Form factors in GL(3)-invariant integrable models

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in collaboration with

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RAQIS'14

1–5 September 2014

We consider form factors of the monodromy matrix entries

$$\langle\psi(\bar u)|T_{ij}(z)|\psi'(\bar u')\rangle$$

Here $\langle\psi(\bar u)|$ and $|\psi'(\bar u')\rangle$ are eigenstates of the transfer matrix $\mathcal{T}(z) = \mathrm{tr}\, T(z)$. The monodromy matrix $T(z)$ is 3×3

$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) & T_{13}(z) \\ T_{21}(z) & T_{22}(z) & T_{23}(z) \\ T_{31}(z) & T_{32}(z) & T_{33}(z) \end{pmatrix}$$

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$$\begin{pmatrix} ? & ? & ? \\ ? & T_{22}(z) & ? \\ ? & ? & ? \end{pmatrix}$$

ANGERS'12 (Part I) [S. Belliard, S. Pakuliak, E. Ragoucy, N.S.]

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DIJON'14 (Part III) [S. Pakuliak, E. Ragoucy, N.S., in preparation]

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Similar statement is true for the $GL(N)$ case.

Algebraic Bethe Ansatz for $GL(3)$-invariant models

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) & T_{13}(z) \\ T_{21}(z) & T_{22}(z) & T_{23}(z) \\ T_{31}(z) & T_{32}(z) & T_{33}(z) \end{pmatrix}$$

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$GL(3)$-invariant R-matrix

$$R(u,v) = \mathbf{I} + g(u,v)\mathbf{P}, \qquad g(u,v) = \frac{c}{u-v}$$

$$(s_1 s_2)\ R_{12}\ (s_1 s_2)^{-1} = R_{12}, \qquad \forall s \in GL(3)$$

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$GL(3)$-invariant R-matrix

$$R(u,v) = \mathbf{I} + g(u,v)\mathbf{P}, \qquad g(u,v) = \frac{c}{u-v}$$

$$f(u,v) = 1 + g(u,v) = \frac{u-v+c}{u-v}$$

Sets of variables

$$\bar{u} = \{u_1, \ldots, u_a\}, \qquad \bar{v} = \{v_1, \ldots, v_b\}, \qquad \bar{w} = \{w_1, \ldots, w_n\}$$

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Products over the sets

$$T_{\epsilon,\epsilon'}(\bar{w}) = \prod_{w_k \in \bar{w}} T_{\epsilon,\epsilon'}(w_k)$$

$$g(\bar{u}, v_j) = \prod_{u_k \in \bar{u}} g(u_k, v_j)$$

$$f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k)$$

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Special subsets

$$\bar{u}_j = \bar{u} \setminus u_j \qquad\qquad f(\bar{u}_j, u_j) = \prod_{\substack{u_k \in \bar{u} \\ u_k \neq u_j}} f(u_k, u_j)$$

Bethe vectors

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

We consider a generalized model with a pseudovacuum vector $|0\rangle$ and dual pseudovacuum vector $\langle 0|$

$$T_{jj}(u)|0\rangle = r_j(u)|0\rangle, \qquad T_{jk}(u)|0\rangle = 0, \quad j > k$$

$$\langle 0|T_{jj}(u) = r_j(u)\langle 0|, \qquad \langle 0|T_{jk}(u) = 0, \quad j < k$$

One can set one of $r_j(u)$ equals to 1 without loss of generality. Other $r_j(u)$ remain free functional parameters (generalized model). We set $r_2(u) = 1$.

Bethe vectors

We look for the eigenvectors of the transfer matrix

$$\mathcal{T}(w) = \mathsf{tr}\, T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w)$$

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The first step is to construct special polynomials in creation operators (T_{12}, T_{13}, T_{23}) applied to the pseudovacuum $|0\rangle$ (Bethe vectors).

