m - x_n - ib) \prod_{m \in l} \frac{s_{\delta}(x_m - x_n)}{s_{\delta}(x_m - x_n)} \exp(-ia_{-\delta} \partial x_m),$$

where $k = 1, \ldots, N$, $\delta = +, -$, and

$$s_{\delta}(z) \equiv \sinh(\pi z/a_{\delta}).$$

Physical picture: The imaginary periods and shift step sizes are determined by the length parameters

$$a_+ \equiv 2\pi/\mu, \quad \text{(interaction length)},$$

$$a_- \equiv \hbar/mc, \quad \text{(Compton wave length)},$$

with $\hbar$ Planck's constant, $m$ particle mass and $c$ speed of light.
Relativistic generalisation

Tools

- The hyperbolic gamma function

\[ G(a_+, a_-, z) \equiv \exp(ig(a_+, a_-; z)), \]

with

\[ g(z) \equiv \int_0^\infty \frac{dy}{y} \left( \frac{\sin 2yz}{2 \sinh(a_+y) \sinh(a_-y)} - \frac{z}{a_+a_-y} \right), \]

for \(|\text{Im } z| < (a_+ + a_-)/2\).

- The kernel function

\[ S_{N}^\sharp(b; x, y) \equiv \prod_{j=1}^{N} \prod_{k=1}^{N-1} \frac{G(x_j - y_k - ib/2)}{G(x_j - y_k + ib/2)}, \]

which satisfies the 2\(N\) kernel identities

\[ A_{k,\delta}^{(N)}(x)S_{N}^\sharp(b; x, y) = \left( A_{k,\delta}^{(N-1)}(-y) + A_{k-1,\delta}^{(N-1)}(-y) \right) S_{N}^\sharp(b; x, y). \]
Relativistic generalisation

Sketch of results

For $N = 1$ we set

$$J_1(x, y) \equiv \exp(i\alpha xy), \quad \alpha \equiv \frac{2\pi}{a_+a_-}.$$

For $N > 1$ we construct joint eigenfunctions $J_N(x, y), x, y \in \mathbb{C}^N$, recursively according to

$$J_N(x, y) \equiv e^{i\alpha_{y_N}(x_1 + \cdots + x_N)} \int_{\mathbb{R}^{N-1}} dz W_{N-1}(z) S_N^\#(x, z)$$

$$\times J_{N-1}(z, (y_1 - y_N, \ldots, y_{N-1} - y_N))$$

with

$$W_M(z) \equiv \prod_{1 \leq m < n \leq M} w(z_m - z_n),$$

$$w(z) \equiv 1/c(z)c(-z), \quad c(z) \equiv G(z + ia - ib)/G(z + ia).$$
References


