APPLICATIONS OF ALGEBRAIC BETHE ANSATZ MATRIX ELEMENTS TO SPIN CHAINS

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SPIN-1/2 XXZ CHAIN

$$H_{\text{XXZ}} = J\sum_{j=1}^{N}\left[S^x_jS^x_{j+1} + S^y_jS^y_{j+1} + \Delta\left(S^z_jS^z_{j+1} - \frac{1}{4}\right)\right]$$

SPIN-1 BABUJAN-TAKHTAJAN CHAIN

$$H_{\text{BT}} = \frac{J}{4}\sum_{j=1}^{N}\left[\hat{\boldsymbol{S}}_j\cdot\hat{\boldsymbol{S}}_{j+1} - (\hat{\boldsymbol{S}}_j\cdot\hat{\boldsymbol{S}}_{j+1})^2\right]$$

ABSTRACT

Algebraic Bethe ansatz based techniques at finite size afford a way into computing observables for integrable models. Dealing with Bethe roots explicitly as deviated string-solutions is necessary. Dynamical correlations of the Babujan-Tahktajan spin-1 chain are obtained by a higher spin generalisation of this method. The obtained real-space spin-spin correlation displays asymptotics fitting predictions from conformal field theory. Moreover, prepared out-of-equilibrium initial states for the anisotropic spin chain can be addressed as well. For the Néel state in particular as an initial state, contributions from coinciding deviated strings at zero are challenging to implement.

MAGNONS, SPINONS AND STRINGS

- **String hypothesis**
  - Bethe roots organise themselves in strings of length n

$$\lambda^{n,a}_j = \lambda^n_j + \frac{i}{2}(n+1-2a)$$

- **Magnons**
  - Excitations on ferromagnetic GS
  - Real rapidities $\cdots\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\cdots$
  - Bound states of n magnons built from string solutions
  - Strings of length n $\cdots\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\cdots$
- **Spinons**
  - Excitations on antiferromagnetic GS
  - Propagating domain walls

$$\cdots\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$$
$$\cdots\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$$

BETHE STATES

The exact eigenstates of the integrable spin-s chains are constructed via the Bethe ansatz,

$$|\{\lambda\}\rangle = \sum_j\sum_Q A_Q(\{\lambda\})\prod_{a=1}^{M} e^{ij_a p(\lambda_{Q_j})}S^-_{j_a}|\Uparrow\Uparrow\ldots\Uparrow\rangle$$

where the set of M complex rapidities obeying Bethe equations completely determines the state. The logarithmic Bethe equations for the integrable spin-s chain are given by

$$\theta_{2s}(\lambda_j) - \frac{1}{N}\sum_{k\neq j}^{M}\theta_2(\lambda_j-\lambda_k) = \frac{2\pi}{N}J_j$$

where the Bethe quantum numbers J are integers for N+M odd and half-odd integers for N+M even.

For the isotropic spin-s case, the kernels are given by

$$\theta_n(\lambda) = 2\text{atan}\left(\frac{2\lambda}{n}\right)$$

MATRIX ELEMENTS

The matrix elements of operators are given by the normalised determinant expressions obtained from algebraic Bethe ansatz [1,2],

$$\langle\psi_m|S^z_j|\psi_n\rangle = \frac{F^z_j(\{\mu\}_{\psi_m},\{\lambda\}_{\psi_n})}{\sqrt{\mathcal{N}(\{\mu\}_{\psi_m})\mathcal{N}(\{\lambda\}_{\psi_n})}}$$

$$F^z_j(\{\lambda\},\{\mu\}) = \frac{\varphi_j(\{\lambda\})}{\varphi_j(\{\mu\})}\frac{s_j\det H - \sum_{p=1}^{l}\prod_{k=1}^{l}(\lambda_k-\lambda_p+\eta)\det\mathcal{Z}}{\prod_{j<j}(\mu_i-\mu_j)(\lambda_j-\lambda_i)}$$

