Multipoint correlation functions in critical quantum integrable models

Véronique TERRAS

LPTMS, Univ. Paris Sud

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References:

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- *Form factor approach to dynamical correlation functions in critical models*, N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. Slavnov and V. T., J. Stat. Mech. (2012) P09001, **arXiv:1206.2630**
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Two simple *critical* quantum integrable models

The XXZ spin-1/2 Heisenberg chain

$$H_{\mathrm{XXZ}} = \sum_{m=1}^{L} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\} - h \sum_{m=1}^{L} \sigma_m^z$$

$\sigma_m^{x,y,z}$: local spin-1/2 operators (Pauli matrices) at site m

Δ : anisotropy parameter ($-1 < \Delta < 1$);

h : magnetic field ($0 < h < h_c$)

The Quantum Non-Linear Schrödinger model (1D Bose gas)

$$H_{\mathrm{NLS}} = \int_0^L \left\{ \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger \Psi^\dagger \Psi \Psi - h \Psi^\dagger \Psi \right\} dx$$

$\Psi(x), \Psi^\dagger(x)$: canonical quantum Bose fields $[\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$

coupling constant $c > 0$ (repulsive regime)

chemical potential $h > 0$

- periodic boundary conditions
- in the thermodynamic limit $L \to +\infty$ the spectrum is gapless

Form factor approach to correlation functions

Our goal is to study the **large distance asymptotic behavior** $|x_i - x_j| \to \infty$ of $(T = 0)$ multipoint correlation functions in **critical** integrable models (such as NLS, XXZ...) using their **form factor expansion**:

$$\langle \prod_{j=1}^{r} \mathcal{O}_j(x_j) \rangle = \sum_{|\psi_1\rangle,\ldots,|\psi_{r-1}\rangle} \langle \psi_g | \mathcal{O}_1(x_1) | \psi_1 \rangle \times \langle \psi_1 | \mathcal{O}_2(x_2) | \psi_2 \rangle \ \ldots \langle \psi_{r-1} | \mathcal{O}_r(x_r) | \psi_g \rangle$$

with $\mathcal{O}_k(x)$: local operator at position x

Main difficulty : form factors scale to zero in the large-size limit $L \to \infty$ for critical models:

$$\langle \psi_i | \mathcal{O}(x) | \psi_j \rangle = L^{-\theta_{ij}} \, e^{ix(\mathcal{P}_i - \mathcal{P}_j)} \, \mathcal{A}(\psi_i, \psi_j)$$

$\leadsto$ Analyze the form factor series for large (but finite) system size L.

Hence we need

- to describe states that will contribute to the leading behavior of the series in the limits $|x_i - x_j| \to \infty$ and $L \to \infty$ with $|x_i - x_j| << L$
- to compute the corresponding form factors and their behavior in these limits
- to sum up the corresponding series

Outline of the talk

1 General setting

- $\rightsquigarrow$ the (contributing) spectrum of the model is of particle-hole type with a finite Fermi zone $[-q, q]$
- $\rightsquigarrow$ the form factors admits a large-size behavior which has a particular "discrete" form when particles and holes are in the vicinities of the Fermi boundaries (i.e. the singular part of the f.f. is of Cauchy type)

2 Recall of the summation process for the 2-point case

Involves a particular combinatorial identity

- $\rightsquigarrow$ Large distance asymptotic behavior of static ($T = 0$) spin-spin correlation functions (XXZ chain) $\langle \sigma_1^\alpha \sigma_m^\beta \rangle \underset{m\to\infty}{\sim} ?$
- $\rightsquigarrow$ Long time/Large distance asymptotic behavior of dynamical two-point functions (1D Bose gas) $\langle \mathcal{O}^\dagger(x,t)\, \mathcal{O}(0,0) \rangle \underset{\substack{x,t\to\infty \\ t/x=const}}{\sim} ?$
- $\rightsquigarrow$ Behavior of dynamical response functions near the excitation dispersion curves (1D Bose gas)

3 Generalization to (static) multipoint correlation functions

Involves a multidimensional generalization of the 2-point combinatorial identity

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3. **Generalization to (static) multipoint correlation functions**