Nested algebraic Bethe ansatz

P. Kulish, N. Reshetikhin, '83

V. Tarasov, A. Varchenko '95

S. Belliard, S. Khoroshkin, S. Pakuliak, E. Ragoucy '08, '10

Bethe vectors

We look for the eigenvectors of the transfer matrix

$$\mathcal{T}(w) = \mathrm{tr}\, T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w)$$

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big)|0\rangle, \qquad i < j$$

$\bar{u} = u_1, \dots, u_a$

$\bar{v} = v_1, \dots, v_b$

$a, b = 0, 1 \dots$

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$$\bar{u} = u_1, \dots, u_a$$
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$$a, b = 0, 1 \dots$$

Normalization

$$P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big) = f^{-1}(\bar{v},\bar{u})\; T_{12}(\bar{u})T_{23}(\bar{v}) + \dots$$

We say that $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ is a Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ are generic complex numbers.

Bethe vectors

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$$\mathcal{T}(w) = \operatorname{tr} T(w) = T_{11}(w) + T_{22}(w) + T_{33}(w)$$

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big)|0\rangle, \qquad i < j$$

We say that $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ is an on-shell Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ satisfy the system of Bethe equations

$$r_1(u_k) = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)} f(\bar{v},u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)} f(v_k,\bar{u})$$

Recall: $\bar{u}_k = \bar{u} \setminus u_k, \qquad f(u_k,\bar{u}_k) = \prod_{\substack{u_s\in\bar{u} \\ u_s\neq u_k}} f(u_k,u_s)$

Dual Bethe vectors

Dual Bethe vectors are special polynomials in annihilation operators (T_{21}, T_{31}, T_{32}) applied to the dual pseudovacuum $\langle 0|$.

$$\mathbb{C}^{a,b}(\bar{u};\bar{v}) = \langle 0|P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big), \qquad i > j$$

$$\bar{u} = u_1, \dots, u_a$$
$$\bar{v} = v_1, \dots, v_b$$
$$a, b = 0, 1 \dots$$

Normalization

$$P\Big(T_{ij}(u_k), T_{ij}(v_k)\Big) = f^{-1}(\bar{v}, \bar{u})\; T_{21}(\bar{u})T_{32}(\bar{v}) + \dots$$

We say that $\mathbb{C}^{a,b}(\bar{u};\bar{v})$ is a dual on-shell Bethe vector, if the parameters $\bar{u}$ and $\bar{v}$ satisfy the system of Bethe equations

$$r_1(u_k) = \frac{f(u_k, \bar{u}_k)}{f(\bar{u}_k, u_k)} f(\bar{v}, u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k, v_k)}{f(v_k, \bar{v}_k)} f(v_k, \bar{u})$$

Transfer matrix eigenvalues

On-shell Bethe vectors are eigenvectors of the transfer matrix $\mathcal{T}(w) = \mathrm{tr}\, T(w)$.

$$\mathcal{T}(w)\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \Lambda(w|\bar{u},\bar{v})\ \mathbb{B}^{a,b}(\bar{u};\bar{v})$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})\mathcal{T}(w) = \Lambda(w|\bar{u},\bar{v})\ \mathbb{C}^{a,b}(\bar{u};\bar{v})$$

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$$\Lambda(w|\bar{u},\bar{v}) = r_1(w)f(\bar{u},w) + f(w,\bar{u})f(\bar{v},w) + r_3(w)f(w,\bar{v})$$

$$r_1(u_k) = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)}f(\bar{v},u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)}f(v_k,\bar{u})$$

Form factors of $T_{\epsilon,\epsilon'}(z)$

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)\, T_{\epsilon,\epsilon'}(z)\, \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$$

Here both $\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors.