GROUND STATE AND SPINONS

- **Spin-1 Babujan-Takhtajan chain**
  - Antiferromagnetic ground state built up from a sea of two-strings (bound states of magnons).
  - Excitations (spinons) constructed by breaking up two-strings into a real rapidity (two-spinon excitations) or into three-strings (four-spinon excitations)

HANDLING BETHE ROOTS EXPLICITLY

- Start with Bethe state defined from string quantum numbers
- Solve Bethe equations by iteration
  - Parameterised in terms of string deviations
- Rapidities of deviated string configurations enter in matrix element expressions directly
- **Problem:** Determinants and prefactors diverge for small string deviations, so high numerical precision is required
  - **Solution:** Use reduced form of determinants whenever small string deviations are encountered

DEVIATED STRINGS

- Two-strings

$$\lambda^{(2),\pm}_j = \lambda^{(2)}_j \pm \frac{i}{2}\left(1+2\delta^{(2)}_j\right)$$

- Three-strings

$$\lambda^{(3),\pm}_j = \lambda^{(3)}_j + \epsilon^{(3)}_j \pm i\left(1+\delta^{(3)}_j\right)$$
$$\lambda^{(3),0}_j = \lambda^{(3)}_j$$

- Insert deviated strings and add/subtract Bethe equations in order to obtain a parameterised set of Bethe equations for string deviations, which can be solved by iteration. [3]

APPLICATIONS

TIME EVOLUTION

Explicit unitary time evolution by using Bethe state matrix elements of operators

$$\langle\mathcal{O}(t)\rangle = \sum_{n,m}e^{-i(E_n-E_m)t}c_nc^*_m\langle\psi_m|\mathcal{O}|\psi_n\rangle$$

INITIALLY PREPARED STATES

- Construction of initial state in Bethe state basis $|\Psi\rangle = \sum_n c_n|\psi_n(\{\lambda\})\rangle$
  - Wave-packets of (bound) magnons from Bethe momenta

$$c_n = e^{-ik_n\bar{j}-\frac{1}{4}\alpha(k_n-\bar{k})^2}$$

  - Real space tracking of (bound) magnons

QUENCHES

- Néel to XXX/XXZ quench, overlap formula known [4]

$$c_n = \langle\Psi_{\text{Néel}}|\psi_n(\{\lambda\})\rangle$$

- Only non-zero overlap with parity invariant states
- Contributions from exotic states
  - Multiple deviated strings centered at zero pushing each other apart

DYNAMICAL STRUCTURE FACTOR

The dynamical structure factor is directly measured in inelastic neutron scattering experiments, and is defined as the Fourier Transform of the connected spin-spin correlation function,

$$S^{a\bar{a}}_{\text{DSF}}(q,\omega) = \frac{1}{N}\sum_{j,j'}^{N}e^{iq(j-j')}\int_{-\infty}^{\infty}\text{d}t\; e^{i\omega t}\langle S^a_j(t)S^{\bar{a}}_{j'}(0)\rangle_c$$
$$= 2\pi\sum_\alpha|\langle\text{GS}|S^a_q|\alpha\rangle|^2\delta(\omega-\omega_\alpha)$$