   Involves a multidimensional generalization of the 2-point combinatorial identity

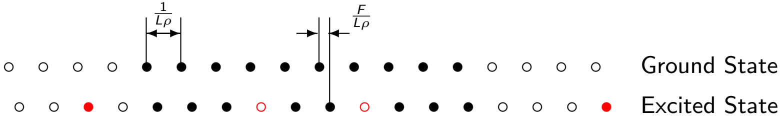
The particle-hole spectrum

Eigenstates parametrized by solutions of the logarithmic Bethe equations:

$$L p_0(\mu_{\ell_j}) + \sum_{k=1}^{N} \theta(\mu_{\ell_j} - \mu_{\ell_k}) = 2\pi\Big(\ell_j - \frac{N+1}{2}\Big), \quad j = 1, \ldots, N \quad (\ell_j \in \mathbb{Z})$$

- **Ground state $|\,\psi_g\,\rangle$:** $N = N_0$, $\ell_j = j$, $j = 1, \ldots, N_0$
  G.S. Bethe roots λ_j are real. In the thermodynamic limit, they densely fill the Fermi zone $[-q, q]$ with a density $\rho(\lambda)$ solution of a linear integral eq.

- **"particle-hole" excitations:** roots $\mu_{\ell_j} \in \mathbb{R}$ corresponding to
  $\ell_j = j$ for $j \in \{1, \ldots, N\} \setminus \{h_1, \ldots, h_n\}$ and $\ell_{h_a} = p_a \notin \{1, \ldots, N\}$
  $\rightsquigarrow$ associated particle rapidities μ_{p_a} and hole rapidities μ_{h_a}



$\hookrightarrow$ Excited state roots μ_j infinitesimally shifted from G.S. roots λ_j, with a shift function $F(\lambda)$ solution of a linear integral eq.

- We don't consider **complex solutions** for XXZ (open problem, can contribute for the dynamical correlation functions)

Large-size behavior of particle-hole form factors

singularities of the form factors contained in Cauchy determinant
(can be extracted from the **determinant representation** of form factors):

$$\langle\psi(\{\mu\})|\,\mathcal{O}(0)\,|\psi(\{\lambda\})\rangle = \det_N \frac{1}{\lambda_a - \mu_b} \times \text{Smooth part}$$

⇝ **Large L behavior of form factors:**

$$|\langle\psi(\{\mu\})|\,\mathcal{O}(0)\,|\psi_g\rangle|^2 \underset{L\to\infty}{\sim} L^{-\theta}\,\mathcal{S}(\{\mu_p\},\{\mu_h\})\,\mathcal{D}(\{p\},\{h\})$$

- $\mathcal{S}$ - **smooth part** (model dependent, explicit expression is rather complicated)
  It depends continuously on the particle/hole rapidities μ_{p_j} and μ_{h_j}
- $\mathcal{D}$ - **discrete part** (universal and rather simple)
  It depends on the set of integers p_a and h_a labeling the particles and holes
  ⇝ in the vicinity of the Fermi boundaries $\pm q$, a microscopic (of order $1/L$) deviation of a particle (or hole) rapidity leads to a macroscopic change in the expression of $\mathcal{D}$
- the **exponent** θ can be written in terms of the shift function

It is **solely** the discrete part $\mathcal{D}$ (together with the values of θ for the various form factors) that drives the asymptotic behavior, while the smooth part $\mathcal{S}$ enters only the corresponding amplitude.

Outline of the talk

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2. **Recall of the summation process for the 2-point case**

   Involves a particular combinatorial identity

   ⇝ Large distance asymptotic behavior of static ($T = 0$) spin-spin correlation functions (XXZ chain) $\langle \sigma_1^\alpha \sigma_m^\beta \rangle \underset{m\to\infty}{\sim} ?$

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3. **Generalization to (static) multipoint correlation functions**

   Involves a multidimensional generalization of the 2-point combinatorial identity

Sum over particle-hole form factors

$$\langle \mathcal{O}^\dagger(x,t)\,\mathcal{O}(0,0)\rangle = \lim_{L\to\infty} \sum_{\substack{\text{particles } p \\ \text{holes } h}} L^{-\theta}\, e^{ix\mathcal{P}_{ex} - it\mathcal{E}_{ex}}\, \mathcal{S}(\{\mu_p\},\{\mu_h\})\, \mathcal{D}(\{p\},\{h\})$$

