Correlations in massive phase

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The aim

Evaluate (various) Form Factors in spin $\frac{1}{2}$ XXZ model in (AF) massive phase at arbitrary size or arbitrary temperatures.

The Hamiltonian (PBC)

$$H_{\mathrm{X}XZ} = J\sum_{i=1}^{M}(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{\cosh\eta}{2}(\sigma_i^z\sigma_{i+1}^z + 1) + \frac{h}{2}\sigma_i^z). \qquad q = \mathrm{e}^{-\eta}$$

We assume $\eta, h > 0$

The aim

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Why do we want to evaluate it?

- The exact evaluation of correlation functions is possible only for small separations. For $\Delta > 1$, few terms in FF (or spectral) expansion is enough to obtain good asymptotic behaviors
- The vertex operator method yields explicit results in the infinite volume limit.
- The bootstrap method yields explicit results in the QFT limit.

Our idea to attack the problem for finite T
$=$ "fuse" two ingredients.

- Quantum Transfer Matrix for the thermodynamics of 1D quantum spin chains. (M. Suzuki, "Wuppertal group")
- Quantum Inverse Scattering Method for matrix elements ("LOMI group" and "Lyon group")

The outline of talk

- QTM for the thermodynamics of 1D quantum spin chains
- formal formulae for simpe FFs
- The behavior of Bethe ansatz roots in low T limit.
- outlook

- QTM method

M. Suzuki, PRB 31 (1985) 2957. J. S., Y.Akutsu and M.Wadati, JPSJ 59 (1990) 2667. A.Klümper, Ann. Physik 1 (1992) 540.

main idea

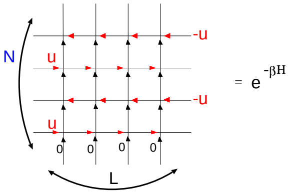
Use the equivalence of 1D quantum system at finite T and 2D classical system of finite size N

use a simple identity

$$\lim_{N\to\infty} Z_{\rm 2D}(N, L) = \lim_{N\to\infty} \mathrm{tr}\mathcal{T}_{\rm RTR}(u)^{\frac{N}{2}}$$

$$= \lim_{N\to\infty} \mathrm{tr}(1+u\mathcal{H})^{N} \qquad u = -\frac{\beta}{N}$$

$$= \mathrm{tr}e^{-\beta\mathcal{H}} = Z_{\text{1D quantum}}(\beta, L)$$



Theorem

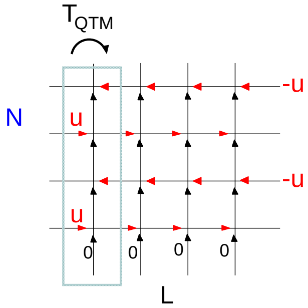
Consider the transfer matrix $T_{\mathrm{QTM}}(u)$ defined in fictitious dimension and propagating horizontally. The free energy per site f is given only(!) by its largest eigenvalue Λ_0

$$f = -\lim_{N\to\infty} \frac{1}{\beta} \ln \Lambda_0(u = -\frac{\beta}{N})$$

$$T_{\mathrm{QTM}}\Psi_0 = \Lambda_0\Psi_0$$

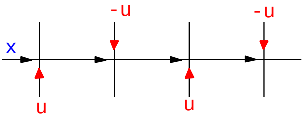
note

- real dimension $L \to \infty$
- T_{QTM} acts on $V^{\otimes N}$
- fine tuning is necessary in Trotter limit ($N \to \infty$)



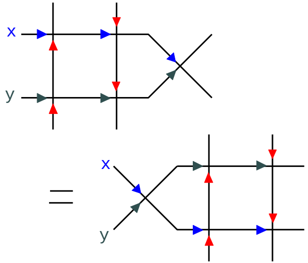
Commuting QTM (Klümper)

