Phase topology of generalized two-field gyrostat (the GTFG-system)

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Abstract

We consider the integrable system with three degrees of freedom for which Sokolov and Tsiganov specified a Lax representation [1]. This representation generalizes the $\boldsymbol{L}$–$\boldsymbol{A}$ pair of the Kowalevski gyrostat in two constant fields found by Reyman and Semenov-Tian-Shansky [2]. The authors of [1] called their case the *generalized two fields gyrostat (the GTFG-system)*. We give explicit formulas for two almost everywhere independent additional first integrals $\boldsymbol{K}$ and $\boldsymbol{G}$. These integrals are functionally connected with the coefficients of the spectral curve of the $\boldsymbol{L}$–$\boldsymbol{A}$ pair by Sokolov and Tsiganov. In the talk, we suggest an approach to describe phase topology of a new integrable system with three degrees of freedom, using the method of critical subsystems. The notion of a critical subsystem was formed in the beginning of 2000-ies in the problem of study of phase topology of irreducible systems with three degrees of freedom. Due to the obtained form of additional integrals $\boldsymbol{K}$ and $\boldsymbol{G}$, we managed to find analytically invariant four-dimensional submanifolds on which the induced dynamic system is almost everywhere Hamiltonian with two degrees of freedom [3], [4]. The system of equations that describes one of these invariant submanifolds is a generalization of the invariant relations of the corresponding integrable Bogoyavlensky case for the rotation of a magnetized rigid body with a fixed point in homogeneous magnetic and gravitational fields [5]. For each subsystem, we construct the bifurcation diagrams and specify the bifurcations of Liouville tori both in subsystems, and in the system as a whole.

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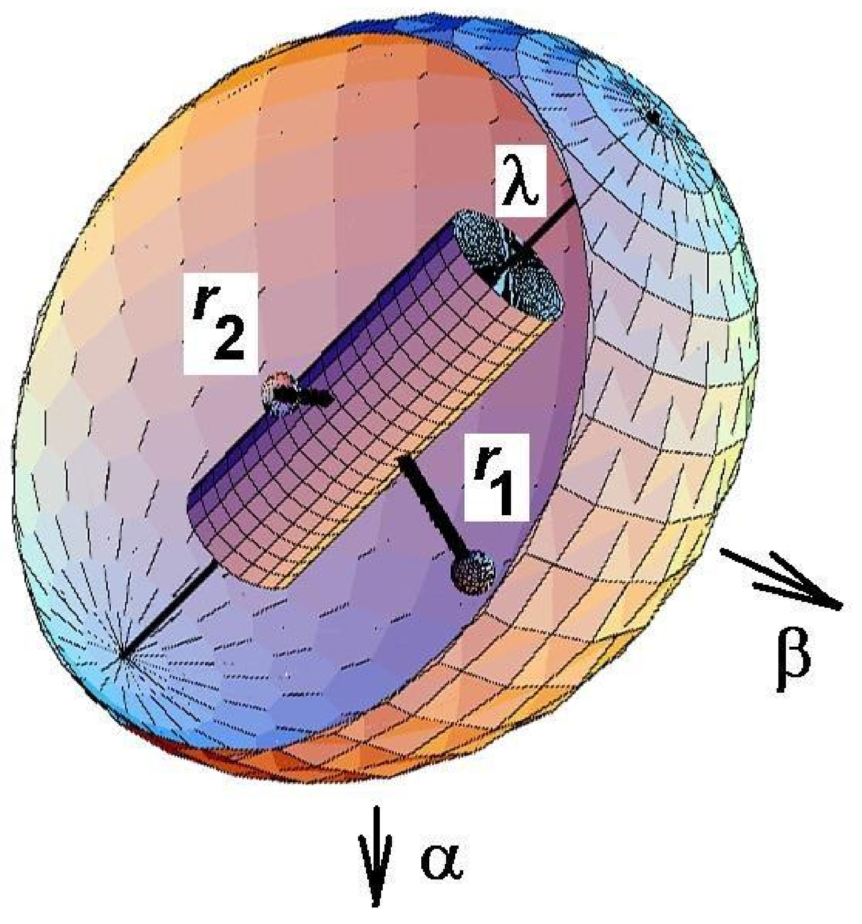
Model



Notation

- The diagonal inertia tensor with the principal moments of inertia satisfying: $\boldsymbol{A} = \boldsymbol{B} = 2\boldsymbol{C}$.
- The gyrostatic momentum λ is directed along the dynamical symmetry axis $\lambda = \lambda \boldsymbol{e}_3$.
- Constant vectors $\boldsymbol{r}_k$ points from $\boldsymbol{O}$ to the center of application of the fields, $\boldsymbol{r}_k \perp \boldsymbol{e}_3$, $k = 1, 2$, α, β are the field's intensity.
- Let $\boldsymbol{M}_F = \boldsymbol{r}_1 \times \alpha + \boldsymbol{r}_2 \times \beta$ denote the moment of external forces with respect to $\boldsymbol{O}$.

