A recursive construction of joint eigenfunctions for the commuting hyperbolic Calogero-Moser Hamiltonians

(Joint work with M. Hallnäs)

Simon Ruijsenaars

School of of Mathematics
University of Leeds, UK

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Introduction

Reminders

- The hyperbolic (A_{N-1}) Calogero-Moser systems are integrable systems describing N particles on the line with hyperbolic pair interaction.
- The nonrelativistic quantum version is defined by the Hamiltonian

$$H \equiv -\frac{\hbar^2}{2} \sum_{j=1}^{N} \partial_{x_j}^2 + g(g-\hbar) \sum_{1 \leq j < l \leq N} V(x_j - x_l),$$

  with $\hbar > 0$ (Planck's constant), $g \in \mathbb{R}$ (coupling constant), and pair potential

$$V(x) \equiv \mu^2 / 4 \sinh^2(\mu x/2), \quad \mu > 0.$$

- The $N = 2$ Schrödinger equation can be solved via the conical function, a specialization of the Gauss hypergeometric function.

- Associated integrable system (N commuting PDOs):

$$H_1 \equiv -i\hbar \sum_{j=1}^{N} \partial_{x_j}, \; H_2 \equiv -\hbar^2 \sum_{1\le j_1<j_2\le N} \partial_{x_{j_1}} \partial_{x_{j_2}} - g(g-\hbar) \sum_{1\le j<l\le N} V(x_j - x_l),$$

$$H_k \equiv (-i\hbar)^k \sum_{1\le j_1<\cdots<j_k\le N} \partial_{x_{j_1}} \cdots \partial_{x_{j_k}} + \text{l. o.}, \quad k = 3, \ldots, N,$$

where l.o. = lower order in partials. Thus, the defining Hamiltonian is given by

$$H = H_1^2/2 - H_2.$$

- Integrable versions exist for Lie algebras $B_N, \ldots, E_8, F_4, G_2$ (Olshanetsky/Perelomov, Oshima) and BC_N (Inozemtsev, Oshima).
- $N > 2$ eigenfunctions: Harish-Chandra, Heckman/Opdam, Felder/Varchenko, Chalykh,...

Introduction

Goal

- By proceeding recursively in N, construct joint eigenfunctions $\Psi_N(x, p)$ of the Hamiltonians H_k:

$$H_k(x)\Psi_N(x,p) = S_k(p)\Psi_N(x,p), \quad k = 1, \ldots, N,$$

where

$$S_k(p) := \sum_{1 \le j_1 < \cdots < j_k \le N} p_{j_1} \cdots p_{j_N}.$$

- It is crucial to switch from Ψ_N to F_N given by

$$\Psi_N(g; (x_1, \ldots, x_N), (p_1, \ldots, p_N)) =: W_N(g/\hbar; \mu x/2)^{1/2} \\ \times F_N(g/\hbar; (\mu x_1/2, \ldots, \mu x_N/2), (2p_1/\hbar\mu, \ldots, 2p_N/\hbar\mu)),$$

with

$$W_N(\lambda; t) := \prod_{1 \le j < k \le N} [4 \sinh^2(t_j - t_k)]^{\lambda}.$$

- Assuming $\mathrm{Re}\,\lambda \geq 1$, $u \in \mathbb{R}^N$ and $|\mathrm{Im}\, t_j| < \pi/2$, we obtain

$$F_N(\lambda; t, u) = \int_{\mathbb{R}^{N(N-1)/2}} \prod_{n=1}^{N-1} \frac{\prod_{1\leq j<k\leq n}[4\sinh^2(t_{nj}-t_{nk})]^\lambda}{n! \prod_{j=1}^{n+1}\prod_{k=1}^{n}\left[2\cosh(t_{n+1,j}-t_{nk})\right]^\lambda} \times \exp\left(i\sum_{n=1}^{N} u_n\left(\sum_{j=1}^{n} t_{nj} - \sum_{j=1}^{n-1} t_{n-1,j}\right)\right)\prod_{n=1}^{N-1}\prod_{j=1}^{n} dt_{nj},$$

