The inhomogeneous T-Q relation and its applications

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Outline

I. Introduction

II. The inhomogeneous T-Q relation

III.The XYZ model

IV.The antiperiodic XXZ chain

  1. The ODBA solution

  2. Retrieve the Bethe state

V. Concluding remarks & perspective

I. Introduction

Two classes of integrable models

With U(1) symmetry

↑

Periodic boundary
Parallel boundary
.....

↑

Coordinate BA,
Baxter's T-Q,
Algebraic BA

Without U(1) symmetry

↑

XYZ spin chain (odd N)
Antiperiodic boundary
Non-diagonal boundary fields
......

↑

SoV, q-Onsager etc
Off-diagonal Bethe Ansatz

II. The inhomogeneous T-Q relation

Absence of reference state !

The eigenvalue of the transfer matrix is a degree N polynomial!

$$\Lambda(u) = c\prod_{j=1}^{N} f(u - z_j)$$

N+1 equations or values at N+1 points $\Lambda(\theta_j)$ **determine it completely!**

II. The inhomogeneous T-Q relation

CYSW: PRL 111, Nucl. Phys. B 875&879 (2013)

Two key invariants of the monodromy matrix:

(1) Trace $t(u)$ **(2)Quantum determinant** $\Delta_q(u)$

Intrinsic relationship between them!

QH edge
TI surface
Open string

$$t(\theta_j)t(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta)\sim\Delta_q(\theta_j),\quad j=1,\cdots,N.$$

C(u) encodes the boundary

$$\Lambda(u)=e^{i\phi(u)}a(u)\frac{Q(u-\eta)Q_1(u-\eta)}{Q(u)Q_2(u)}+e^{-i\phi(u+\eta)}d(u)\frac{Q(u+\eta)Q_2(u+\eta)}{Q(u)Q_1(u)}+c(u)\frac{a(u)d(u)}{\bar{Q}(u)Q_1(u)Q_2(u)};$$

$$a(\theta_j-\eta)=d(\theta_j)=0$$

Regularity → **BAE**

II. The inhomogeneous T-Q relation

Why?

$$d(\theta_j) = a(\theta_j - \eta) = 0$$

$$\Lambda(\theta_j) = e^{i\phi(\theta_j)} a(\theta_j) \frac{Q(\theta_j - \eta) Q_1(\theta_j - \eta)}{Q(\theta_j) Q_2(\theta_j)}$$

$$\Lambda(\theta_j - \eta) = e^{-i\phi(\theta_j)} d(\theta_j - \eta) \frac{Q(\theta_j) Q_2(\theta_j)}{Q(\theta_j - \eta) Q_1(\theta_j - \eta)}$$

$$\Lambda(\theta_j)\Lambda(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \cdots, N$$

Frahm 09
Niccoli 11
Reshtikhin 83

For arbitrary Q's and C(u)

A minimal T-Q is enough to give the complete spectrum because we need only N Bethe roots to parameterize the degree N polynomial

III. The XYZ model

Baxter 71
Faddeev & Takhatajian 79

$$H = \frac{1}{2}\sum_{n=1}^{N}(J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z)$$

$$J_x = e^{i\pi\eta}\frac{\sigma(\eta+\frac{\tau}{2})}{\sigma(\frac{\tau}{2})}, \quad J_y = e^{i\pi\eta}\frac{\sigma(\eta+\frac{1+\tau}{2})}{\sigma(\frac{1+\tau}{2})}, \quad J_z = \frac{\sigma(\eta+\frac{1}{2})}{\sigma(\frac{1}{2})},$$

$$\theta\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}(u,\tau) = \sum_{m=-\infty}^{\infty}\exp\left\{i\pi\left[(m+a_1)^2\tau + 2(m+a_1)(u+a_2)\right]\right\}$$

$$\sigma(u) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,\tau), \quad \zeta(u) = \frac{\partial}{\partial u}\{\ln\sigma(u)\}.$$

III. The XYZ model

Initial condition : $R_{1,2}(0) = P_{1,2}$,
Unitarity relation : $R_{1,2}(u)R_{2,1}(-u) = -\xi(u) \times \mathrm{id}$,

$$\xi(u) = \frac{\sigma(u-\eta)\sigma(u+\eta)}{\sigma(\eta)\sigma(\eta)},$$

Crossing relation : $R_{1,2}(u) = V_1 R^{t_2}_{1,2}(-u-\eta)V_1, \quad V = -i\sigma^y$