Define a commuting family of QTM, $T_{\rm QTM}(u,x)$

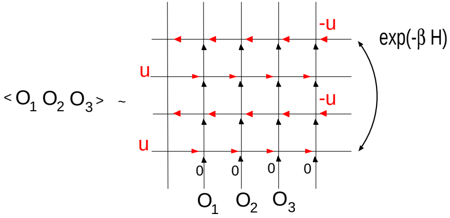


$$[T_{\rm QTM}(u,x), T_{\rm QTM}(u,y)] = 0$$

More precisely, intertwined by same R with $T_{\rm RTR}$



Graphic representation of $\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle$

- 

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \frac{\mathrm{Tr}\, \mathcal{O}_1 \cdots \mathcal{O}_m \, \mathrm{e}^{-\beta \mathcal{H}}}{Z_{1\mathrm{D}}(\beta)}$$

$$= \frac{\langle \Psi_0 | \mathrm{tr}(\mathcal{O}_1 \mathcal{T}_{\mathsf{QTM}}(0)) \cdots \mathrm{tr}(\mathcal{O}_m \mathcal{T}_{\mathsf{QTM}}(0)) | \Psi_0 \rangle}{\Lambda_0^m \langle \Psi_0 | \Psi_0 \rangle}$$

In terms of Density Matrix Elements,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_{\alpha,\beta} D^{\alpha_1 \cdots, \alpha_m}_{\beta_1, \cdots, \beta_m} \mathcal{O}^{\beta_1}_{\alpha_1} \cdots \mathcal{O}^{\beta_m}_{\alpha_m}$$

$$\boxed{\Big(D\Big)^{\alpha_1,\cdots,\alpha_m}_{\beta_1,\cdots,\beta_m}(\xi_1,\cdots,\xi_m) = \frac{\langle \Psi_0 | (\mathcal{T}_{\mathrm{QTM}})^{\alpha_1}_{\beta_1}(\xi_1) \cdots (\mathcal{T}_{\mathrm{QTM}})^{\alpha_m}_{\beta_m}(\xi_m) | \Psi_0 \rangle}{\Lambda_0(\xi_1) \cdots \Lambda_0(\xi_m) \langle \Psi_0 | \Psi_{\rangle}}}$$

example

$$\langle \sigma_1^+ \sigma_2^- \rangle = \frac{\langle \Psi_0 | B(0) C(0) | \Psi_0 \rangle}{\Lambda_0(0)^2 ||\Psi_0||^2}$$

If $T_{\rm QTM} \to T_{\rm RTR}$, the technology in evaluation of DME is well developed.

Kitanine et al, NPB 554 (1999) 647, NPB 567(2000) 55, NPB 641 (2002) 487, NPB 712 (2005) 600, JSTAT (2009) P04003, JMP 50(2009) 095209, JSTAT (2011) P05028

The essential tool is the QISM algebra

$$R(x-y)[\mathcal{T}(x)\otimes\mathcal{T}(y)] = [\mathcal{T}(y)\otimes\mathcal{T}(x)]R(x-y)$$

$$\mathcal{T}(\xi) = \begin{pmatrix} A(\xi) & B(\xi) \\ C(\xi) & D(\xi) \end{pmatrix}$$

Important observation

T_{QTM} shares same R with T_{RTR} thus satisfies same QISM algebra. Thus algebraically finite T problem remains the same with $T = 0$ case.

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There are differences:

- Thermodynamic limit ($M \to \infty$). The system behaves "smoothly". The BAE root distribution is smooth.
- The Trotter limit ($N \to \infty$). The coupling constant u depends on fictitious size N. The BAE root distribution is singular.

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Algebraic structure is same. Analytic properties are different.