  where

$$t_{Nj} := t_j, \quad j = 1, \ldots, N.$$

- This integral can also be written

$$\exp\left(iu_N \sum_{j=1}^{N} t_j\right) \prod_{n=1}^{N-1} \int_{t_{nn}<\cdots<t_{n1}} \exp\left(i(u_n - u_{n+1})(t_{n1} + \cdots + t_{nn})\right) \times \frac{\prod_{1\leq j<k\leq n}[2\sinh(t_{nj}-t_{nk})]^{2\lambda}}{\prod_{j=1}^{n+1}\prod_{k=1}^{n}\left[2\cosh(t_{n+1,j}-t_{nk})\right]^\lambda} \prod_{j=1}^{n} dt_{nj}.$$

Introduction

Tools

We proceed with an explicit description of the eigenvalue equations for F_N.

- The starting point consists of the Lax matrix

$$\mathcal{L}(t,u)_{jk} \equiv \delta_{jk} u_j + (1-\delta_{jk}) \frac{i\lambda}{\sinh(t_j - t_k)},$$

and the diagonal matrix

$$\mathcal{E}(t) \equiv \mathrm{diag}\,(w_1(t), \dots, w_N(t)),$$

with

$$w_j(t) \equiv -i\lambda \sum_{k \neq j} \coth(t_j - t_k).$$

- We let $:\hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t):$ denote the normal-ordered PDOs obtained from the symmetric functions

$$\Sigma_k(\mathcal{L}(t,u)+\mathcal{E}(t)) \equiv \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=k}} \det(\mathcal{L}(t,u)+\mathcal{E}(t))_I$$

by performing the substitutions

$$u_j \to -i\partial_{t_j}, \quad j=1,\dots,N.$$

- The Hamiltonians $\mathcal{H}_k(\lambda;t) \equiv (2/\hbar\mu)^k H_k(\lambda\hbar; 2t/\mu)$ are then given by (S. R., 1994)

$$\mathcal{H}_k(\lambda;t) = W(t)^{1/2} :\hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t): W(t)^{-1/2}.$$

- It follows that $F_N(t,u)$ should satisfy the eigenvalue equations

$$:\hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t): F_N(t,u) = S_k(u)F_N(t,u), \quad k=1,\dots,N.$$

Another key tool is given by so-called kernel functions.

- Given a pair of operators $H_1(v)$ and $H_2(w)$, a kernel function is a function $\Psi(v, w)$ satisfying
$$H_1(v)\Psi(v, w) = H_2(w)\Psi(v, w).$$
Here, v and w may vary over spaces of different dimension.
- There exist elementary kernel functions that connect the PDOs $:\hat{\Sigma}_k(\mathcal{L} + \mathcal{E})(t):$ to a sum of PDOs in variables $s_1, \ldots, s_{N-\ell}$. (Langmann for k=2, Hallnäs/S. R. for k>2.)
- For $\ell = 1$ this connection can be used to set up a recursive scheme yielding the above explicit integral representations of the joint eigenfunctions F_N.
- For $\lambda = 1/2$ recursive H-eigenfunctions were previously found by Gerasimov/Kharchev/Lebedev, and for $\lambda = -1, -2, \ldots$ by Felder/Veselov. (Relation unclear to date.)