Initial →

$$t(\theta_j) = R_{j\,j-1}(\theta_j - \theta_{j-1}) \ldots R_{j\,1}(\theta_j - \theta_1) \times R_{j\,N}(\theta_j - \theta_N) \ldots R_{j\,j+1}(\theta_j - \theta_{j+1}).$$

Crossing →

$$t(\theta_j - \eta) = (-1)^N R_{j\,j+1}(-\theta_j + \theta_{j+1}) \ldots R_{j\,N}(-\theta_j + \theta_N) \times R_{j\,1}(-\theta_j + \theta_1) \ldots R_{j\,j-1}(-\theta_j + \theta_{j-1}).$$

Unitary →

$$t(\theta_j)t(\theta_j - \eta) = a(\theta_j)d(\theta_j - \eta) \sim \Delta_q(\theta_j), \quad j = 1, \cdots, N.$$

$$a(u) = \prod_{l=1}^{N} \frac{\sigma(u - \theta_l + \eta)}{\sigma(\eta)}, \quad d(u) = a(u - \eta) = \prod_{l=1}^{N} \frac{\sigma(u - \theta_l)}{\sigma(\eta)}$$

III. The XYZ model: Functional relations

Analyticity →

$\Lambda(u)$ is an entire function of u.

Periodicity →

$$\Lambda(u+1) = (-1)^N \Lambda(u),$$

$$\Lambda(u+\tau) = (-1)^N e^{-2\pi i \{Nu + N(\frac{\eta+\tau}{2}) - \sum_{j=1}^{N} \theta_j\}} \Lambda(u)$$

Identities →

$$\Lambda(\theta_j)\Lambda(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta),\ j = 1, \ldots, N,$$

$$\prod_{j=1}^{N} \Lambda(\theta_j) = \prod_{j=1}^{N} a(\theta_j).$$

III. The XYZ model: Inhomogeneous T-Q

$$\begin{aligned}\Lambda(u) &= e^{2i\pi l_1 u+i\phi}a(u)\frac{Q_1(u-\eta)Q(u-\eta)}{Q_2(u)Q(u)} + e^{-2i\pi l_1(u+\eta)-i\phi}d(u)\frac{Q_2(u+\eta)Q(u+\eta)}{Q_1(u)Q(u)} \\ &\quad +c\,\frac{\sigma^m(u+\frac{\eta}{2})a(u)d(u)}{\sigma^m(\eta)Q_1(u)Q_2(u)Q(u)},\end{aligned}$$

$$\begin{aligned}Q_1(u) &= \prod_{j=1}^{M}\frac{\sigma(u-\mu_j)}{\sigma(\eta)},\quad Q_2(u)=\prod_{j=1}^{M}\frac{\sigma(u-\nu_j)}{\sigma(\eta)}, \\ Q(u) &= \prod_{j=1}^{M_1}\frac{\sigma(u-\lambda_j)}{\sigma(\eta)}.\end{aligned}$$

$$N+m=2M+M_1.$$

The minimal T-Q ← m=0

III. The XYZ model: Bethe ansatz equations

$$(\frac{N}{2} - M - M_1)\eta - \sum_{j=1}^{M}(\mu_j - \nu_j) = l_1\tau + m_1, \quad l_1, m_1 \in Z,$$

$$\frac{N}{2}\eta + \sum_{j=1}^{M}(\mu_j + \nu_j) + \sum_{j=1}^{M_1}\lambda_j = m_2, \quad m_2 \in Z,$$

$$\frac{c\, e^{2i\pi(l_1\mu_j + l_1\eta) + i\phi}\sigma^N(\mu_j + \eta)}{\sigma^N(\eta)} = -Q_2(\mu_j)Q_2(\mu_j + \eta)Q(\mu_j + \eta), \quad j = 1, \ldots, M,$$

$$\frac{c\, e^{-2i\pi l_1\nu_j - i\phi}\sigma^N(\nu_j)}{\sigma^N(\eta)} = -Q_1(\nu_j)Q_1(\nu_j - \eta)Q(\nu_j - \eta), \quad j = 1, \ldots, M,$$