The phase factor is additive w.r.t. particles and holes:

$$\mathcal{P}_{\text{ex}} - \frac{t}{x}\mathcal{E}_{\text{ex}} \underset{L\to\infty}{\longrightarrow} \sum_{a=1}^{n} [p(\mu_{p_a}) - p(\mu_{h_a})] - \frac{t}{x}[\varepsilon(\mu_{p_a}) - \varepsilon(\mu_{h_a})]$$

p: dressed momentum; ε: dressed energy

- **Equal time correlation functions:** In the large distance limit $x \to \infty$, the oscillatory character of the form factor sum localizes the particle and hole rapidities, in the absence of any other saddle point of the oscillating exponent, around the Fermi boundaries $\pm q$.

  $\hookrightarrow$ (infinite) sum over critical states

- **Time-dependent correlation functions:** asymptotic regime $x, t \to +\infty$ and t/x fixed.

  the oscillating phase $xp(\lambda) - t\varepsilon(\lambda)$ has a unique simple saddle-point λ_0:

  $$xp'(\lambda_0) - t\varepsilon'(\lambda_0) = 0$$

  $\rightsquigarrow$ the particle and hole rapidities localize around the saddle point and around the Fermi boundaries $\pm q$

  $\hookrightarrow$ sum over quasi-critical states

$\hookrightarrow$ these **restricted sums** can be exactly computed thanks to a purely combinatorial **multiple sum identity**

Details of the computation for (static) 2-point functions

For equal time correlation functions, we have to sum over critical form factors corresponding to excited states with particles and holes on the Fermi boundaries $\pm q$:

- **critical excited states of class ℓ** : contain $n_p^{\pm}$ particles, resp. $n_h^{\pm}$ holes, with rapidities equal to $\pm q$ such that

$$n_p^+ - n_h^+ = n_h^- - n_p^- = \ell, \qquad \ell \in \mathbb{Z}.$$

Define $p_j = p_j^+ + N$ if $\mu_{p_j} = q$, $\quad p_j = 1 - p_j^-$ if $\mu_{p_j} = -q$

$h_j = N + 1 - h_j^+$ if $\mu_{h_j} = q$, $\quad h_j = h_j^-$ if $\mu_{h_j} = -q$

- inside a given **class ℓ of critical form factors**:

$\rightsquigarrow$ smooth parts $\mathcal{S}(\{\mu_p\};\{\mu_h\})[F]$ are all the same

$\rightsquigarrow$ critical exponents θ_ℓ are all the same

$\rightsquigarrow$ phase factors $\mathcal{P}_{\text{ex}}$ and finite discrete parts depend on the particular state we consider (they are expressed in terms of particle/hole integers $p_j^{\pm}, h_j^{\pm}$ around the Fermi zone)

$\hookrightarrow$ all critical form factors inside a same class ℓ can be expressed in terms of the simplest form factor of the class (the ℓ-shifted state $|\,\psi_\ell\,\rangle$ with integers $\ell_j = j + \ell$) by just taking in consideration the modification of the discrete part

Details of the computation for (static) 2-point functions

We sum over all classes of **critical form factors**:

$$\langle \mathcal{O}^{\dagger}(x)\,\mathcal{O}(0)\rangle_{cr} = \lim_{L\to\infty} \sum_{\ell=-\infty}^{\infty} L^{-\theta_\ell}\, e^{2ix\ell k_F}\, |\mathcal{F}_\ell|^2\; f_\ell(F_\ell^+, w)\, f_\ell(F_\ell^-, w)\Big|_{w=\exp\left(\frac{2\pi i x}{L}\right)}$$

- $|\mathcal{F}_\ell|^2$ is the special renormalized form factor of class ℓ associated to the ℓ-shifted state $|\,\psi_\ell\,\rangle$ with integers $\ell_j = j+\ell$ (ℓ-**Umklapp excited state**, with ℓ particles and ℓ holes located on the opposite ends of the Fermi zone $\pm q$):

$$|\mathcal{F}_\ell|^2 = \lim_{L\to+\infty}\left\{ L^{\theta_\ell} \left|\langle\, \psi_\ell\,|\mathcal{O}|\,\psi_g\,\rangle\right|^2 \right\}$$

