Poisson Structure of Difference Lax Equations, Exchange Algebras and Difference Galois Theory

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Introduction

- *Basic question to discuss:* How to extend the Poisson structure on the space of differential/difference operators to the space of wave functions.
- Special cases:
  - First order matrix differential/difference operators
  - Higher order differential/difference operators.
  - In particular, second order differential/difference operators related to Virasoro algebra
- Two approaches to second order operators:
  - Virasoro algebra
  - Drinfeld–Sokolov theory.

  Both approaches need a non-trivial generalization in the difference case related to lattice and q-deformed versions of Virasoro algebra.

Exchange algebra

The case of current algebra on the line is already non-trivial as it leads to a peculiar symmetry breaking (first noticed by Babelon) Let $\mathcal{C}(\mathfrak{g})$ be the space of connections on the line with values in a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate invariant inner product. $\mathcal{C}(\mathfrak{g})$ may be identified with (a hyperplane in) the dual space of the central extension of the current algebra $C^\infty(\mathbb{R};\mathfrak{g})$ and hence carries a natural Poisson structure (the so called *Schwinger Poisson bracket*). The current group $C^\infty(\mathbb{R};G)$ acts on $\mathcal{C}(\mathfrak{g})$ by gauge transformations,

$$g : L \mapsto \mathrm{Ad}g \cdot L + \partial_x g \cdot g^{-1}. \tag{1}$$

Let $W(\mathfrak{g})$ be the space of wave functions, i. e., of G-valued solutions of the differential equation

$$\partial_x \psi = L\psi.$$

Exchange algebra

We want to equip $W(\mathfrak{g})$ with a Poisson structure such that the natural mapping

$$W(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}) : \psi \mapsto \partial_x \psi \cdot \psi^{-1}$$

is Poisson. Since the gauge action (1) is Hamiltonian with respect to the natural Poisson structure on $\mathcal{C}(\mathfrak{g})$, the Poisson structure on $W(\mathfrak{g})$ should be left-invariant. Let us choose this Poisson structure in the following form:

$$\{\psi_1(x), \psi_2(y)\} = \psi_1(x)\psi_2(y)r_{12}(x-y), \tag{2}$$

One would like to make this Poisson structure also invariant with respect to constant right shifts $\psi \mapsto \psi \cdot h$ which amount to the change of basis in the space of solutions. A natural guess is $r_{12}(x-y) = t_{12}\epsilon(x-y)$, where $\epsilon(x-y)$ is the sign function (distribution kernel of ∂^{-1}).

Exchange algebra

It is instructive to write the r-matrix associated with $\epsilon(x-y) = \partial^{-1}$ as a singular integral operator given formally by

$$rX(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{X}(k)}{ik} e^{ikx}\, dk, \tag{3}$$

This operator needs to be regularized, which introduces a peculiar symmetry breaking and brings into play a finite dimensional r-matrix action on constant functions:

$$rX(x) = r_0(\hat{X}(0)) + \text{v.p.}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{X}(k)}{ik} e^{ikx}\, dk, \tag{4}$$

Exchange algebra

We have ${}^0r_+ -^0 r_- = t$; this yields

$$r'_x(x-y) = -r'_y(x-y) = t\delta(x-y),\ r''_{xy}(x-y) = -t\delta'(x-y), \quad (5)$$

and hence

$$\{L_1(x), L_2(y)\} = [t,\ L_1(x) - L_2(y)]\delta(x-y) - t\delta'(x-y), \quad (6)$$

i. e., the correct Schwinger Poisson structure on $\mathcal{C}(\mathfrak{g})$. Note that r_0 cancells out!

Difference case

We shall generalize Exchange algebra to the case of difference operators.
To set up a general framework for the difference case let us assume that $\mathbb{G}$ is a Lie group equipped with an automorphism τ Let $G = \mathbb{G}^\tau$ be the group of "quasi-constants", $G = \{g \in \mathbb{G}; g^\tau = g\}$.
The "auxiliary linear problem" reads:

$$\psi^\tau \psi^{-1} = L.$$

There is a natural action of $\mathbb{G}$ on itself by left multiplication which induces gauge transformations for L:

$$g\colon \psi \mapsto g \cdot \psi, L \mapsto g^\tau L g^{-1}.$$

The quasi-constants act by right multiplications, $\psi \mapsto \psi h$ and leave L invariant.