$$\frac{e^{2i\pi l_1(2\lambda_j + \eta) + 2i\phi}\sigma^N(\lambda_j + \eta)}{\sigma^N(\lambda_j)} + \frac{Q_2(\lambda_j)Q_2(\lambda_j + \eta)Q(\lambda_j + \eta)}{Q_1(\lambda_j)Q_1(\lambda_j - \eta)Q(\lambda_j - \eta)}$$
$$= \frac{-c\, e^{2i\pi l_1(\lambda_j + \eta) + i\phi}\sigma^N(\lambda_j + \eta)}{Q_1(\lambda_j)Q_1(\lambda_j - \eta)Q(\lambda_j - \eta)\sigma^N(\eta)}, \quad j = 1, \ldots, M_1,$$

BAE

$$\Lambda(0) = e^{i\phi}\left\{\prod_{j=1}^{M}\frac{\sigma(\mu_j + \eta)}{\sigma(\nu_j)}\right\}\left\{\prod_{j=1}^{M_1}\frac{\sigma(\lambda_j + \eta)}{\sigma(\lambda_j)}\right\} = e^{\frac{2i\pi k}{N}}, \quad k = 1, \ldots, N.$$

Selection rule

$$E = \frac{\sigma(\eta)}{\sigma'(0)}\left\{\sum_{j=1}^{M}[\zeta(\nu_j) - \zeta(\mu_j + \eta)] + \sum_{j=1}^{M_1}[\zeta(\lambda_j) - \zeta(\lambda_j + \eta)] + \frac{1}{2}N\zeta(\eta) + 2i\pi l_1\right\}$$

III. The XYZ model: Even N case

For even N and generic coupling constants, we have c=0 solutions.

$$l_1 = 0, \quad N = 2M, \quad \{\mu_j\} = \{\nu_j\} \equiv \{\lambda_j\}$$

$$\Lambda(u) = e^{i\phi}\frac{\sigma^N(u+\eta)}{\sigma^N(\eta)}\frac{Q(u-\eta)}{Q(u)} + e^{-i\phi}\frac{\sigma^N(u)}{\sigma^N(\eta)}\frac{Q(u+\eta)}{Q(u)}$$

$$Q(u) = \prod_{l=1}^{M}\frac{\sigma(u-\lambda_l)}{\sigma(\eta)}.$$

Baxter's solution Faddeev & Takhtajian

$$\frac{\sigma^N(\lambda_j+\eta)}{\sigma^N(\lambda_j)} = -e^{-2i\phi}\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}, \quad j = 1, \cdots, M,$$

$$e^{i\phi}\prod_{j=1}^{M}\frac{\sigma(\lambda_j+\eta)}{\sigma(\lambda_j)} = e^{\frac{2i\pi k}{N}}, \quad k = 1, \cdots, N.$$

III. The XYZ model: Even N case

Table 3.1 Numerical solutions of the BAEs (3.2.52)-(3.2.53) for $N = 4$, $\eta = 0.4$, $\tau = i$. The eigenvalues E_n calculated from (3.2.54) are exactly the same to those from the exact diagonalization of the Hamiltonian. n denotes the number of the energy levels. (depicted from [17])

λ_1	λ_2	k	E_n	n
$0.80000+0.11349i$	$0.80000+0.88651i$	2	-3.21353	1
$0.80000+0.00000i$	$0.80000+0.50000i$	1	-2.34227	2
$0.80000+0.00000i$	$0.30000+0.50000i$	1	-1.71217	3
$0.30000+0.00000i$	$0.80000+0.00000i$	0	-0.61387	4
$0.30000+0.70000i$	$0.80000+0.80000i$	3	0.00000	5
$0.30000+0.30000i$	$0.80000+0.20000i$	1	0.00000	5
$0.30000+0.86676i$	$0.80000+0.13324i$	2	0.00000	5
$0.30000+0.13324i$	$0.80000+0.86676i$	2	0.00000	5
$0.62340+0.25000i$	$0.97660+0.25000i$	1	0.00000	5
$0.62340+0.75000i$	$0.97660+0.75000i$	3	0.00000	5
0.6	1.0	0	0.00000	5
$0.03367+0.50000i$	$0.56633+0.50000i$	2	0.58230	6
$0.30000+0.50000i$	$0.80000+0.50000i$	2	0.61387	7
$0.30000+0.00000i$	$0.80000+0.50000i$	1	1.71217	8
$0.30000+0.00000i$	$0.30000+0.50000i$	1	2.34227	9
$0.30000+0.16022i$	$0.30000+0.83978i$	2	2.63122	10