- The sum over integers on the right and left Fermi boundaries factorizes in two decoupled sums $f_\ell(F_\ell^+, w)$ and $f_\ell(F_\ell^-, w)$:

$$f_\ell(\nu, w) \equiv \sum_{\substack{n_p,n_h=0\\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_a\in\mathbb{N}^*}} w^{\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h} h_k} \left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h}$$
$$\times \frac{\prod_{j>k}^{n_p}(p_j-p_k)^2 \prod_{j>k}^{n_h}(h_j-h_k)^2}{\prod_{j=1}^{n_p}\prod_{k=1}^{n_h}(p_j+h_k-1)^2} \prod_{j=1}^{n_p} \frac{\Gamma^2(p_j+\nu)}{\Gamma^2(p_j)} \prod_{k=1}^{n_h} \frac{\Gamma^2(h_k-\nu)}{\Gamma^2(h_k)}$$

Details of the computation for (static) 2-point functions

Main identity

$$f_\ell(\nu, w) = w^{\ell(\ell-1)/2} \frac{G^2(1+\ell+\nu)}{G^2(1+\nu)} (1-w)^{-(\nu+\ell)^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z) G(z)$

and $$f_\ell(\nu, w) \equiv \sum_{\substack{n_p, n_h = 0 \\ n_p - n_h = \ell}}^{\infty} \sum_{\substack{p_1 < \cdots < p_{n_p} \\ p_a \in \mathbb{N}^*}} \sum_{\substack{h_1 < \cdots < h_{n_h} \\ h_a \in \mathbb{N}^*}} w^{\sum_{j=1}^{n_p}(p_j - 1) + \sum_{k=1}^{n_h} h_k} \left(\frac{\sin \pi\nu}{\pi} \right)^{2n_h}$$
$$\times \frac{\prod_{j>k}^{n_p} (p_j - p_k)^2 \prod_{j>k}^{n_h} (h_j - h_k)^2}{\prod_{j=1}^{n_p} \prod_{k=1}^{n_h} (p_j + h_k - 1)^2} \prod_{j=1}^{n_p} \frac{\Gamma^2(p_j + \nu)}{\Gamma^2(p_j)} \prod_{k=1}^{n_h} \frac{\Gamma^2(h_k - \nu)}{\Gamma^2(h_k)}$$

- $\ell = 0$ case Z-measures on partitions (**Kerov-Vershik, Borodin-Olshanski, Okounkov**);
- generalization to $\ell \neq 0$ and alternative proof at $\ell = 0$ (**'11, KKMST**).

Results for the (static) 2-point functions

The **thermodynamic limit** becomes easy to handle leading to the **asymptotic results**:

$$\left\langle \mathcal{O}(x)\,\mathcal{O}^{\dagger}(0)\right\rangle \;=\; \sum_{\ell\in\mathbb{Z}} \frac{\mathrm{e}^{i2x\ell k_F}\cdot|\mathcal{F}_{\ell}|^{2}}{(-ix)^{\Delta_{\ell;+}}\cdot(ix)^{\Delta_{\ell;-}}}(1+o(1))\;.$$

$\rightsquigarrow$ leading asymptotic behavior of each oscillating harmonic

Structure of the asymptotics

- Asymptotics indexed by Umklapp excitations ℓ ;
- the amplitudes $|\mathcal{F}_{\ell}|^{2}$ are model-dependent **but** have a universal interpretation ;
- the critical exponent $\Delta_{\ell;+}=(F_{\ell}^{+}+\ell)^{2}$ and $\Delta_{\ell;-}=(F_{\ell}^{-}+\ell)^{2}$ are given in terms of the values $F_{\ell}^{\pm}$ of the shift function on the left and right Fermi boundaries

Results for the XXZ chain

leading asymptotic terms for the 2-point functions

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{cr} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2\sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2 \, \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \, |\mathcal{F}_\ell^+|^2 \, \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