$N = 2$ case

From $N = 1$ to $N = 2$

- For $N = 1$ we set
$$F_1(t, u) \equiv \exp(itu),$$
which obviously satisfies
$$-i\partial_t F_1(t, u) = uF_1(t, u).$$
- Now consider
$$F_2(\lambda; t, u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\sharp(\lambda; t, s) F_1(s, u_1 - u_2)$$
with kernel function
$$\mathcal{K}_2^\sharp(\lambda; t, s) \equiv \prod_{j=1}^{2} [2\cosh(t_j - s)]^{-\lambda}.$$

- If $\operatorname{Re}\lambda > 0$ and $u \in \mathbb{R}^2$, then the integrand decays exponentially as $|s| \to \infty$. It has singularities only at

$$s = t_j \pm \frac{i\pi}{2}(2n+1), \quad j = 1, 2, \quad n \in \mathbb{N}.$$

- Hence $F_2(\lambda; t, u)$ is well defined as long as

$$\operatorname{Re}\lambda > 0, \quad u \in \mathbb{R}^2,$$

and $t \in \mathbb{C}^2$ satisfies

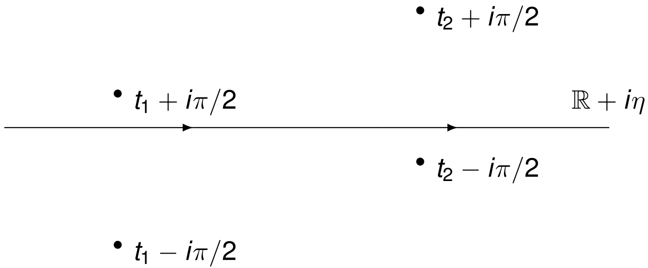
$$|\operatorname{Im} t_j| < \pi/2, \quad j = 1, 2.$$

$N = 2$ case
Holomorphy

- $F_2(\lambda; t, u)$ has analytic continuation in (λ, t) to

$$\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\} \times \{t \in \mathbb{C}^2 \mid |\mathrm{Im}\,(t_1 - t_2)| < \pi\}.$$

- Follows via contour shifts:



where we can choose $\eta = \mathrm{Im}\,(t_1 + t_2)/2$.

- Can allow $u \in \mathbb{C}^2$ such that $|\mathrm{Im}\,(u_1 - u_2)| < 2\mathrm{Re}\,\lambda$.

$N = 2$ case

Eigenfunction property

We claim that $F_2(\lambda; t, u)$ is a joint eigenfunction of the PDOs

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) := -i(\partial_{t_1} + \partial_{t_2}),$$

$$: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) := -\partial_{t_1}\partial_{t_2} + \lambda \coth(t_1 - t_2)(\partial_{t_1} - \partial_{t_2}) + \lambda^2.$$

- Key point: we have kernel identities

  $$: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_2^{\sharp}(t, s) = 0,$$

  and

  $$: \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t) : \mathcal{K}_2^{\sharp}(t, s) =: \hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(-s) : \mathcal{K}_2^{\sharp}(t, s) = i\partial_s \mathcal{K}_2^{\sharp}(t, s).$$

- By analyticity, need only consider $t \in \mathbb{R}$ and $\lambda > 0$ (say).

To establish the eigenfunction property for $:\hat{\Sigma}_k^{(2)}(\mathcal{L}+\mathcal{E})(t):$, $k=1,2$, we take 7 steps. First let $k=1$.

1. Recall that

$$F_2(\lambda;t,u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda;t,s) F_1(s,u_1-u_2).$$

2. Act with $:\hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t):$, and shift through plane wave:

$$e^{iu_2(t_1+t_2)} : \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E}+u_2\mathbf{1}_2)(t) : \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda;t,s) F_1(s,u_1-u_2).$$

3. Use determinant expansion to get

$$:\hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E}+u_2\mathbf{1}_2)(t) :=: \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t) : +2u_2.$$

4. Act with PDO under integral sign and invoke second kernel identity:

$$e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds F_1(s,u_1-u_2) : \hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(-s) : \mathcal{K}_2^{\sharp}(\lambda;t,s)$$

$$+ 2u_2 e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda;t,s) F_1(s,u_1-u_2).$$

5. Use : $\hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(-s) := i\partial_s$, and integrate by parts:

$$e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) : \hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(s) : F_1(s, u_1 - u_2) \\ + 2u_2 e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) F_1(s, u_1 - u_2).$$

6. Invoke eigenfunction property for F_1 to obtain

$$(u_1 - u_2 + 2u_2) e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) F_1(s, u_1 - u_2).$$

7. Conclude that

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t) : F_2(t, u) = S_1^{(2)}(u) F_2(t, u),$$

with $S_1^{(2)}(a_1, a_2) \equiv a_1 + a_2$.