Equations of motion

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \alpha \times \frac{\partial H}{\partial \alpha} + \beta \times \frac{\partial H}{\partial \beta}, \quad \dot{\alpha} = \alpha \times \frac{\partial H}{\partial \boldsymbol{M}}, \quad \dot{\beta} = \beta \times \frac{\partial H}{\partial \boldsymbol{M}}. \tag{1}$$

The Hamiltonian function and additional integrals

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + \lambda M_3 - \varepsilon_1 \boldsymbol{M} \cdot (\boldsymbol{r}_1 \times \alpha + \boldsymbol{r}_2 \times \beta) - \varepsilon_2(\boldsymbol{r}_1 \cdot \alpha + \boldsymbol{r}_2 \cdot \beta). \tag{2}$$

Here, $\boldsymbol{M}$ is the projection of the kinetic momentum, the parameters $\varepsilon_1, \varepsilon_2$ are called the deformation parameters.

$$\begin{aligned}
K = {} & \boxed{Z_1^2 + Z_2^2} - \lambda[(M_3 + \lambda)(M_1^2 + M_2^2) + 2\varepsilon_2(\alpha_3 M_1 + \beta_3 M_2)] \\
& + \lambda\varepsilon_1^2(\alpha^2 + \beta^2)M_3 + 2\lambda\varepsilon_1[\alpha_2 M_1^2 - \beta_1 M_2^2 - (\alpha_1 - \beta_2)M_1 M_2] - 2\lambda\varepsilon_1^2\omega_\gamma, \\
G = {} & \omega_\alpha^2 + \omega_\beta^2 + 2(M_3 + \lambda)\omega_\gamma - 2\varepsilon_2(\alpha^2\beta_2 + \beta^2\alpha_1) \\
& + 2\varepsilon_1[\beta^2(M_2\alpha_3 - M_3\alpha_2) - \alpha^2(M_1\beta_3 - M_3\beta_1)] \\
& + 2(\alpha \cdot \beta)[\varepsilon_2(\alpha_2 + \beta_1) + \varepsilon_1(\alpha_3 M_1 - \alpha_1 M_3 + \beta_2 M_3 - \beta_3 M_2)].
\end{aligned}$$

Here

$$\begin{aligned}
Z_1 &= \tfrac{1}{2}(M_1^2 - M_2^2) + \varepsilon_2(\alpha_1 - \beta_2) + \varepsilon_1[M_3(\alpha_2 + \beta_1) - M_2\alpha_3 - M_1\beta_3] + \tfrac{1}{2}\varepsilon_1^2(\beta^2 - \alpha^2), \\
Z_2 &= M_1 M_2 + \varepsilon_2(\alpha_2 + \beta_1) - \varepsilon_1[M_3(\alpha_1 - \beta_2) + \beta_3 M_2 - \alpha_3 M_1] - \varepsilon_1^2(\alpha \cdot \beta), \\
\omega_\alpha &= M_1\alpha_1 + M_2\alpha_2 + M_3\alpha_3, \quad \omega_\beta = M_1\beta_1 + M_2\beta_2 + M_3\beta_3, \\
\omega_\gamma &= M_1(\alpha_2\beta_3 - \alpha_3\beta_2) + M_2(\alpha_3\beta_1 - \alpha_1\beta_3) + M_3(\alpha_1\beta_2 - \alpha_2\beta_1).
\end{aligned}$$

The phase space $\mathcal{P}$

$$\alpha^2 = a^2, \quad \beta^2 = b^2, \quad \alpha \cdot \beta = c, \quad (0 < b < a, |c| < ab).$$

Lie–Poisson brackets

$$\begin{gathered}
\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \alpha_j\} = \varepsilon_{ijk}\alpha_k, \quad \{\alpha_i, \alpha_j\} = 0, \\
\{M_i, \beta_j\} = \varepsilon_{ijk}\beta_k, \quad \{\alpha_i, \beta_j\} = 0, \quad \{\beta_i, \beta_j\} = 0, \\
\varepsilon_{ijk} = \tfrac{1}{2}(i - j)(j - k)(k - i), \quad 1 \leqslant i, j, k \leqslant 3.
\end{gathered}$$

The spectral curve $\mathcal{E}(z, \zeta)$ for GTFG-system

$$\mathcal{E}(z, \zeta) \ : \ d_4\zeta^4 + d_2\zeta^2 + d_0 = 0,$$

where

$$\begin{aligned}
d_4 &= -z^4 - \varepsilon_1^2(\alpha^2 + \beta^2)z^2 - \varepsilon_1^4[\alpha^2\beta^2 - (\alpha \cdot \beta)^2], \\
d_2 &= 2z^6 + [\varepsilon_1^2(\alpha^2 + \beta^2) - h - \lambda^2]z^4 + [\varepsilon_2^2(\alpha^2 + \beta^2) - \varepsilon_1^2 g]z^2 + 2\varepsilon_1^2\varepsilon_2^2[\alpha^2\beta^2 - (\alpha \cdot \beta)^2], \\
d_0 &= -z^8 + hz^6 + f_{\varepsilon_1,\varepsilon_2} z^4 + \varepsilon_2^2 g z^2 - \varepsilon_2^4[\alpha^2\beta^2 - (\alpha \cdot \beta)^2].
\end{aligned}$$