Even with singular BAE root distribution, define Auxiliary function

$$\hat{\mathfrak{a}}_n(\hat{\lambda}|\alpha) = \mathrm{e}^{-\frac{h}{T}+2\eta\alpha}\left(\frac{s(\hat{\lambda}-u)s(\hat{\lambda}+u+\eta)}{s(\hat{\lambda}+u)s(\hat{\lambda}+u-\eta)}\right)^N \prod_{k=1}^{M} \frac{s(\hat{\lambda}-\hat{\mu}_k-\eta)}{s(\hat{\lambda}-\hat{\mu}_k+\eta)}$$

$$\hat{\mathfrak{a}}_n(\hat{\lambda}) = \hat{\mathfrak{a}}_n(\hat{\lambda}|0) \qquad \alpha = \text{extra}\, U(1)\, \text{twist}$$

$$u \propto -\frac{1}{TN} \qquad s = \sinh(\text{massive}) \quad \sin(\text{massless})$$

Always possible to write Non Linear Integral Equation $(\lambda = i\hat{\lambda}, \mathfrak{a}_n(\lambda) = \hat{\mathfrak{a}}_n(\hat{\lambda})\ )$

$$\ln \mathfrak{a}_n(\lambda|\alpha) = -\frac{h}{T} + d(\lambda, T, N, \alpha) - \int_{\mathcal{C}_n} \frac{d\mu}{2\pi i} K(\lambda-\mu) \ln(1+\mathfrak{a}_n(\mu|\alpha))$$

Contour $\mathcal{C}_n$ specifies excited states s.t. BAE roots are inside, holes are outside.

Simplest FF(longitudinal)

$$|\langle\Psi_n|\sigma^z|\Psi_0\rangle|^2 \propto \frac{\partial^2}{\partial\alpha^2}A_n(\alpha)|_{\alpha=2\pi} \qquad A_n(\alpha) = \frac{\langle\Psi_0|\Psi_n^\alpha\rangle\langle\Psi_n^\alpha|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle\langle\Psi_n^\alpha|\Psi_n^\alpha\rangle}$$

Slavnov formula $\rightarrow$ a compact formula (Dugave et al, ArXiv:1305.0118)

$$A_n(\alpha) = \mathrm{e}^W \frac{\det_{dm_+^\alpha,\mathcal{C}_n}(1-\mathcal{K}_{-\alpha})\det_{dm_-^\alpha,\mathcal{C}_n}(1-\mathcal{K}_\alpha)}{\det_{dm_0^\alpha,\mathcal{C}_n}(1-\mathcal{K}_0)\det_{dm,\mathcal{C}_n}(1-\mathcal{K}_0)}$$

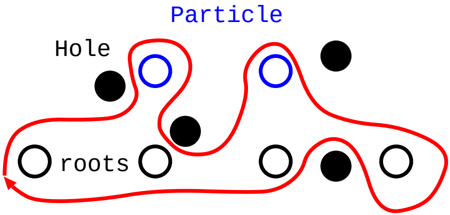
$$W = \int_{\mathcal{C}_n} \frac{d\lambda}{2\pi i}(\ln\frac{\Lambda_n(\lambda|\alpha)}{\Lambda_0})' \ln\frac{1+\mathfrak{a}_n(\lambda|\alpha)}{1+\mathfrak{a}_0(\lambda)}$$

$\det_{dm,\mathcal{C}_n}(1-K)$ is a Fredholm determinant with measure dm

$$\det_{dm,\mathcal{C}_n}(1-K) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Big(\prod_{j=1}^{k} \int_{\mathcal{C}_n} dm(\lambda_j) \Big) \det_{k\times k} K(\lambda_i - \lambda_j)$$

$$dm_+^{\alpha}(\lambda) = \frac{d\lambda \Lambda_n(\lambda|\alpha)}{2\pi i \Lambda_0(\lambda)(1+\mathfrak{a}_n(\lambda|\alpha))} \qquad \text{etc}$$

Contour can be very complicated.



To do (low T analysis)

1. Identify important ecitations and separate contributions from "Fermi sea".
2. Asymptotic analysis on auxiliary functions, eigenvalues and amplitudes.
3. Summation over leading excitations.

Most important is the 1st step.