III. The XYZ model: Odd N case

For generic coupling constants, no c=0 solution

Table 3.2 Numerical solutions of the BAEs (3.2.64)-(3.2.69) for $N=3$, $\eta=0.20$, $\tau=i$, $l_1=m_1=m_2=0$. The eigenvalues E_n calculated from (3.2.70) are exactly the same to those from the exact diagonalization of the Hamiltonian. n denotes the number of the energy levels. (depicted from [17])

μ_1	ν_1	λ_1	c	ϕ	k	E_n	n
$0.35000+0.02632i$	$0.45000+0.02632i$	$-1.10000-0.05263i$	$-0.08948+0.00000i$	$-0.08501-0.00000i$	1	-1.40865	1
$0.35000-0.02632i$	$0.45000-0.02632i$	$-1.10000+0.05263i$	$-0.08948+0.00000i$	$0.08501-0.00000i$	2	-1.40865	1
$-0.15000+0.08693i$	$-0.05000+0.08693i$	$-0.10000-0.17387i$	$3.04065+0.00000i$	$4.10893-0.00000i$	2	-1.40865	1
$-0.15000-0.08693i$	$-0.05000-0.08693i$	$-0.10000+0.17387i$	$3.04065-0.00000i$	$-4.10893-0.00000i$	1	-1.40865	1
$-0.65000-0.27875i$	$-0.55000-0.27875i$	$0.90000+0.55749i$	$-0.28951-0.00000i$	$0.35925-0.00000i$	0	1.18468	2
$-0.28066+0.31196i$	$-0.18066+0.31196i$	$0.16133-0.62392i$	$-0.61188+0.36729i$	$-0.27657+0.04967i$	0	1.18468	2
$0.15828+0.12139i$	$0.25828+0.12139i$	$-0.71655-0.24279i$	$-0.09303-0.16695i$	$-0.29190+0.31832i$	0	1.63263	3
$-0.42198+0.50000i$	$-0.32198+0.50000i$	$0.44397-1.00000i$	$3.33371-7.57925i$	$-0.94248-0.14392i$	0	1.63263	3

IV. The antiperiodic XXZ model

Cao et. al, PRL 111, 137201(2013)

$$H=-\sum_{j=1}^{N}\left[\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cosh\eta\,\sigma_j^z\sigma_{j+1}^z\right]$$

$$\sigma_{N+1}^{\alpha}=\sigma_1^x\sigma_1^{\alpha}\sigma_1^x$$

Monodromy matrix

$$T_0(u)=\sigma_0^x R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1)=\begin{pmatrix} C(u) & D(u)\\ A(u) & B(u)\end{pmatrix}$$

Transfer matrix

$$t(u)=tr_0 T_0(u)=B(u)+C(u)$$

$$t(\theta_j)t(\theta_j-\eta)=-a(\theta_j)d(\theta_j-\eta),\quad j=1,\cdots,N,$$

$$d(u)=a(u-\eta)=\prod_{j=1}^{N}\frac{\sinh(u-\theta_j)}{\sinh\eta}$$

IV. The antiperiodic XXZ model: T-Q (I)

Functional relation

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta)=-a(\theta_j)d(\theta_j-\eta),\quad j=1,\cdots,N.$$

Periodicity

$$\Lambda(u+i\pi)=(-1)^{N-1}\Lambda(u)$$

$$\Lambda(u)=e^{u}a(u)\frac{Q_1(u-\eta)}{Q_2(u)}-e^{-u-\eta}d(u)\frac{Q_2(u+\eta)}{Q_1(u)}-c(u)\frac{a(u)d(u)}{Q_1(u)Q_2(u)}$$

Degree N-1 trigonometric polynomial

$$Q_1(u)=\prod_{j=1}^{M}\sinh(u-\mu_j),\quad Q_2(u)=\prod_{j=1}^{M}\sinh(u-\nu_j)$$

$$c(u)=\sinh^N\eta\,[e^{i\phi_1+u}-e^{i\phi_2-u-\eta}]$$

Even N, M=N/2

$$c(u)=\frac{1}{2}\sinh^N\eta\,\left[e^{i\phi_1+2u}+e^{i\phi_2-2u-2\eta}\right]$$