Natural realizations

There are three natural realizations of this scheme:

- $\mathbb{G}$ consists of functions on a lattice $\mathbb{Z}$ with values in a matrix group G and τ is a shift operator; it is also possible to introduce multi-dimensional lattices with several commuting shift automorphisms.
- $\mathbb{G}$ consists of functions of the line with $g^{\tau}(x) = g(x+1)$.
- $\mathbb{G}$ consists of functions which are meromorphic in $\mathbb{C}^*$ and τ acts by $g^{\tau}(z) = g(qz), q \neq 1$.

In the 1st case, the group of quasi-constants consists of genuine constant functions on the lattice with values in G, in the 2nd case it consists of G-valued periodic functions on the line, in the 3d, it consists of elliptic functions on the elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Choice of a Poisson structure

In the differential case, the gauge action is Hamiltonian. In the difference case it is not; rather, the gauge group itself carries the structure of a Poisson Lie group and the gauge action is Poisson; the same applies to the right multiplications. This suggests the following

Definition. For $f \in \mathbf{Fun}(\mathbb{G})$ we denote by ∇_f, ∇'_f its left and right gradients defined by

$$\langle \nabla_f(\psi), \xi \rangle = \frac{d}{dt}\bigg|_{t=0} f(e^{t\xi}\psi), \quad \langle \nabla'_f(\psi), \xi \rangle = \frac{d}{dt}\bigg|_{t=0} f(\psi e^{t\xi}).$$

We put

$$\{f_1, f_2\} = \langle l(\nabla_{f_1}), \nabla_{f_2} \rangle + \langle r(\nabla'_{f_1}), \nabla'_{f_2} \rangle.$$

Choice of a Poisson structure – 2

Here l and r are two (a priori, different) classical r-matrices. In tensor form this formula may be written as

$$\{\psi_1, \psi_2\} = l_{12}\psi_1\psi_2 + \psi_1\psi_2 r_{12}, \quad \text{where} \quad \psi_1 = \psi \otimes I, \psi_2 = I \otimes \psi. \tag{7}$$

(7) is an abstract version of the Exchange algebra. The main question, of course, is to restrict and explain the choice of l and r.

Easy observations.

- For a given l the gauge action becomes Poisson if the gauge group carries the Sklyanin bracket associated with l.
- Left and right bracket are almost independent, but they are linked via the Jacoby/Yang–Baxter identity.

Contribution of l

A simple computation. Recall that $l, r \in \mathrm{End}\,\mathfrak{g}$, where is the Lie algebra of $\mathbb{G}$. We assume that they commute with τ (which we now regard as an automorphism of $\mathfrak{g}$). Suppose that $f(\psi) = F(\psi^\tau \psi^{-1})$. We denote by X_F, X'_F left and right gradients of F. Then

$$\{f_1, f_2\}^l(\psi) = \langle l(X_1), X_2\rangle + \langle l(X'_1), X'_2\rangle - \langle l \circ \tau^{-1}(X_1), X'_2\rangle - \langle \tau \circ l(X'_1), X_2\rangle.$$

Equivalently, this yields for $L = \psi^\tau \psi^{-1}$:

$$\{L_1, L_2\}^l = lL_1L_2 + L_1L_2l - L_1l^\tau L_2 - L_2l^{\tau^{-1}}L_1.$$

This formula resembles the “lattice current algebra”, but it differs from it in some crucial terms and so is not yet satisfactory. The Jacobi identity fails!

The role of r

The rescue comes through the choice of r which appears to be very rigid.

Key observation. The mapping $\psi \mapsto \psi^\tau \psi^{-1}$ is Poisson if and only if :

$r = r_0 + \frac{\tau + I}{\tau - I},$ where r_0 acts in the subspace of quasi-constants.

Explanation. Left gradient of $f(\psi) = F(\psi^\tau \psi^{-1})$ depends only on left and right gradients of F (regarded as a function of $L = \psi^\tau \psi^{-1}$). By contrast, its right gradient depends on ψ. After some calculations, one gets explicitly:

$$\{f_1, f_2\}^r(\psi) = \langle (r - \tau \cdot r + r - r \cdot \tau^{-1}) \mathrm{Ad}\psi^{-1} X_1', \mathrm{Ad}\psi^{-1} X_2' \rangle.$$

Our mapping is Poisson if and only if $\mathrm{Ad}\psi^{-1}$ cancels.

The role of r

Remarkably, this cancellation is achieved by a simple and unique choice $r = \frac{\tau+I}{\tau-I}$ if we assume that $\tau - I$ is *invertible*.