- $\mathcal{Z} = Z(q)$ where $Z(\lambda)$ is the dressed charge

  $$Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda-\mu)\, Z(\mu) = 1$$

- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1-\langle \sigma^z \rangle}{2} = \frac{k_F}{\pi}$

- $$\left|\mathcal{F}_\ell^z\right|^2 = \lim_{L\to\infty} L^{2\ell^2 \mathcal{Z}^2} \, |\langle \psi_g \, |\sigma_1^z| \psi_\ell \rangle|^2 \,,$$

  $|\,\psi_\ell\,\rangle$ being the ℓ-shifted ground state

- $$|\mathcal{F}_\ell^+|^2 = \lim_{L\to\infty} L^{(2\ell^2 \mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2})} \left|\langle \psi_g \, |\sigma_1^+| \psi'_\ell \rangle\right|^2,$$

  $|\,\psi'_\ell\,\rangle$ being the ℓ-shifted ground state in the N_0+1 sector

The time-dependent case in NLS model

Example: **Density-density function** $\langle \psi_g \,|\, j(x,t)\, j(0,0) \,|\, \psi_g \rangle$, $j = \psi^\dagger \psi$ when $x, t \to +\infty$ (x/t fixed)

Let λ_0 be the (unique) saddle-point of $p(\lambda) - \frac{t}{x}\epsilon(\lambda)$

- **space-like regime** ($|x/t| > v_F$ i.e. $\lambda_0 \notin [-q, q]$):

$$\langle j(x,t)\, j(0,0)\rangle = \left(\frac{p_F}{\pi}\right)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2 + t^2 v_F^2}{\left[x^2 - t^2 v_F^2\right]^2} + \frac{2\cos(2xp_F)\cdot\left|\mathcal{F}_{-q}^{q}\right|^2}{[-i(x - v_F t)]^{\mathcal{Z}^2}[i(x + v_F t)]^{\mathcal{Z}^2}}$$

$$+ \frac{\sqrt{2\pi}\, e^{-i\frac{\pi}{4}} p'(\lambda_0)}{[t\epsilon''(\lambda_0) - x p''(\lambda_0)]^{1/2}} \frac{e^{ix[p(\lambda_0) - p_F] - it\epsilon(\lambda_0)}\cdot\left|\mathcal{F}_{q}^{\lambda_0}\right|^2}{[-i(x - v_F t)]^{[F_q^{\lambda_0}(q) - 1]^2}[i(x + v_F t)]^{F_q^{\lambda_0}(-q)^2}}$$

$$+ \frac{\sqrt{2\pi}\, e^{-i\frac{\pi}{4}} p'(\lambda_0)}{[t\epsilon''(\lambda_0) - x p''(\lambda_0)]^{1/2}} \frac{e^{ix[p(\lambda_0) + p_F] - it\epsilon(\lambda_0)}\cdot\left|\mathcal{F}_{-q}^{\lambda_0}\right|^2}{[-i(x - v_F t)]^{F_{-q}^{\lambda_0}(q)^2}[i(x + v_F t)]^{[F_{-q}^{\lambda_0}(-q) + 1]^2}} + \dots$$

  - $p_F = p(q)$: Fermi momentum; $v_F = \frac{\epsilon'(q)}{p'(q)}$: Fermi velocity
  - $\mathcal{Z} = Z(q)$ where $Z(\lambda)$ is the dressed charge (solution of an integral equation)
  - $F_{\mu_h}^{\mu_p}(\lambda)$ (resp. $\mathcal{F}_{\mu_h}^{\mu_p}$): shift function (resp. properly normalized form factor of density) between the ground state and an excited state with one particle at μ_p and one hole at μ_h

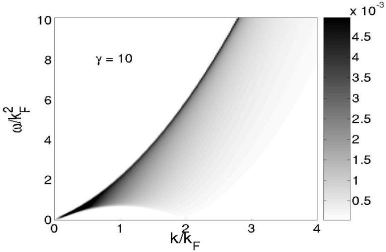
- **time-like regime** ($|x/t| < v_F$ i.e. $\lambda_0 \in]-q, q[$): Similar type of formula

Singularities of dynamical response functions in NLS model

Example: **Dynamical structure factor**

$$S(k,\omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\ e^{i(\omega t - kx)} \langle\ j(x,t)\, j(0,0)\ \rangle$$

$\rightsquigarrow$ probability to excite the ground state with momentum and energy transfer (k,ω), can be experimentally measured.