Odd N, M=(N+1)/2

IV. The antiperiodic XXZ model: T-Q (II)

$$\Lambda(u) = a(u)e^{u}\frac{Q(u-\eta)}{Q(u)} - e^{-u-\eta}d(u)\frac{Q(u+\eta)}{Q(u)} - c(u)\frac{a(u)d(u)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{N}\sinh(u-\lambda_j)$$

$$c(u) = \sinh^N\eta\left[e^{u-N\eta+\sum_{j=1}^{N}(\theta_j-\lambda_j)} - e^{-u-\eta-\sum_{j=1}^{N}(\theta_j-\lambda_j)}\right].$$

Table 4.3 Numerical solutions of the BAEs (4.4.18) for $N=3$, $\eta=\ln 2$. E_n is the n-th eigenenergy and n indicates the number of the energy levels. The eigenvalues calculated from (4.4.19) are exactly the same to those given in Table 4.1.

λ_1	λ_2	λ_3	E_n	n
-0.97964	-0.34657	0.28649	-3.02200	1
$-1.11092+0.50000i\pi$	$-0.34657+0.50000i\pi$	$0.41778+0.50000i\pi$	-3.02200	1
$-0.72948+0.43156i\pi$	$-0.34657+0.21409i\pi$	$0.03633+0.43156i\pi$	-1.25000	2
$-0.72948-0.43156i\pi$	$-0.34657-0.21409i\pi$	$0.03633-0.43156i\pi$	-1.25000	2
$-0.72959-0.09417i\pi$	$-0.34657-0.45213i\pi$	$0.03644-0.09417i\pi$	-1.25000	2
$-0.72959+0.09417i\pi$	$-0.34657+0.45213i\pi$	$0.03644+0.09417i\pi$	-1.25000	2
$-0.34657-0.50000i\pi$	$-0.34657-0.10517i\pi$	$-0.34657+0.10517i\pi$	5.52200	3
$-0.71841-0.50000i\pi$	$-0.34657-0.00000i\pi$	$0.02526+0.50000i\pi$	5.52200	3

IV. The antiperiodic XXZ model: T-Q (II)

Table 4.4 Numerical solutions of the BAEs (4.4.18) for $N = 4$, $\eta = 1$. E_n is the n-th eigenenergy and n indicates the number of the energy levels. The eigenvalues calculated from (4.4.19) are exactly the same to those given in Table 4.2.

λ_1	λ_2	λ_3	λ_4	E_n	n
$-2.12754+0.50000i\pi$	$-1.03389+0.50000i\pi$	$0.03389+0.50000i\pi$	$1.12754+0.50000i\pi$	-4.27591	1
-1.83685	-1.00000	0.00000	0.83685	-4.27591	1
$-1.74449+0.15319i\pi$	$-0.98935+0.15137i\pi$	$-0.01065+0.15137i\pi$	$0.74449+0.15319i\pi$	-3.67755	2
$-1.74449-0.15319i\pi$	$-0.98935-0.15137i\pi$	$-0.01065-0.15137i\pi$	$0.74449-0.15319i\pi$	-3.67755	2
$-2.03215+0.42071i\pi$	$-0.96024+0.36347i\pi$	$-0.03976+0.36347i\pi$	$1.03215+0.42071i\pi$	-3.67755	2
$-2.03215-0.42071i\pi$	$-0.96024-0.36347i\pi$	$-0.03976-0.36347i\pi$	$1.03215-0.42071i\pi$	-3.67755	2
$-1.86358+0.31377i\pi$	$-1.02758+0.24134i\pi$	$0.02758+0.24134i\pi$	$0.86358+0.31377i\pi$	-3.46526	3
$-1.86358-0.31377i\pi$	$-1.02758-0.24134i\pi$	$0.02758-0.24134i\pi$	$0.86358-0.31377i\pi$	-3.46526	3
$-1.31784-0.50000i\pi$	-1.00000	0.00000	$0.31784+0.50000i\pi$	0.27591	4
$-1.60669-0.50000i\pi$	$-0.50000-0.25533i\pi$	$-0.50000+0.25533i\pi$	$0.60669+0.50000i\pi$	0.27591	4
$-1.03760-0.12735i\pi$	$-0.50000-0.41963i\pi$	$-0.50000+0.15711i\pi$	$0.03760-0.12735i\pi$	3.67755	5
$-1.03760+0.12735i\pi$	$-0.50000+0.41963i\pi$	$-0.50000-0.15711i\pi$	$0.03760+0.12735i\pi$	3.67755	5
$-1.81891-0.48351i\pi$	$-0.50000-0.47875i\pi$	$-0.50000-0.06781i\pi$	$0.81891-0.48351i\pi$	3.67755	5
$-1.81891+0.48351i\pi$	$-0.50000+0.47875i\pi$	$-0.50000+0.06781i\pi$	$0.81891+0.48351i\pi$	3.67755	5
$-1.53412-0.41713i\pi$	$-0.50000-0.21809i\pi$	$-0.50000+0.02400i\pi$	$0.53412-0.41713i\pi$	7.46526	6
$-1.53412+0.41713i\pi$	$-0.50000+0.21809i\pi$	$-0.50000-0.02400i\pi$	$0.53412+0.41713i\pi$	7.46526	6