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Here both $\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors.

$$r_1(u_k^B) = \frac{f(u_k^B, \bar{u}_k^B)}{f(\bar{u}_k^B, u_k^B)} f(\bar{v}^B, u_k^B), \qquad r_3(v_k^B) = \frac{f(\bar{v}_k^B, v_k^B)}{f(v_k^B, \bar{v}_k^B)} f(v_k^B, \bar{u}^B)$$

$$r_1(u_k^C) = \frac{f(u_k^C, \bar{u}_k^C)}{f(\bar{u}_k^C, u_k^C)} f(\bar{v}^C, u_k^C), \qquad r_3(v_k^C) = \frac{f(\bar{v}_k^C, v_k^C)}{f(v_k^C, \bar{v}_k^C)} f(v_k^C, \bar{u}^C)$$

Generically $\{\bar{u}^C, \bar{v}^C\}$ and $\{\bar{u}^B, \bar{v}^B\}$ are different solutions of Bethe equations.

Form factors of $T_{\epsilon,\epsilon'}(z)$

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)\, T_{\epsilon,\epsilon'}(z)\, \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$$

The integers a and b are fixed. Then

$a' = a + \delta_{\epsilon,1} - \delta_{\epsilon',1}$,

$b' = b + \delta_{\epsilon',3} - \delta_{\epsilon,3}$.

The parameter z is an arbitrary complex.

Form factors and morphisms

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)T_{\epsilon,\epsilon'}(z)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$$

There exist 9 matrix elements $T_{\epsilon,\epsilon'}(z)$, thus there exist 9 form factors. However not all of them are independent due to symmetries of the R-matrix and morphisms of the $RTT = TTR$ relation.

$$\mathcal{F}(z) = \begin{pmatrix} \mathcal{F}^{1,1}(z) & \mathcal{F}^{1,2}(z) & \mathcal{F}^{1,3}(z) \\ \mathcal{F}^{2,1}(z) & \mathcal{F}^{2,2}(z) & \mathcal{F}^{2,3}(z) \\ \mathcal{F}^{3,1}(z) & \mathcal{F}^{3,2}(z) & \mathcal{F}^{3,3}(z) \end{pmatrix}$$

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$$\psi \,: T_{ij}(u) \quad \mapsto \quad T_{ji}(u)$$

$$\varphi \,: T_{ij}(u) \quad \mapsto \quad T_{4-j,4-i}(-u)$$

Form factors and morphisms

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$$\mathcal{F}(z) = \begin{pmatrix} \mathcal{F}^{1,1}(z) & \mathcal{F}^{1,2}(z) & \textbf{?} \\ \mathcal{F}^{2,1}(z) & \mathcal{F}^{2,2}(z) & \mathcal{F}^{2,3}(z) \\ \mathcal{F}^{3,1}(z) & \mathcal{F}^{3,2}(z) & \mathcal{F}^{3,3}(z) \end{pmatrix}$$

The key tool for the calculation of $\mathcal{F}^{1,1}(z)$, $\mathcal{F}^{2,2}(z)$, and $\mathcal{F}^{1,2}(z)$ is a formula for the scalar product of generic Bethe vectors (N. Reshetikhin'86). This formula describes the scalar product as a sum over partitions of Bethe parameters.

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Restriction:

$$r_1(z) = 1 + O(z^{-1}), \qquad z \to \infty$$

$$r_3(z) = 1 + O(z^{-1}), \qquad z \to \infty$$

Zero modes

We consider the standard generating series of Yangian

$$T_{\epsilon,\epsilon'}(z) = \delta_{\epsilon,\epsilon'} + \sum_{n=1}^{\infty} T^{(n)}_{\epsilon,\epsilon'} z^{-n}$$

and study the action of the zero modes $T^{(1)}_{\epsilon,\epsilon'}$ on Bethe vectors $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ and $\mathbb{C}^{a,b}(\bar{u};\bar{v})$

$$T^{(1)}_{\epsilon,\epsilon'}\mathbb{B}^{a,b}(\bar{u};\bar{v}) \qquad\qquad \mathbb{C}^{a,b}(\bar{u};\bar{v})T^{(1)}_{\epsilon,\epsilon'}$$

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The action of zero modes on Bethe vectors in $GL(N)$
is known (E. Mukhin, V. Tarasov, A. Varchenko'06)