For $|\Delta| < 1$, sum of particle-hole excitations $\Rightarrow$ Cardy's formula
(Dugave et al, ArXiv 1305.0118, 1401.4132)

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle \sim -\frac{2\mathcal{Z}^2}{\pi^2} \Big(\frac{\frac{\pi T}{v_F}}{\sinh(m\frac{\pi T}{v_F})} \Big)^2 + \text{const.} (\cos 2mk_F) \Big(\frac{\frac{\pi T}{v_F}}{\sinh(m\frac{\pi T}{v_F})} \Big)^{2\mathcal{Z}^2}$$

- At $T > 0$ one is usually only interested in the largest eigenvalue state of T_{QTM}.
- Now we want to know the important excitations
- To know them, we solve the T-Q relation (Nepomechie's talk in this conference).

T=0 BAE roots

M : System Size
N_r: number of BAE roots in the ground state:
H: magnetic field
H_ℓ, H_u: The lower and the upper critical field

$$H_\ell = \frac{2J \sinh \eta}{\pi} k' K(k) \qquad H_u = \frac{J \sinh^2 \eta}{\cosh \eta - 1} \qquad \eta = \frac{\pi K(k')}{K(k)}$$

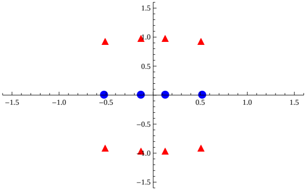
Three phases

$H < H_\ell$	$N_r = \frac{M}{2}$	excitation gap
$H_\ell < H < H_u$	$N_r < \frac{M}{2}$	gapless
$H > H_u$	$N_r = 0$	excitation gap

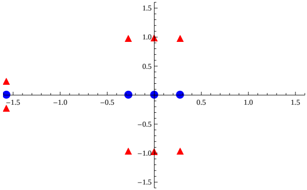
$M = 8, N_r = 4, \alpha = H = 0$, 70 states in total

Blue: BAE roots Red: T zeros

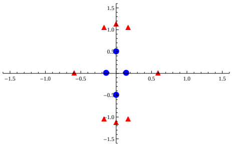
g.s.



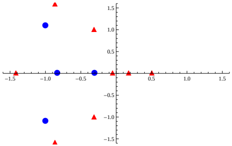
2nd



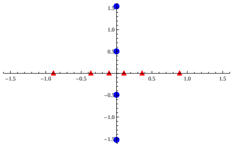
3rd .



43rd



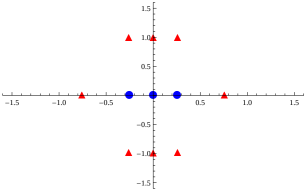
63rd



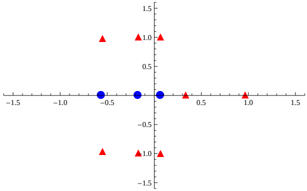
$M = 8, N_r = 3, \alpha = H = 0$, 56 states in total

Blue: BAE roots Red: T zeros

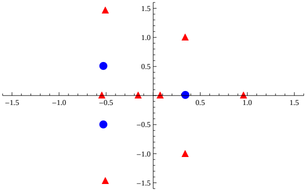
g.s.



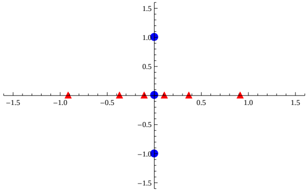
2nd



30th.



49th



$T = 0$ summary

- BAE root locations are independent of h.
- Hole locations are independent of h.
- Zeeman term shifts energy levels .
- the number of BAE roots in the ground state depends on h.

If $H \ll H_\ell$

- the ground state and 1 st excited state almost degenerated
- the 3 rd excitation $=$ 2-string
- higher excitation contains more holes
- most of these excitations $=$ m-strings

$T > 0$ BAE roots : comparison

$T = 0$

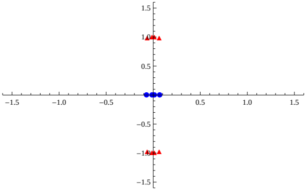
- BAE roots locations are independent of h.
- Hole locations are independent of h.
- Zeeman term shifts energy levels .
- the number of BAE roots in the ground state depends on h.