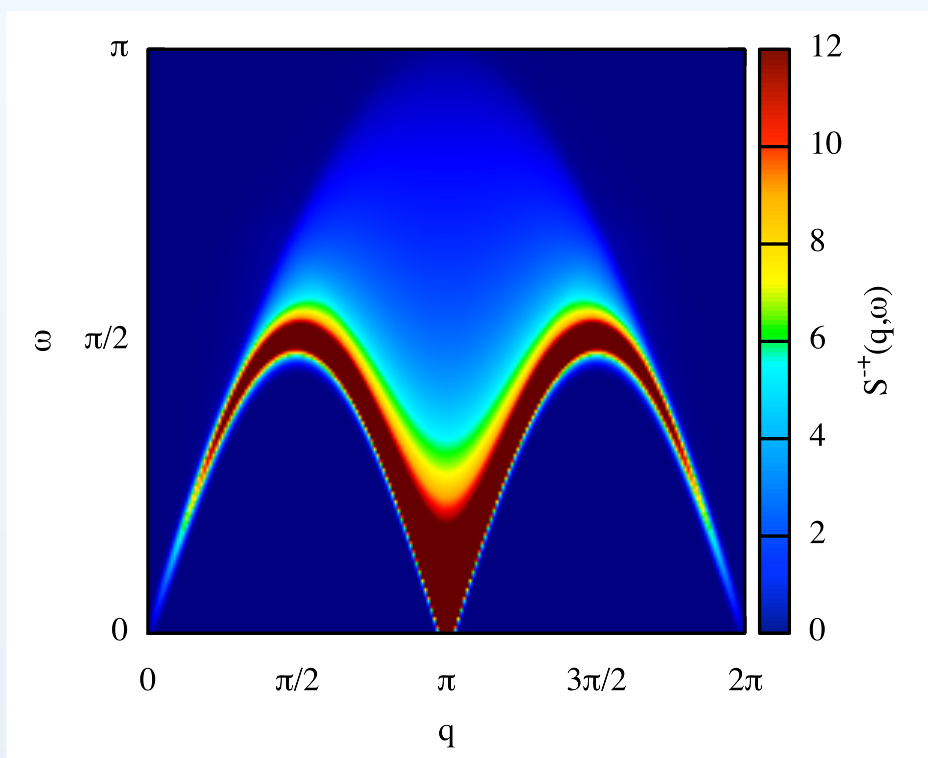


Figure Transverse dynamical structure factor of the Babujan-Takhtajan spin-1 chain at N=200, including two-spinon and four-spinon contributions, with a sum rule saturation of 99.16%. [5]

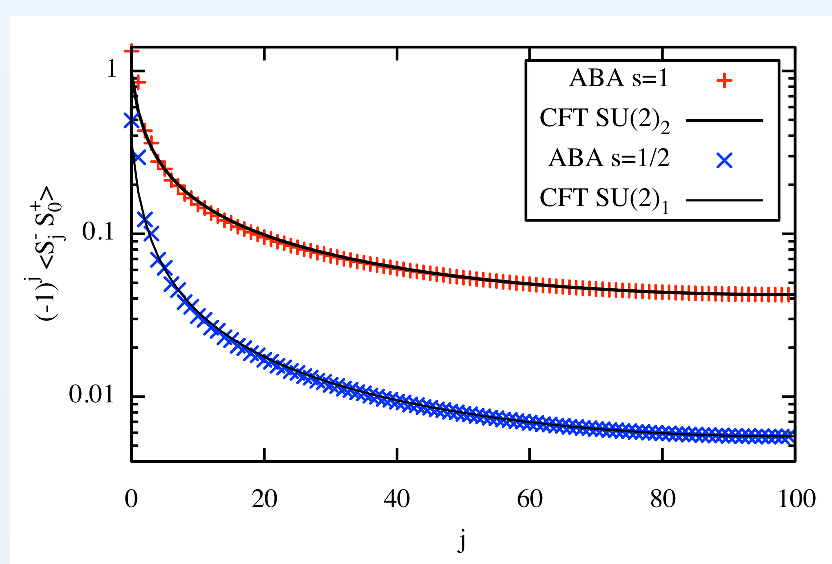
REAL SPACE CORRELATIONS & CONFORMAL FIELD THEORY

We compare the ABA results on the (inverse Fourier transformed) dynamical structure factor to the predictions on the asymptotics of the spin-spin correlator from the continuum limit described by conformal field theory.

The critical low-energy sector of an integrable spin-s chain is described by the SU(2) level-2s Wess-Zumino-Novikov-Witten models. The dominant antiferromagnetic correlations decay as a power law,

$$\langle S^a_jS^{\bar{a}}_0\rangle \sim \frac{(-1)^j}{|j|^{2\Delta}},$$

Figure Equal time spin-spin correlation at N=200 with sum rule saturation 99.16% for spin-1, compared to the prediction from the SU(2) level-2 WZW-model [5]. The same comparison is shown for spin-1/2 data from ABACUS at N=200 (sum rule saturation 99.24%) and the SU(2) level-1 WZW-model.



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