Numerical computations (Calabrese, Caux 06)

DSF exhibit power-law singularities along the one-particle (upper) and one-hole (lower) dispersion curves

Edge exponents and amplitudes can be computed for both thresholds from the analytic study of the form factor series:

$$S(k,\omega) = \sum_{|\,\psi_j\,\rangle} \delta(\omega - \mathcal{E}_{ex})\ \delta(k - \mathcal{P}_{ex})\ |\langle\psi_j|\, j(0,0)\,|\psi_g\rangle|^2$$

Example: DSF around the hole threshold

Let $k_h = p_F - p(\lambda)$ (resp. $\epsilon_h = -\epsilon(\lambda)$) be the momentum (resp. energy) of the excitation corresponding a particle at q and a hole at $\lambda \in]-q, q[$

We are interested in the $\delta\omega \to 0$ behavior of Dynamical Structure Factor when
$k = k_h$ and $\omega = \epsilon_h + \delta\omega$

- ⇝ one can restrict the sum over form factors to excited states with one hole in a vicinity of λ + one particle at q
  + an arbitrary number of additional particle-hole excitations with rapidities accumulating on the Fermi boundaries $\pm q$ (with zero total momentum and energy)
- ⇝ same kind of summation identity as in the previous case

$$S(k,\omega)_{\text{hole}} = H(\delta\omega)\,\frac{\left|\mathcal{F}_\lambda^q\right|^2}{\Gamma(\alpha_+ + \alpha_-)\,(v - v_F)^{\alpha_+}\,(v + v_F)^{\alpha_-}}\left(\frac{\delta\omega}{2\pi}\right)^{\alpha_+ + \alpha_- - 1}$$

$v = \frac{\epsilon'(\lambda)}{p'(\lambda)}$: sound velocity; $v_F = \frac{\epsilon'(q)}{p'(q)}$: Fermi velocity

$\mathcal{F}_\lambda^q$: properly normalized form factor of density between the ground state and an excited state with one particle at q and one hole at λ

the exponents $\alpha_\pm$ are given in terms of the corresponding shift functions

⇝ confirms predictions obtains from the non-linear Luttinger liquid approach (Imambekov & Glazman 08)

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   Involves a particular combinatorial identity

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   - ⇝ Long time/Large distance asymptotic behavior of dynamical two-point functions (1D Bose gas) $\langle \mathcal{O}^\dagger(x,t)\, \mathcal{O}(0,0) \rangle \underset{\substack{x,t\to\infty \\ t/x=const}}{\sim} ?$
   - ⇝ Behavior of dynamical response functions near the excitation dispersion curves (1D Bose gas)

3. **Generalization to (static) multipoint correlation functions**

   Involves a multidimensional generalization of the 2-point combinatorial identity

Multipoint correlation functions

We consider the r-point function

$$C\left(\mathbf{x}_r;\mathbf{o}_r\right) = \langle\, \Psi_g\, | \mathcal{O}_1(x_1)\ldots\mathcal{O}_r(x_r) |\, \Psi_g\,\rangle$$

where local operators $\mathcal{O}_a(x)$ connect states with N and $N+o_a$ quasi-particles

$\rightsquigarrow$ large-distance asymptotic behavior in the regime $1 << |x_l - x_k|$ $(l \neq k)$ and $x_k << L$ $(k = 1,\ldots,r)$?

Form factor expansion:

multiple sum over intermediate **normalized** states $|\,\Psi\big(\mathcal{I}_n^{(s)}\big)\,\rangle$, $s = 1,\ldots r-1$, labelled by sets of integers $\mathcal{I}_n^{(s)} = \big\{\{p_a^{(s)}\}_1^n\,;\,\{h_a^{(s)}\}_1^n\big\}$ corresponding to particles and holes excitations :

$$C\left(\mathbf{x}_r;\mathbf{o}_r\right) = \prod_{s=1}^{r-1}\Big\{\sum_{\{\mathcal{I}_{n^{(s)}}^{(s)}\}}\Big\}\prod_{s=1}^{r-1}\Big\{e^{i(x_{s+1}-x_s)\,\Delta\mathcal{P}(\mathcal{I}_{n^{(s)}}^{(s)})}\Big\}$$
$$\times\prod_{s=1}^{r}\langle\,\Psi\big(\mathcal{I}_m^{(s-1)}\big)\,|\mathcal{O}_s(0)|\,\Psi\big(\mathcal{I}_n^{(s)}\big)\,\rangle$$