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$$T^{(1)}_{\epsilon,\epsilon'}\mathbb{B}^{a,b}(\bar{u};\bar{v}) \qquad\qquad \mathbb{C}^{a,b}(\bar{u};\bar{v})T^{(1)}_{\epsilon,\epsilon'}$$

This action follows from the one of $T_{\epsilon,\epsilon'}(z)$
on $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ and $\mathbb{C}^{a,b}(\bar{u};\bar{v})$

Action of zero modes

In particular:

$$T_{12}^{(1)}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lim_{w\to\infty} w\,\mathbb{B}^{a+1,b}(\{\bar{u},w\};\bar{v})$$

$$T_{23}^{(1)}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lim_{w\to\infty} w\,\mathbb{B}^{a,b+1}(\bar{u};\{\bar{v},w\})$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T_{21}^{(1)} = \lim_{w\to\infty} w\,\mathbb{C}^{a+1,b}(\{\bar{u},w\};\bar{v})$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T_{32}^{(1)} = \lim_{w\to\infty} w\,\mathbb{C}^{a,b+1}(\bar{u};\{\bar{v},w\})$$

Action of zero modes

In particular:

$$T^{(1)}_{12}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lim_{w\to\infty} w\,\mathbb{B}^{a+1,b}(\{\bar{u},w\};\bar{v})$$

$$T^{(1)}_{23}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lim_{w\to\infty} w\,\mathbb{B}^{a,b+1}(\bar{u};\{\bar{v},w\})$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T^{(1)}_{21} = \lim_{w\to\infty} w\,\mathbb{C}^{a+1,b}(\{\bar{u},w\};\bar{v})$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T^{(1)}_{32} = \lim_{w\to\infty} w\,\mathbb{C}^{a,b+1}(\bar{u};\{\bar{v},w\})$$

If Bethe vectors are normalized by 1, then one can simply write $\mathbb{B}^{a+1,b}(\{\bar{u},\infty\};\bar{v})$, $\mathbb{C}^{a,b+1}(\bar{u};\{\bar{v},\infty\})$ and so on.

Action of zero modes

In particular:

$$T_{21}^{(1)}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = c\sum_{i=1}^{a}\Big\{\frac{r_1(u_i)f(\bar{u}_i,u_i)}{f(\bar{v},u_i)} - f(u_i,\bar{u}_i)\Big\}\mathbb{B}^{a-1,b}(\bar{u}_i;\bar{v})$$

$$T_{32}^{(1)}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = -c\sum_{i=1}^{b}\Big\{\frac{r_3(v_i)f(v_i,\bar{v}_i)}{f(v_i,\bar{u})} - f(\bar{v}_i,v_i)\Big\}\mathbb{B}^{a,b-1}(\bar{u};\bar{v}_i)$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T_{12}^{(1)} = c\sum_{i=1}^{a}\Big\{\frac{r_1(u_i)f(\bar{u}_i,u_i)}{f(\bar{v},u_i)} - f(u_i,\bar{u}_i)\Big\}\mathbb{C}^{a-1,b}(\bar{u}_i;\bar{v})$$

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T_{23}^{(1)} = -c\sum_{i=1}^{b}\Big\{\frac{r_3(v_i)f(v_i,\bar{v}_i)}{f(v_i,\bar{u})} - f(\bar{v}_i,v_i)\Big\}\mathbb{C}^{a,b-1}(\bar{u};\bar{v}_i)$$