Now take $k = 2$. Then steps 1 and 2 are clear, and step 3 yields

$$:\hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E} + u_2 \mathbf{1}_2)(t) :=: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : + u_2 : \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) : + u_2^2.$$

In step 4 we use both kernel identities. Then $:\hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t):$ drops out and we can take step 5 just as for $k = 1$. As a result, we obtain as the counterpart of step 6

$$\left(u_2(u_1 - u_2) + u_2^2\right) e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) F_1(s, u_1 - u_2).$$

Hence step 7 gives

$$:\hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : F_2(t, u) = S_2^{(2)}(u) F_2(t, u),$$

with $S_2^{(2)}(a_1, a_2) \equiv a_1 a_2$. Thus we have proved the claim and obtained both eigenvalues.

□

- Let $u \in \mathbb{R}^2$. For $\mathrm{Re}\,\lambda > 0$ and $|\mathrm{Im}\,(t_1 - t_2)| < \pi$, we have the F_2-bound

$$|F_2(\lambda; t, u)| < C(\lambda, |\mathrm{Im}\,(t_1 - t_2)|) \times \exp(-\mathrm{Im}\,(t_1 + t_2)(u_1 + u_2)/2) \frac{\mathrm{Re}\,(t_1 - t_2)}{\sinh(\mathrm{Re}\,\lambda \mathrm{Re}\,(t_1 - t_2))}.$$

- We need this bound to get analytic control on the step $N = 2 \to N = 3$ in the recursive scheme we now turn to. The bound readily follows from the integral evaluation

$$\int_{\mathbb{R}} \frac{ds}{\prod_{\delta=+,-} 2\cosh(s + \delta z/2)} = \frac{z}{2\sinh z}.$$

Recursion scheme

Kernel function

- The function

  $$\mathcal{K}_N^\sharp(\lambda; t, s) \equiv \prod_{j=1}^{N} \prod_{k=1}^{N-1} [2\cosh(t_j - s_k)]^{-\lambda}, \quad N > 1,$$

  satisfies the kernel identities

  $$: \hat{\Sigma}_N^{(N)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_N^\sharp(t, s) = 0,$$

  and

  $$\Big(: \hat{\Sigma}_k^{(N)}(\mathcal{L}+\mathcal{E})(t) : - : \hat{\Sigma}_k^{(N-1)}(\mathcal{L}+\mathcal{E})(-s) : \Big)\mathcal{K}_N^\sharp(t, s) = 0, \quad k < N.$$

- This connection between the N and $N-1$ variable cases can be used to recursively construct the joint eigenfunctions F_N of the N PDOs $: \hat{\Sigma}_k^{(N)}(\mathcal{L} + \mathcal{E})(t) :$, $k = 1, \ldots, N$.

Recursion scheme

Formal structure

- Assume the function $F_{N-1}(t, u)$, $t, u \in \mathbb{C}^{N-1}$, has been constructed.
- Then $F_N(t, u)$, $t, u \in \mathbb{C}^N$, is formally given by

$$F_N(t,u) \equiv \frac{e^{iu_N \sum_{j=1}^{N} t_j}}{(N-1)!} \int_{\mathbb{R}^{N-1}} ds W_{N-1}(s) \mathcal{K}_N^{\sharp}(t,s) \times F_{N-1}(s, (u_1 - u_N, \ldots, u_{N-1} - u_N)).$$

- Using explicit bounds we have shown $F_N(\lambda; t, u)$ is well defined for $\mathrm{Re}\,\lambda \geq 1$ and $t \in \mathbb{C}^N$ such that $|\mathrm{Im}\, t_j| < \pi/2$ (and $u \in \mathbb{R}^N$), and continues analytically to

$$\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 1\} \times \{t \in \mathbb{C}^N \mid \max_{1 \leq j < k \leq N} |\mathrm{Im}\,(t_j - t_k)| < \pi\}.$$