The most complicated coefficient $f_{\varepsilon_1,\varepsilon_2}$ at z^4 in d_0 is expressed in terms of the integral constants h, k, and g as follows:

$$f_{\varepsilon_1,\varepsilon_2} = \varepsilon_1^2 g + k - \varepsilon_1^4(\alpha \cdot \beta)^2 - \frac{1}{4}[h^2 + 2\varepsilon_1^2(\alpha^2 + \beta^2)h + \varepsilon_1^4(\alpha^2 - \beta^2)^2] - \varepsilon_2^2(\alpha^2 + \beta^2).$$

The first system is the generalization of the integrable O. I. Bogoyavlensky case

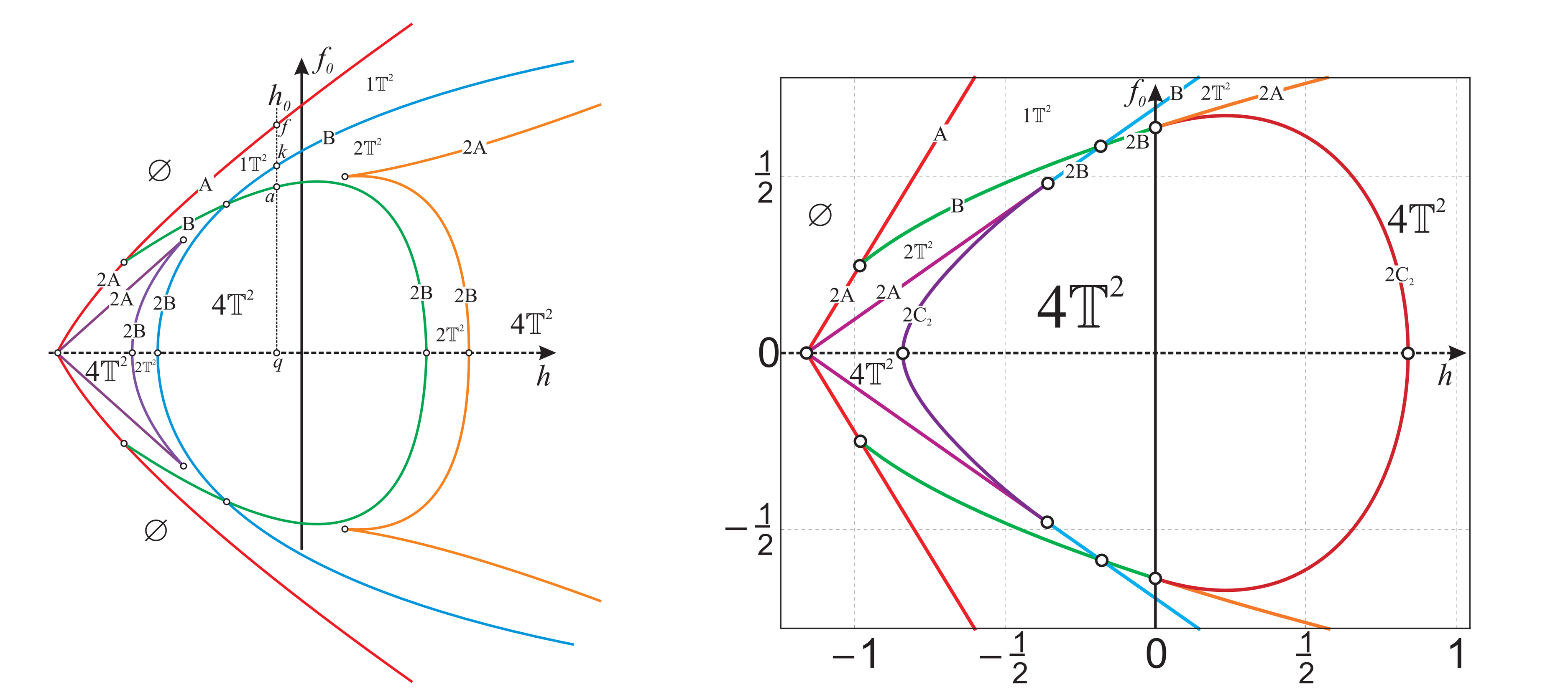
Let us $\lambda = 0$.

Proposition. System of relations

$$\boxed{Z_1 = 0, \quad Z_2 = 0} \tag{3}$$

defines the four-dimensional invariant submanifold of $\mathcal{M}_1$ of the phase space $\mathcal{P}$ of the equations (1) with Hamiltonian (2). The function $F_0 = \{Z_1, Z_2\}$ is a first integral on the submanifold of $\mathcal{M}_1$ given by the equations (3).

The bifurcations of Liouville tori



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