Action of zero modes

In particular:

$$T^{(1)}_{21}\mathbb{B}^{a,b}(\bar u;\bar v) = c\sum_{i=1}^{a}\left\{\frac{r_1(u_i)f(\bar u_i,u_i)}{f(\bar v,u_i)} - f(u_i,\bar u_i)\right\}\mathbb{B}^{a-1,b}(\bar u_i;\bar v) \quad = 0$$

$$T^{(1)}_{32}\mathbb{B}^{a,b}(\bar u;\bar v) = -c\sum_{i=1}^{b}\left\{\frac{r_3(v_i)f(v_i,\bar v_i)}{f(v_i,\bar u)} - f(\bar v_i,v_i)\right\}\mathbb{B}^{a,b-1}(\bar u;\bar v_i) \quad = 0$$

$$\mathbb{C}^{a,b}(\bar u;\bar v)T^{(1)}_{12} = c\sum_{i=1}^{a}\left\{\frac{r_1(u_i)f(\bar u_i,u_i)}{f(\bar v,u_i)} - f(u_i,\bar u_i)\right\}\mathbb{C}^{a-1,b}(\bar u_i;\bar v) \quad = 0$$

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if $\mathbb{B}^{a,b}(\bar u;\bar v)$ and $\mathbb{C}^{a,b}(\bar u;\bar v)$ are on-shell

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In particular:

$$T_{21}^{(1)}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = c\sum_{i=1}^{a}\left\{\frac{r_1(u_i)f(\bar{u}_i,u_i)}{f(\bar{v},u_i)} - f(u_i,\bar{u}_i)\right\}\mathbb{B}^{a-1,b}(\bar{u}_i;\bar{v}) \quad = 0$$

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$$\mathbb{C}^{a,b}(\bar{u};\bar{v})T_{12}^{(1)} = c\sum_{i=1}^{a}\left\{\frac{r_1(u_i)f(\bar{u}_i,u_i)}{f(\bar{v},u_i)} - f(u_i,\bar{u}_i)\right\}\mathbb{C}^{a-1,b}(\bar{u}_i;\bar{v}) \quad = 0$$

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if $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ and $\mathbb{C}^{a,b}(\bar{u};\bar{v})$ are on-shell and $\infty \notin \{\bar{u},\bar{v}\}$.

Special on-shell vectors

If $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ is an on-shell Bethe vector, then $\mathbb{B}^{a,b}(\{\bar{u},\infty\};\bar{v})$ and $\mathbb{B}^{a,b}(\bar{u};\{\bar{v},\infty\})$ also are on-shell Bethe vectors.

If $\mathbb{C}^{a,b}(\bar{u};\bar{v})$ is a dual on-shell Bethe vector, then $\mathbb{C}^{a,b}(\{\bar{u},\infty\};\bar{v})$ and $\mathbb{C}^{a,b}(\bar{u};\{\bar{v},\infty\})$ also are dual on-shell Bethe vectors.

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$$r_1(u_k) = \frac{f(u_k,\bar{u}_k)}{f(\bar{u}_k,u_k)} f(\bar{v},u_k), \qquad r_3(v_k) = \frac{f(\bar{v}_k,v_k)}{f(v_k,\bar{v}_k)} f(v_k,\bar{u})$$

$$f(v,u) = \frac{v-u+c}{v-u}, \qquad f(v,\infty) = f(\infty,u) = 1$$

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$$r_k(\infty) = 1,$$

Calculation of form factors

We have from RTT-relation

$$[T_{ij}(z_1), T_{kl}(z_2)] = g(z_1, z_2)\Big(T_{kj}(z_2)\, T_{il}(z_1) - T_{kj}(z_1)\, T_{il}(z_2)\Big)$$

Sending here one of z_k to infinity we obtain commutation relations involving zero modes.

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Sending here one of z_k to infinity we obtain commutation relations involving zero modes. In particular

$$[T_{12}(z_1), T_{23}(z_2)] = g(z_1, z_2)\Big(T_{22}(z_2)\, T_{13}(z_1) - T_{22}(z_1)\, T_{13}(z_2)\Big)$$

$$[T_{12}(z_1), T_{23}^{(1)}] = -cT_{13}(z_1)$$

Calculation of form factors

$$c\, T_{13}(z) = [T_{23}^{(1)}, T_{12}(z)]$$

Calculation of form factors

$$c\, T_{13}(z) = [T^{(1)}_{23}, T_{12}(z)]$$

$$c\, \mathbb{C}^{a+1,b+1}(\bar{u}^C; \bar{v}^C) T_{13}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \mathbb{C}^{a+1,b+1}(\bar{u}^C; \bar{v}^C) [T^{(1)}_{23}, T_{12}(z)] \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$$

where both $\mathbb{C}^{a+1,b+1}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors.