Large-L expansion of form factors connecting critical states

$$\langle \Psi(\mathcal{I}_m^{(s-1)}) \,|\mathcal{O}_s(0)|\, \Psi(\mathcal{I}_n^{(s)}) \rangle \underset{L\to\infty}{\sim} L^{-\rho_s(\nu_s^+)-\rho_s(\nu_s^-)} \, \mathcal{F}_{\mathcal{O}_s}(\ell_{s-1},\ell_s) \cdot C^{(\ell_{s-1};\ell_s)}(\nu_s^+,\nu_s^-)$$
$$\times f^{(+)}\Big[\mathcal{J}^{(s-1)}_{m_{p;+};m_{h;+}}\,;\,\mathcal{J}^{(s)}_{n_{p;+};n_{h;+}} \mid \nu_s^+\Big] \cdot f^{(-)}\Big[\mathcal{J}^{(s-1)}_{m_{p;-};m_{h;-}}\,;\,\mathcal{J}^{(s)}_{n_{p;-};n_{h;-}} \mid \nu_s^-\Big] \,.$$

- the quantity $\mathcal{F}_{\mathcal{O}_s}(\ell_{s-1},\ell_s)$ represents the properly normalized form factor of the operator $\mathcal{O}_s$ taken between fundamental representatives of the ℓ_s and ℓ_{s-1} critical classes:

$$\mathcal{F}_{\mathcal{O}_s}(\ell_{s-1},\ell_s) = \lim_{L\to+\infty}\Big\{ L^{\rho_s(\nu_s^+)+\rho_s(\nu_s^-)} \langle \Psi_{\ell_{s-1}} \,|\mathcal{O}_s(0)|\, \Psi_{\ell_s} \rangle \Big\}$$

- $C^{(\ell_{s-1};\ell_s)}$ is a normalization constant (written in terms of Barnes G-function)
- $\nu_s^+ = \nu_s(q) - o_s$ and $\nu_s^- = \nu_s(-q)$ are given in terms of the values that the relative shift function $\nu_s(\lambda) = F_{s-1}(\lambda) - F_s(\lambda)$ between the ℓ_s, ℓ_{s-1} critical states takes on the right/left endpoints of the Fermi zone

- $f^{(\pm)}\left[\mathcal{J}^{(s-1)}_{m_{p;\pm};m_{h;\pm}};\mathcal{J}^{(s)}_{n_{p;\pm};n_{h;\pm}}\mid\nu_s^{\pm}\right]$ correspond to the contributions of the excitations on the right/left Fermi boundary of the model (discrete part)

  ⇝ depend on the sets of integers $\mathcal{J}^{(s-1)}_{m_{p;\pm};m_{h;\pm}}$ and $\mathcal{J}^{(s)}_{n_{p;\pm};n_{h;\pm}}$ parametrizing the excitations on the right/left boundary for the $s-1$ and s excited states.

$$f^{(+)}\left[\mathcal{J}^{(s-1)}_{n_p;n_h};\mathcal{J}^{(s)}_{n_k;n_t}\mid\nu\right]=(-1)^{n_t}\left(\frac{\sin[\pi\nu]}{\pi}\right)^{n_t+n_h}\varpi\left(\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t}\mid\nu\right)$$
$$\times\frac{\prod_{a<b}^{n_p}(p_a-p_b)\prod_{a<b}^{n_h}(h_a-h_b)}{\prod_{a=1}^{n_p}\prod_{b=1}^{n_h}(p_a+h_b-1)}\cdot\frac{\prod_{a<b}^{n_k}(k_a-k_b)\prod_{a<b}^{n_t}(t_a-t_b)}{\prod_{a=1}^{n_k}\prod_{b=1}^{n_t}(k_a+t_b-1)}$$
$$\times\Gamma\left(\begin{matrix}\{p_a+\nu\} & \{h_a-\nu\} & \{k_a-\nu\} & \{t_a+\nu\}\\ \{p_a\} & \{h_a\} & \{k_a\} & \{t_a\}\end{matrix}\right),$$