$T > 0$

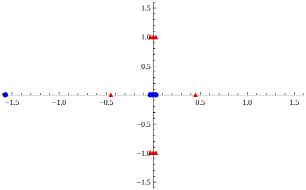
- BAE roots locations depend on h.
- Hole locations depend on h.
- Zeeman term induces nontrivial level crossings .
- the number of BAE roots in the ground state is always $\frac{N}{2}$.

if $H = 0$, even for $T > 0$, BAE roots look similar : more densely distributed
(blue BAE roots, Red T zero)

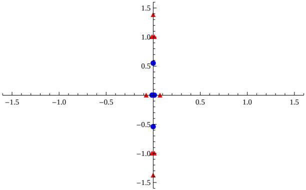
g.s.



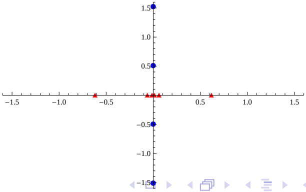
2nd



7th



64th

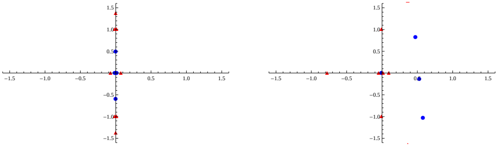


if $H \neq 0$ for $T > 0$, situation changes drastically.

Conjecture

For sufficiently low T, lower excitations do not involve strings. Instead, the distribution of complex rapidities looks like Free Fermion case.

However, for $T \gg 1$, strings do exist.



How these strings disappear with decrease in T?

Mechanism of truncation of longer strings

Reconsider BAE

$$-1 = \mathrm{e}^{-2\frac{H}{T}} \frac{\phi(\xi_k - i(u+1))\phi(\xi_k + iu)q(\xi_k + i)}{\phi(\xi_k + i(u+1))\phi(\xi_k - iu)q(\xi_k - i)}$$

Observation

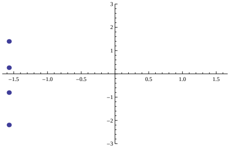
Suppose $T = \frac{H}{m\eta}$ and m roots $\rightarrow \infty$ then the BAE among finite roots $\bar{\xi}$ is identical to BAE with $N_r - m$ roots (magnetic sector) without H.

$$-1 = \frac{\phi(\bar{\xi}_k - i(u+1))\phi(\bar{\xi}_k + iu)\bar{q}(\bar{\xi}_k + i)}{\phi(\xi_k + i(u+1))\phi(\bar{\xi}_k - iu)\bar{q}(\bar{\xi}_k - i)}.$$

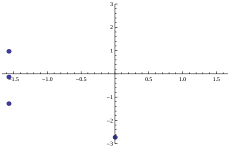
This is one of the possible senarios, but it does happen.