Calculation of form factors

$$c\,T_{13}(z) = [T_{23}^{(1)}, T_{12}(z)]$$

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$$\mathbb{C}^{a+1,b+1}(\bar{u}^C;\bar{v}^C)T_{23}^{(1)} = 0$$

if $\infty \notin \{\bar{u}^C, \bar{v}^C\}$

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$$T_{23}^{(1)}\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lim_{w\to\infty} w\,\mathbb{B}^{a,b+1}(\bar{u};\{\bar{v},w\})$$

and $\mathbb{B}^{a,b+1}(\bar{u}^B;\{\bar{v}^B,\infty\})$ is on-shell Bethe vector.

Calculation of form factors

$$c\,\mathbb{C}^{a+1,b+1}(\bar u^C;\bar v^C)T_{13}(z)\mathbb{B}^{a,b}(\bar u^B;\bar v^B) = -\mathbb{C}^{a+1,b+1}(\bar u^C;\bar v^C)T_{12}(z)T^{(1)}_{23}\mathbb{B}^{a,b}(\bar u^B;\bar v^B)$$

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$$\mathbb{C}^{a+1,b+1}(\bar u^C;\bar v^C)T_{13}(z)\mathbb{B}^{a,b}(\bar u^B;\bar v^B)$$

$$= -\lim_{w\to\infty}\frac{w}{c}\,\mathbb{C}^{a+1,b+1}(\bar u^C;\bar v^C)T_{12}(z)\mathbb{B}^{a,b}(\bar u^B;\{\bar v^B,w\})$$

Calculation of form factors

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$$\mathcal{F}^{1,3}_{a,b}(z|\bar u^C,\bar v^C;\bar u^B,\bar v^B) = -\lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{1,2}_{a,b+1}(z|\bar u^C,\bar v^C;\bar u^B,\{\bar v^B,w\})$$

Calculation of form factors

In order to compute the form factor of $T_{13}(z)$ one should take the determinant formula for the form factor of $T_{12}(z)$ and to send there one of Bethe parameters to infinity

$$\mathcal{F}_{a,b}^{1,3}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = -\lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}_{a,b+1}^{1,2}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\{\bar{v}^B,w\})$$

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Too good to be true?

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Too good to be true?

Using RTT-relations one can derive formulas of the similar type for other form factors, whose determinant representations are already known.

Relations between form factors

$$\mathcal{F}^{(1,2)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = -\lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{(1,1)}_{a+1,b}(z|\bar{u}^C,\bar{v}^C;\{\bar{u}^B,w\},\bar{v}^B)$$

$$\mathcal{F}^{(1,2)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{(2,2)}_{a+1,b}(z|\bar{u}^C,\bar{v}^C;\{\bar{u}^B,w\},\bar{v}^B)$$

$$\mathcal{F}^{(1,1)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) - \mathcal{F}^{(2,2)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B)$$

$$= \lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{(1,2)}_{a,b}(z|\{\bar{u}^C,w\},\bar{v}^C;\bar{u}^B,\bar{v}^B)$$

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$$\mathcal{F}^{(1,1)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) - \mathcal{F}^{(2,2)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B)$$

$$= \lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{(1,2)}_{a,b}(z|\{\bar{u}^C,w\},\bar{v}^C;\bar{u}^B,\bar{v}^B)$$

Explicit representations for all these form factors were obtained by straightforward calculations without any assumptions on the analytical structure of $T(z)$. One can check that all the relations above are indeed valid.

$$\mathcal{F}^{1,3}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = -\lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{1,2}_{a,b+1}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\{\bar{v}^B,w\})$$

This formula was checked by independent straightforward calculation in the particular cases:

- $a \geq 0, \qquad b = 0;$
- $a = b = 1.$

$$\mathcal{F}^{1,3}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = -\lim_{w\to\infty}\frac{w}{c}\,\mathcal{F}^{1,2}_{a,b+1}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\{\bar{v}^B,w\})$$

This formula was checked by independent straightforward calculation in the particular cases:

- $a \geq 0, \qquad b = 0;$
- $a = b = 1.$

We believe that it remains valid in the general case independently of the analytical structure of $T(z)$.