Proposition. Assume that $\tau - I$ is invertible. Then $r = \frac{\tau+I}{\tau-I}$ satisfies the modified classical Yang–Baxter identity

$$[rX, rY] - r([rX, Y] + [X, rY]) + [X, Y] = 0;$$

It is skew iff τ is orthogonal. With our choice of r, the contribution of the right bracket finally becomes

$$\{f_1, f_2\}^r(\psi) = \langle X_1, \tau X_2' \rangle - \langle \tau X_1',\ X_2 \rangle.$$

- The above formula precisely provides the missing terms to convert the left bracket for L's into the correct lattice algebra.

The formula we derived fixes the choice of r up to the subspace of quasi-constants.

r as a singular integral operator

When τ is a translation operator, $g^\tau(x) = g(x+1)$, the Cayley transform $r = (1+\tau)(1-\tau)^{-1}$ is a singular integral operator formally given by

$$(rf)(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \operatorname{cotan}(k/2)\hat{f}(k)e^{ikx}\,dk. \tag{8}$$

For $f \in C_0^\infty(\mathbb{R};\mathfrak{g})$ we set

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n);$$

clearly, F is 1-periodic and hence lies in the kernel of $1-\tau$.

r as a singular integral operator

In order to regularize (8) we use the standard decomposition of $\mathrm{cotan}(k/2)$ into simple fractions and the Poisson formula

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n) e^{2\pi i n x}.$$

The regularization of (8) is given by

$$(rf)(x) = r_0(F(x)) + \mathrm{v.p.}\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{cotan}(k/2)\hat{f}(k)e^{ikx}\,dk, \quad (9)$$

where r_0 is acting pointwise in the subspace of quasiconstants. This is very similar to the case of exchange algebra; the choice of r is now fully explained.

q-difference case

In q-difference case, when $g^\tau(z) = g(qz)$, the Cayley transform $r = (1+\tau)(1-\tau)^{-1}$ is completely characterized by its distribution kernel given by the formal series

$$r(z, z') = r\delta(z/z'),$$

where

$$\delta(z/z') = \sum_{n=-\infty}^{\infty} (z/z')^n$$

is the Dirac delta function.

q-difference case

Set $z/z' = t$; we get

$$r(z,z') = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1+q^n}{1-q^n} t^n = \sum_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} t^n + \frac{1+q^{-n}}{1-q^{-n}} t^{-n} \right) =$$

$$\sum_{n=1}^{\infty} (t^n - t^{-n}) + 2 \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} (t^n - t^{-n}) = \frac{t+1}{t-1} + 2 \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} (t^n - t^{-n}).$$

Put $z = e^{ix}, z' = e^{ix'}$; we get a Fourier series expansion of $r(z,z')$ on the unit circle,

$$r(z,z') = \frac{1}{i} \operatorname{cotan}(x - x'/2) + 4i \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin n(x-x'). \quad (10)$$

q-difference case

It is easy to see that the r.h.s. is essentially the logarithmic derivative of Jacobi's theta function θ_1 (the difference is due to the fact that in the standard definition the (quasi)-periods of θ_1 are 2 and $q^2 = e^{2\pi i\tau}$, while in our case they are 2π and q). Note also that $r(z, z')$ is not quite 'elliptic', since the logarithmic derivative of θ_1 is invariant with respect to translations by $2\pi i\tau$ only up to an additive constant. If we regard (10) as the kernel of a singular integral operator on the unit circle, its regularization may be achieved in the same way as above (now the only pole of the kernel which lies on the unit circle is $x = 0$ (other poles, which coincide with the roots of θ_1 are associated with the points of the multiplicative lattice q^n).

On difference Galois group

Both in rational and in trigonometric case, the regularization introduces just a finite dimensional r-matrix r_0. This prescription is perfectly in line with the difference Galois theory. Consider a typical q-difference equation, $\psi(qz) = L(z)\psi(z)$, where L is a matrix with rational coefficients $L_{ik} \in \mathbb{C}(z)$. A difference Galois extension $F_\psi \supset \mathbb{C}(z)$ is generated by adjoining to $\mathbb{C}(z)$ matrix coefficients of its (particular) solution. According to a theorem of P.Etingof, the corresponding difference Galois group is finite-dimensional; it consists of constant matrices and is a closed subgroup of the infinite-dimensional group of quasi-constants). Generically, this group coincides with the full special linear group $SL(n)$; it naturally inherits the Poisson structure induced by the finite dimensional r-matrix r_0.