A_{N-1} Heckman-Opdam hypergeometric function

- $F_{A_{N-1}}(\tilde{\lambda}, k; h)$ depends on three types of parameters:
  - coupling parameter $k \in \mathbb{C}$,
  - eigenvalue vector $\tilde{\lambda} \in \mathbb{C}^N$,
  - a quantity h, which can be viewed as a diagonal $N \times N$ matrix with $\det(h) = 1$.
- For $t \in \mathbb{C}^N$ such that $\sum t_j = 0$, let

$$h(t) \equiv \mathrm{diag}\left(e^{2t_1}, \ldots, e^{2t_N}\right).$$

- Comparing eigenvalue equations and normalisations, we deduce

$$F_{A_{N-1}}(iu/2, \lambda; h(t)) = \frac{F_N(\lambda; t, u)}{F_N(\lambda; 0, u)},$$

where $\sum t_j = \sum u_j = 0$.

Relativistic generalisation

Reminders

- Given by $2N$ commuting analytic difference operators

$$A_{k,\delta} \equiv \sum_{|I|=k} \prod_{\substack{m\in I \\ n\notin I}} \frac{s_\delta(x_m - x_n - ib)}{s_\delta(x_m - x_n)} \prod_{m\in I} \exp(-ia_{-\delta}\partial_{x_m}),$$

where $k = 1, \ldots, N$, $\delta = +, -$, and

$$s_\delta(z) \equiv \sinh(\pi z / a_\delta).$$

- Physical picture: The imaginary periods and shift step sizes are determined by the length parameters

$$a_+ \equiv 2\pi/\mu, \quad \text{(interaction length)},$$

$$a_- \equiv \hbar/mc, \quad \text{(Compton wave length)},$$

with $\hbar$ Planck's constant, m particle mass and c speed of light.

Relativistic generalisation

Tools

- The hyperbolic gamma function

$$G(a_+, a_-; z) \equiv \exp(ig(a_+, a_-; z)),$$

with

$$g(z) \equiv \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2 \sinh(a_+ y) \sinh(a_- y)} - \frac{z}{a_+ a_- y} \right),$$

for $|\mathrm{Im}\, z| < (a_+ + a_-)/2$.

- The kernel function

$$\mathcal{S}_N^\sharp(b; x, y) \equiv \prod_{j=1}^{N} \prod_{k=1}^{N-1} \frac{G(x_j - y_k - ib/2)}{G(x_j - y_k + ib/2)},$$

which satisfies the $2N$ kernel identities

$$A_{k,\delta}^{(N)}(x) \mathcal{S}_N^\sharp(b; x, y) = \left(A_{k,\delta}^{(N-1)}(-y) + A_{k-1,\delta}^{(N-1)}(-y) \right) \mathcal{S}_N^\sharp(b; x, y).$$

Relativistic generalisation

Sketch of results

- For $N = 1$ we set

$$J_1(x, y) \equiv \exp(i\alpha xy), \quad \alpha \equiv \frac{2\pi}{a_+ a_-}.$$

- For $N > 1$ we construct joint eigenfunctions $J_N(x, y)$, $x, y \in \mathbb{C}^N$, recursively according to

$$J_N(x, y) \equiv e^{i\alpha y_N(x_1+\cdots+x_N)} \int_{\mathbb{R}^{N-1}} dz W_{N-1}(z) \mathcal{S}_N^\sharp(x, z) \times J_{N-1}(z, (y_1 - y_N, \ldots, y_{N-1} - y_N))$$

with

$$W_M(z) \equiv \prod_{1 \le m < n \le M} w(z_m - z_n),$$

$$w(z) \equiv 1/c(z)c(-z), \quad c(z) \equiv G(z + ia - ib)/G(z + ia).$$

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