IV. The antiperiodic XXZ model: Bethe states

Scalar product

$$F_n(\{\mu_j\}) = \langle\Psi|\prod_{j=1}^{n} B(\mu_j)|0\rangle$$

Recursive relation

$$\Lambda(u)F_n = \sum_{l} M_n^l(u)F_{n-1}^l + \sum_{k>l} \tilde{M}_n^{kl}(u)F_{n-1}^{kl} + F_{n+1},$$
$$F_1(u) = \Lambda(u),$$
$$F_{N+1} \equiv 0.$$

$$F_n(\theta_1,\cdots,\theta_n) = \prod_{j=1}^{n} \Lambda(\theta_j)$$

In principle, all the off-shell scalar product can be derived from the inhomogeneous T-Q relation

IV. The antiperiodic XXZ model: Bethe states

A convenient basis

$$\langle\theta_{p_1},\cdots,\theta_{p_n}| = \langle 0|\prod_{j=1}^{n} C(\theta_{p_j}),$$

$$|\theta_{q_1},\cdots,\theta_{q_n}\rangle = \prod_{j=1}^{n} B(\theta_{q_j})|0\rangle,$$

$$D(u)|\theta_{p_1},\cdots,\theta_{p_n}\rangle = d(u)\prod_{j=1}^{n}\frac{\sinh(u-\theta_{p_j}+\eta)}{\sinh(u-\theta_{p_j})}|\theta_{p_1},\cdots,\theta_{p_n}\rangle,$$

$$\langle\theta_{p_1},\cdots,\theta_{p_n}|D(u) = d(u)\prod_{j=1}^{n}\frac{\sinh(u-\theta_{p_j}+\eta)}{\sinh(u-\theta_{p_j})}\langle\theta_{p_1},\cdots,\theta_{p_n}|.$$

$$q_j, p_j \in (1,\cdots,N),\ p_1 < p_2 < \cdots < p_n \text{ and } q_1 < q_2 < \cdots < q_n.$$

$$\langle\theta_{p_1},\cdots,\theta_{p_n}|\theta_{q_1},\cdots,\theta_{q_m}\rangle = f_n(\theta_{p_1},\cdots,\theta_{p_n})\delta_{m,n}\prod_{j=1}^{n}\delta_{p_j,q_j},$$

$$f_n(\theta_{p_1},\cdots,\theta_{p_n}) = \prod_{j=1}^{n} a(\theta_{p_j})d_{p_j}(\theta_{p_j})\prod_{k\neq l}^{n}\frac{\sinh(\theta_{p_k}-\theta_{p_l}+\eta)}{\sinh(\theta_{p_k}-\theta_{p_l})}$$

Orthogonal and complete basis

Niccoli 12
CYSW13

IV. The antiperiodic XXZ model: Bethe states

Expansion of the eigenvector

$$\langle\Psi| = \sum_{n=0}^{N}\sum_{p}\chi_n(\theta_{p_1},\cdots,\theta_{p_n})\langle\theta_{p_1},\cdots,\theta_{p_n}|.$$

$$\chi_n(\theta_{p_1},\cdots,\theta_{p_n}) = \frac{\prod_{j=1}^{n}\Lambda(\theta_{p_j})}{f_n(\theta_{p_1},\cdots,\theta_{p_n})}$$

$$|\Psi\rangle = \sum_{n=0}^{N}\sum_{p}\chi_n(\theta_{p_1},\cdots,\theta_{p_n})|\theta_{p_1},\cdots,\theta_{p_n}\rangle.$$