with

$$\varpi\left(\mathcal{J}_{n_p;n_h};\mathcal{J}_{n_k;n_t}\mid\nu\right)=\prod_{a=1}^{n_h}\left\{\frac{\prod_{b=1}^{n_k}(1-k_b-h_a+\nu)}{\prod_{b=1}^{n_t}(t_b-h_a+\nu)}\right\}\prod_{a=1}^{n_p}\left\{\frac{\prod_{b=1}^{n_t}(p_a+t_b+\nu-1)}{\prod_{b=1}^{n_k}(p_a-k_b+\nu)}\right\}$$

⇝ This ϖ term couples the right and left states particles and holes integers (not present if one of them is the ground state)

⇝ coupling of previous combinatorial sums!

Summation of the large-L form factor series and asymptotic behavior of multipoint correlation functions

⇝ we have to sum up multiple sums of the previous type (obtained for 2-point functions) however highly coupled between themselves by the factors ϖ

It is still possible to do it !
The corresponding identity follows from the identification of two possible representations for the large-size asymptotic behavior of a particular Toeplitz determinant with Fisher-Hartwig singularities

⇝ Taking the thermodynamic limit we arrive at the following r-point correlation function asymptotic behavior :

$$C(\mathbf{x}_r;\mathbf{o}_r) = \sum_{\substack{\kappa_r \in \mathbb{Z}^r \\ \sum \kappa_a = 0}} \prod_{s=1}^{r} \left\{ \mathrm{e}^{2ik_F \kappa_s x_s} \right\} \cdot \prod_{s=1}^{r} \mathcal{F}_{\mathcal{O}_s}(\ell_{s-1},\ell_s)$$

$$\times \prod_{b>a}^{r} \left\{ \left[i(x_b - x_a) \right]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)} \cdot \left[-i(x_b - x_a) \right]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)} \right\} .$$

with $\theta_b^{\pm}(\kappa_b) = \nu_b^{\pm} + \kappa_b$ and $\kappa_s = \ell_{s-1} - \ell_s$.

Four-point function. XXZ chain

Consider a four point function:

$$C_{xxxx} \;=\; \langle\, \Psi_g \,| \sigma^x_{m_1} \;\sigma^x_{m_2} \;\sigma^x_{m_3} \;\sigma^x_{m_4} |\, \Psi_g \,\rangle.$$

The leading term confirms the CFT prediction:

$$C_{xxxx} \;=\; 2\left|\mathcal{F}_0^{+}\right|^4 \cdot \left\{ \left| \frac{(m_2-m_1)\cdot(m_4-m_3)}{(m_3-m_1)\cdot(m_4-m_1)\cdot(m_3-m_2)\cdot(m_4-m_2)} \right|^{\frac{1}{2\mathcal{Z}^2}} \right.$$

$$\left. +\;(2 \leftrightarrow 3) \;+\; (2 \leftrightarrow 4) \right\} \;+\; \dots\,.$$

Conclusion and perspectives

Results

- explicit leading **asymptotic behavior** of static 2-point functions (XXZ), of dynamical 2-point functions (NLS), of static n-point functions + **singularities** of dynamical response functions (NLS)

  $\rightsquigarrow$ reproduces all the predictions (for XXZ and Lieb-Liniger models) from the CFT, Luttinger liquid approach, non-linear Luttinger liquid approach + goes further (time-dependent case, correlation amplitudes. . .)

- The method relies on **simple hypothesis** ($\rightarrow$ easy generalization to other models):
  - Finite Fermi zone + particle-hole spectrum
  - Singularities of the form factors contained in Cauchy determinant (kinematic factor, quite general)

Open problems

- Contribution from the bound states ($\rightarrow$ time-dependent case for XXZ) ?
- Explicit expressions for the amplitudes and limit $h = 0$?
- multipoint time-dependent functions ?