Determinant formulas for form factors

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) \equiv \mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)T_{\epsilon,\epsilon'}(z)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$$

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon')}$$

The pre-factor $H_{a,b}$ is (ϵ,ϵ')-independent.

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The matrices $\mathcal{N}^{(\epsilon,\epsilon')}$ for the form factors $\mathcal{F}_{a,b}^{(\epsilon,\epsilon')}$ with $|\epsilon-\epsilon'|=1$ have the most simple form.

Determinant formulas for form factors

$$\mathcal{F}_{a,b}^{(1,2)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

$$\mathcal{N}^{(1,2)} = \left(\begin{array}{cc} (*) & \dfrac{\partial \Lambda(x_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C} \\ \hline (*) & \dfrac{\partial \Lambda(x_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B} \end{array} \right) \begin{array}{l} \} \, a+1 \\ \} \, b \end{array}$$

$$\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$$

Determinant formulas for form factors

$$\mathcal{F}^{(1,2)}_{a,b}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(1,2)}$$

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$$\Lambda(w|\bar{u}, \bar{v}) = r_1(w) f(\bar{u}, w) + f(w, \bar{u}) f(\bar{v}, w) + r_3(w) f(w, \bar{v})$$

Other $\mathcal{F}^{(\epsilon,\epsilon')}_{a,b}$ can be obtained from $\mathcal{F}^{(1,2)}_{a,b}$ via replacement $\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}$.

Determinant formulas for form factors

$$\mathcal{F}_{a,b}^{(\epsilon,\epsilon)}(z) = H_{a,b} \det_{a+b+1} \mathcal{N}^{(\epsilon,\epsilon)}$$

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$$\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$$

Determinant formulas for form factors

$$\mathcal{F}_{a,b}^{(1,3)}(z) = H_{a,b} \det_{a+b+2} \mathcal{N}^{(1,3)}$$

$$\mathcal{N}^{(1,3)} = \begin{pmatrix} (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^C,\bar{v}^C)}{\partial u_j^C} \\ \hline (*) & \dfrac{\partial\Lambda(x_k|\bar{u}^B,\bar{v}^B)}{\partial v_j^B} \\ \hline \text{additional row} & \end{pmatrix} \begin{matrix} \}\, a+1 \\ \}\, b \\ \end{matrix}$$

$$\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$$

Determinant formulas for form factors

$$\mathcal{F}_{a,b}^{(1,3)}(z) = H_{a,b} \det_{a+b+2} \mathcal{N}^{(1,3)}$$

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$$\mathcal{N}_{a+b+2,k}^{(1,3)} = 0, \qquad k \neq a+b+2$$

$$\mathcal{N}_{a+b+2,a+b+2}^{(1,3)} = \Lambda(z|\bar{u}^C, \bar{v}^C) - \Lambda(z|\bar{u}^B, \bar{v}^B)$$

Calculation of form factors

$\mathcal{F}_{a,b}^{(2,2)}$

Calculation of form factors

$$\mathcal{F}_{a,b}^{(2,2)} \longrightarrow \mathcal{F}_{a,b}^{(1,2)}(\mathcal{F}_{a,b}^{(2,3)}, \mathcal{F}_{a,b}^{(2,1)}, \mathcal{F}_{a,b}^{(3,2)}) \longrightarrow \mathcal{F}_{a,b}^{(1,3)}(\mathcal{F}_{a,b}^{(3,1)})$$