⇐ **SoV state by Niccoli**

The Bethe state for T-Q(II)

Reference state ⇓

$$|\lambda_1,\cdots,\lambda_N\rangle = \prod_{j=1}^{N}\frac{D(\lambda_j)}{d(\lambda_j)}|\omega\rangle$$

$$|\omega\rangle = \sum_{n=0}^{N}\sum_{p} f_n^{-1}(\theta_{p_1},\cdots,\theta_{p_n})e^{\sum_{l=1}^{n}\theta_{p_l}}\prod_{l=1}^{n}a(\theta_{p_l})|\theta_{p_1},\cdots,\theta_{p_n}\rangle$$

$$\langle\theta_{p_1},\cdots,\theta_{p_n}|\lambda_1,\cdots,\lambda_N\rangle = \langle\theta_{p_1},\cdots,\theta_{p_n}|\Psi\rangle = F_n(\theta_{p_1},\cdots,\theta_{p_n})$$

⇐ **Why?**

IV. The antiperiodic XXZ model: Scalar product

Final solution of the recursive relation

$$\langle\theta_1,\cdots,\theta_n|\prod_{k=1}^{m}B(u_k)|0\rangle=\delta_{n,m}\,g_n(\{\theta_j\}|\{u_\alpha\}),\quad g_0=\langle 0|0\rangle=1.$$

$$g_n(\{\theta_j\}|\{u_\alpha\})=\frac{\prod_{j=1}^{n}\prod_{\alpha=1}^{n}\sin(u_\alpha-\theta_j+\eta)\,\det\mathcal{N}(\{u_\alpha\};\{\theta_j\})}{\prod_{j>k}\sinh(\theta_k-\theta_j)\prod_{\alpha>\beta}\sinh(u_\alpha-u_\beta)},$$

$$\mathcal{N}(\{u_\alpha\};\{\theta_j\})_{\alpha,j}=\frac{\sinh\eta\;d(u_\alpha)\,a(\theta_j)}{\sinh(u_\alpha-\theta_j+\eta)\,\sinh(u_\alpha-\theta_j)}.$$

$$F_n(\{\mu_j\})=\sum_{1\le p_1<p_2\cdots<p_n\le N}\left\{\prod_{l=1}^{n}\Lambda(\theta_{p_l})\right\}\frac{g_n(\{\theta_{p_l}\}|\{\mu_j\})}{g_n(\{\theta_{p_l}\}|\{\theta_{p_l}\})}$$

V. Concluding Remarks & Perspective

Yang-Baxter Equation & Reflection Equation

↓

Operator product identities

$$t(\theta_j)t(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta)\sim\Delta_q(\theta_j),\quad j=1,\cdots,N.$$

+ Asymptotic behavior of the polynomial

↓

Polynomialization

$$\Lambda(u)=a(u)\frac{Q_1(u-\eta)}{Q_2(u)}+d(u)\frac{Q_2(u+\eta)}{Q_1(u)}+c(u)\frac{a(u)d(u)}{Q_1(u)Q_2(u)}$$

Regularity → **Bethe ansatz equations**

V. Concluding Remarks & Perspective

- Spin torus: [Phys. Rev. Lett. 111, 137201 (2013)]
- Open XXX: [Nucl. Phys. B 875, 152 (2013)]
- Open XXZ & XYZ: [Nucl. Phys. B 877, 152 (2013)]
- Periodic XYZ: [Nucl. Phys. B 886, 185 (2014)]
- Hubbard: [Nucl. Phys. B 879, 98 (2014)]
- t-J: [JSTAT P04031, (2014)]
- SU(n) & nested ODBA [JHEP 04, 143 (2014)]
- Thermodynamics: [Nucl. Phys. B 884, 17 (2014)]
- Izergin-Korepin: [JHEP 06, 128 (2014)]
- Spin-s Heisenberg: [arXiv:1405.2692]
- Retrieve the eigenstate [arXiv:1407.xxxx]