Lattice case

Important example: Difference operators on a lattice $\Gamma = \mathbb{Z}$ with periodic boundary conditions. This means that we assume $L = \psi^\tau \psi^{-1}$ and the gauge group acting by left translations to be N-periodic.
One can show that in this case the right r-matrix is given by

$$r(n,m) = \frac{1}{2} t \epsilon(n-m) + r_0, \tag{11}$$

where

$$\epsilon(n) = \operatorname{sign} n = \begin{cases} 1, & n > 0, \\ 0, & n = 0, \\ -1, & n < 0. \end{cases}$$

and $r_0 \in \mathfrak{g} \otimes \mathfrak{g}$ is a constant r-matrix independent of n, m.
This brings us back to (lattice) exchange algebra.

Monodromy matrix on the lattice

There are two definitions of the monodromy matrix in this setting:

- $M = \psi_N \psi_0^{-1}$, or
- $\tilde{M} = \psi_0^{-1} \psi_N$.

Of course, M and $\tilde{M}$ are conjugate, but their transformation properties are very different: M is invariant with respect to the right action of the group of quasi-constants (which in this case (almost) coincides with the difference Galois group, while $\tilde{M}$ is gauge invariant. Hence $\tilde{M}$ is adapted to reduction over the subgroups of the gauge group. On the other hand, $M = \overset{\curvearrowleft}{\prod}_n L_n$ and hence the Poisson bracket relations for M are easily computable.

Monodromy matrix on the lattice – 2

We assume that the left r-matrix is constant (does not depend on the argument $n \in \mathbb{Z}$)
Theorem.

- Equip G with the Poisson bracket

$$\{f_1, f_2\}^l(M) = \langle l(X_1), X_2\rangle + \langle l(X_1'), X_2'\rangle - \langle l_+(X_1), X_2'\rangle - \langle l_-(X_1'), X_2\rangle,$$

  where as usual X_1, X_2, X_1', X_2' stand for left and right gradients of f_1, f_2. The mapping

$$\mathbb{G} \to G : (L_n) \mapsto M = \overset{\curvearrowleft}{\prod} L_n$$

  is Poisson.

- Gauge action of the gauge group on the monodromy by conjugation is Poisson.

Monodromy matrix on the lattice – 3

By contrast, Poisson bracket relations for $\tilde{M}$ depend mainly on r. In the lattice case the kernel $r = \frac{I+\tau}{I-\tau} + r_0$ is explicitly computable.

Theorem.

- The mapping $\tilde{M} : \mathbb{G} \to G : \psi \mapsto \psi_0^{-1}\psi_N$ is Poisson with respect to the dual Poisson bracket on G,

$$\{f_1, f_2\}^l(\tilde{M}) = \langle r_0(X_1), X_2\rangle + \langle r(X_1'), X_2'\rangle - \langle r_+^0(X_1), X_2'\rangle - \langle r_-^0(X_1'), X_2\rangle, \tag{12}$$

Monodromy matrix on the lattice – 4

- This bracket is Poisson covariant with respect to the action of the difference Galois group by conjugation.

To assure that Poisson brackets for both versions of the monodromy matrix are the same we may simply assume that $r_0 = l$.

q-deformed Drinfeld–Sokolov theory

Proposition. An n-th order difference equation

$$\tau^n \phi + u_{n-1}\tau^{n-1} \cdot \phi + u_{n-2}\tau^{n-2} \cdot \phi + ... + u_1 \tau \cdot \phi + \phi = 0$$

is associated with the scalar 1st order equation

$$\tau \cdot \psi + L\psi = 0,$$

with $L = Us^{-1}$, where

$$s = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}$$

q-deformed Drinfeld–Sokolov theory

is the Coxeter matrix and

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ u_1 & u_1 & \cdots & u_{n-1} & 1 \end{pmatrix}. \qquad (13)$$

Taking L in this canonical form gives a set of constraints which serve to realize the reduced phase space (over the gauge action of the unipotent upper triangular group); the construction is self-consistent if and only if these constraints are first class, which imposes severe restrictions on the choice of l:

q-deformed Drinfeld–Sokolov theory

$$l = P_+ - P_- + \frac{1 + R_s \tau}{1 - R_s \tau} P_0,$$

where the projection operators $P_+, P_{,} P_0$ are associated with the pointwise triangular decomposition and R_s is the Coxeter automorphism of $\mathfrak{h} = P_0 \mathfrak{g}$ induced by s.