Calculation of form factors

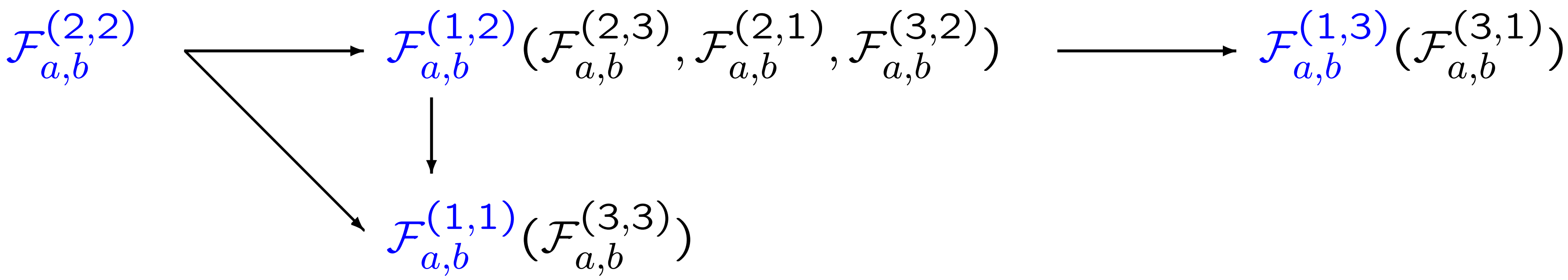
$$
\begin{array}{ccccc}
\mathcal{F}^{(2,2)}_{a,b} & \longrightarrow & \mathcal{F}^{(1,2)}_{a,b}\left(\mathcal{F}^{(2,3)}_{a,b}, \mathcal{F}^{(2,1)}_{a,b}, \mathcal{F}^{(3,2)}_{a,b}\right) & \longrightarrow & \mathcal{F}^{(1,3)}_{a,b}\left(\mathcal{F}^{(3,1)}_{a,b}\right) \\
 & \searrow & \downarrow & & \\
 & & \mathcal{F}^{(1,1)}_{a,b}\left(\mathcal{F}^{(3,3)}_{a,b}\right) & &
\end{array}
$$

Calculation of form factors

$$
\begin{array}{ccccc}
\mathcal{F}_{a,b}^{(2,2)} & \longrightarrow & \mathcal{F}_{a,b}^{(1,2)}\big(\mathcal{F}_{a,b}^{(2,3)},\mathcal{F}_{a,b}^{(2,1)},\mathcal{F}_{a,b}^{(3,2)}\big) & \longrightarrow & \mathcal{F}_{a,b}^{(1,3)}\big(\mathcal{F}_{a,b}^{(3,1)}\big) \\
 & \searrow & \downarrow & & \\
 & & \mathcal{F}_{a,b}^{(1,1)}\big(\mathcal{F}_{a,b}^{(3,3)}\big) & &
\end{array}
$$

All formulas for form factors follow from the determinant representation for form factor $\mathcal{F}_{a,b}^{(2,2)}$. The last one can be derived by the trick with the twisted transfer matrix.

Calculation of form factors



All formulas for form factors follow from the determinant representation for form factor $\mathcal{F}_{a,b}^{(2,2)}$. The last one can be derived by the trick with the twisted transfer matrix.

Similarly, in $GL(N)$ case a determinant representation for one diagonal form factor $\mathcal{F}^{(\epsilon,\epsilon)}$ (if it exists!) yields determinant formulas for all other form factors.

Questions to solve

- Complete proof of the determinant representation for form factor of $T_{13}(z)$.
    - Whether the representation is still valid for arbitrary $r_k(z)$?
    - The role of infinite solutions of Bethe equations.
    - Why straightforward calculations failed?

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- Generalization to the q-deformed case.
  - The form factor $\mathcal{F}_{a,b}^{(2,2)}$ is computed, but there are unexpected problems with other form factors.