an example of a 4-string $\to$ no string

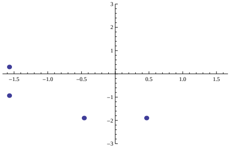
$\eta = 1, H = 5, T = 10$



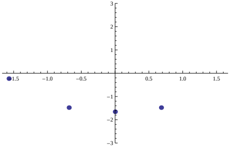
$T = 4$



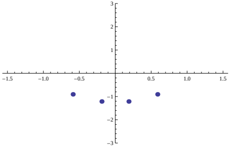
$T = 2$



$T = 1.5$



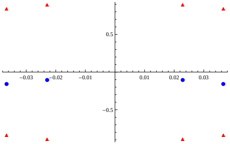
$T = 1.1$



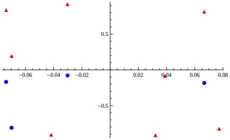
Free Fermion like complex rapidities

$\eta = 1, H = 5, T = 1.1$

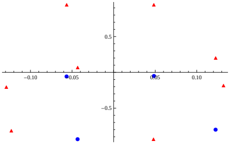
1st



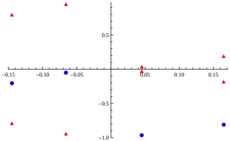
8th



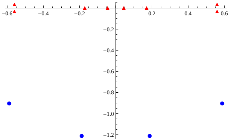
22nd



43th

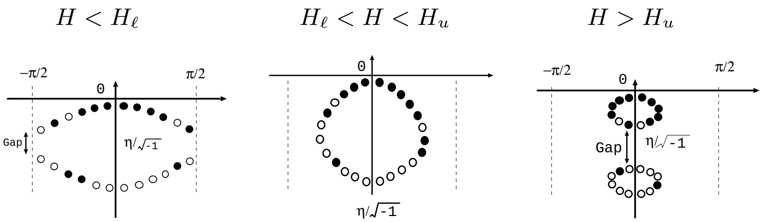


70th



conjecture

For $H \neq 0$, sufficiently low T, roots and holes distribute as one of the following 3 patterns depending in value of $|H|$.



The conjecture is consistent with

- simple 1-body approximation on BAE
- analysis on the higher level Bethe ansatz

conjecture

most important excitations at $H < H_\ell$ are: holes $\{x_a\}$ near $\Im m\lambda \sim 0$ and particles $\{y_a\}$ near $\Im m\lambda \sim -\eta$

$A_n(\alpha)$ is (as always) represented by the product of the discrete and the smooth part.

$$A_n(\alpha) = \mathcal{D}_e[z]\mathcal{A}_e[z]$$

$\mathcal{D}_e[z], \mathcal{A}_e[z]$ are characterized by $\{x_a\}$, $\{y_a\}$, $z(\omega) := \log \frac{1+\mathfrak{a}_n(\omega|\alpha)}{1+\mathfrak{a}_0(\omega)}$

$$\mathcal{D}_e[z] \propto \mathcal{D}(x_a|y_a) \exp\left(\oint_{\mathcal{C}} \frac{d\lambda}{2\pi i} \oint_{\mathcal{C}'} \frac{d\mu}{2\pi i} \frac{z(\lambda)z'(\mu)}{\tan(\lambda-\mu)}\right)$$
$$\times \prod_a \mathrm{e}^{-\mathcal{L}_{\mathcal{C}}[z](x_a)} \frac{(1-\frac{\mathfrak{a}_0}{\mathfrak{a}_n}(x_a))^2}{(\log \mathfrak{a}_n(x_a))'} \prod_a \mathrm{e}^{\mathcal{L}_{\mathcal{C}}[z](y_a)}$$

$$\mathcal{D}(x_a|y_a) = \frac{\prod_{a\neq b}\sin(x_a - x_b)\prod_{a\neq b}\sin(y_a - y_b)}{\prod_{a,b}\sin(x_a - y_b)\sin(y_b - x_a)}$$

$$\mathcal{L}_{\mathcal{C}}[z](\omega) = 2C_{\mathcal{C}}[z](\omega) - C_{\mathcal{C}}[z](\omega + i\eta) - C_{\mathcal{C}}[z](\omega - i\eta)$$

$$C_{\mathcal{C}}[f](\omega) = \oint_{\mathcal{C}} \frac{f(s)}{\tan(\omega - s)}\frac{ds}{2\pi i}$$

Good situation in comparison to massless case:
We perhaps do not have to sum all important contributions.
Now quantitative analysis is possible

summary and outlook

Done

- identify the most important excitations at low T for $H \neq 0$.
- Explicit expressions for longitudinal FF suitable for asymptotic analysis

Massless case $H \to 0$ seems OK.

To do

- Numerical analysis at finite T.
- Comparison (and detect deviation) from results from vertex operator approach
- Finite size case: scaling limit and comparison with bootstrap results.

thank you for your attention