V. Concluding Remarks & Perspective

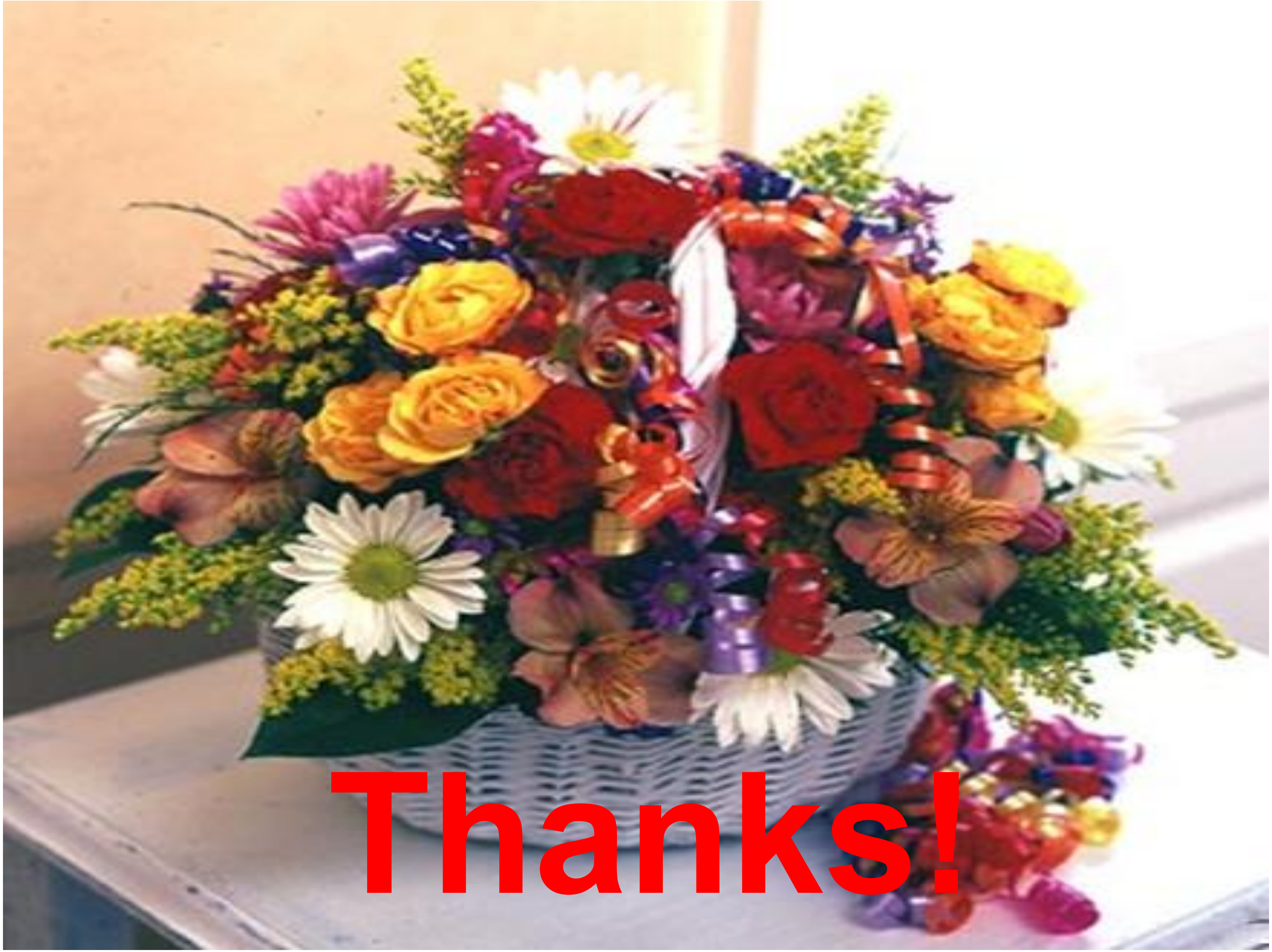
High rank systems: An, Bn, Cn, Dn

$$t^{(1)}(\theta_j)t^{(r)}(\theta_j - \eta) \sim t^{(r+1)}(\theta_j), \quad r = 1, \cdots, N-1.$$

Nested T-Q

The method works for all boundary conditions!

Interesting open problem：How to retrieve the Bethe states?


Thanks!