Poisson bracket for the monodromy

In the difference Drinfeld–Sokolov setting the proper definition of monodromy is $\tilde{M}$, which is gauge invariant and hence survives the reduction. As before, it is natural to choose r_0 which is consistent with the choice of l. This gives

$$r_0 = P_+ - P_- + \frac{1 + R_s}{1 - R_s} P_0$$

Note that the correction term vanishes for $SL(2)$, since in this case $R_s = -I$. With this choice we get a consistent Poisson bracket for the monodromy which is covariant with respect to the action of the difference Galois group (equipped with the Sklyanin bracket associated with r_0).

The space of wave functions

We now come up to the extension of the deformed Poisson Virasoro structure to the space of wave functions. Fundamental solutions of the first order linear problem $\psi^{\tau} = L\psi$ are functions with values in G. By contrast, wave functions of a scalar n-th order difference equation form an n-tuple $(\phi_1, \ldots, \phi_n) \in \mathbb{C}^n \setminus \{0\}$. Up to a natural equivalence this set of wave functions defines a point in $\mathbb{C}P_{n-1}$. To match these descriptions note that the gauge group $\mathbb{N}_-$ acts on wave functions by left translations, $n \cdot \psi(x) = n(x)\psi(x)$. The quotient space $\mathbb{N}_- \backslash \mathbb{G}$ may be identified with the space of functions with values in the principal affine space $N_- \backslash G$. The Cartan subgroup $H \subset G$ normalizes N_- and hence there is a natural action $H \times N_- \backslash G \to N_- \backslash G$ and the associated pointwise action $\mathbb{H} \times \mathbb{N}_- \backslash \mathbb{G} \to \mathbb{N}_- \backslash \mathbb{G}$.

The space of wave functions

When $G = SL(2,\mathbb{C})$, the quotient $HN_-\backslash G$ is isomorphic to the projective space $\mathbb{C}P_1$ and $\mathbb{HN}_-\backslash\mathbb{G}$ is the space of projectivized wave functions of the second order difference equation.

In the general case, when the potential of the first order matrix difference equation in canonical form $L = Us^{-1}$, its matrix wave function ψ has the simple 'Vandermonde' form,

$$\psi = \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1^\tau & \phi_2^\tau & \dots & \phi_n^\tau \\ \dots & \dots & \dots & \\ \phi_1^{\tau^{n-1}} & \phi_2^{\tau^{n-1}} & \dots & \phi_n^{\tau^{n-1}} \end{pmatrix}. \tag{14}$$

and is completely determined by its first row.

The space of wave functions

Let $H_0 \subset H \subset SL(n)$ be the 1-dimensional subgroup consisting of degenerate diagonal matrices $h = \mathrm{diag}\,(t, \ldots, t, s)$, $\det h = 1$. Let M_0' be the centaliser of H_0 in $G = SL(n)$ and M_0 its maximal semisimple subgroup; let $N_0 \subset N_-$ be the unipotent subgroup defined in (13). The subgroup $P_0 = M_0 H_0 N_0 \subset G$ is a maximal parabolic subgroup which contains the standard Borel subgroup $B = HN_-$. We regard $M_0 N_0 \backslash G$ as an affine algebraic variety; its affine ring $\mathbb{A}(M_0 N_0 \backslash G)$ is generated by the matrix coefficients of the first row of the matrix $g \in G$. Since $M_0 N_0 \supset N$, this affine ring is canonically embedded into the affine ring of $N_- \backslash G$. The multi-scaling action of the Cartan subgroup H on the affine ring $\mathbb{A}(N \backslash G)$ induced by the natural action $H \times N_- \backslash G \to N_- \backslash G$ may be restricted to $\mathbb{A}(M_0 N_0 \backslash G)$ and amounts to the scaling action of the rank 1 subgroup $H_0 \subset H$.

The space of wave functions

The quotient $M_0 H_0 N_0 \backslash G$ is isomorphic to the projective space $\mathbb{C}P_{n-1}$; the associated space of functions with values in $M_0 H_0 N_0 \backslash G$ is precisely the space of projectivized wave functions of the n-th order difference equations.

- **Example** For $n = 2$ we get the Poisson structure for the projectivized wave functions:

$$\{\eta(m), \eta(m)\} = \eta(m)^2 - \eta(n)^2 - \operatorname{sign}(m-n)\,(\eta(m) - \eta(n))^2 . \quad (15)$$

  This Poisson bracket is covariant with respect to the projective transformation group $PSL(2)$ equipped with the standard Poisson structure; the Poisson structure in the space of projective invariants yields a lattice version of the Virasoro algebra.