Off-diagonal Bethe ansatz for the Izergin-Korepin model with non-diagonal boundary terms

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- Off-diagonal Bethe ansatz for the the Izergin-Korepin model
  - Periodic boundary condition.
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The Hamiltonian of the closed XXZ chain is

$$H = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\,\sigma_{k+1}^x + \sigma_k^y\,\sigma_{k+1}^y + \cosh\eta\,\sigma_k^z\,\sigma_{k+1}^z\right),$$

where

$$\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar\frac{\partial}{\partial t}h_i = [H, h_i] = 0, \qquad i = 1, \ldots.$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i.$$

Then

$$[t(u),\, t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0,\,\theta_j=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u - \theta_N) \ldots R_{01}(u - \theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \frac{1}{\sinh \eta} \begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & \sin \eta & \\ & \sinh \eta & \sin u & \\ & & & \sinh(u+\eta) \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

Besides the quantum Yang-Baxter equation,

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v) = R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

the R-matrix also satisfies

$$\text{Initial condition}: \; R_{12}(0) = P_{12}, \tag{2}$$

$$\text{Unitarity relation}: \; R_{12}(u)R_{21}(-u) = \rho_1(u) \times \mathrm{id}, \tag{3}$$

$$\text{Crosing-unitarity relation}: \; R_{12}^{t_1}(u)R_{21}^{t_1}(-u-2\eta) = \rho_2(u) \times \mathrm{id}, \tag{4}$$

$$Z_2\text{-symmetry}: \quad \sigma_1^i\sigma_2^i R_{1,2}(u) = R_{1,2}(u)\sigma_1^i\sigma_2^i, \quad \text{for } i = x, y, z, \tag{5}$$

$$\text{(Anti)Symmetry}: \; R_{12}(-\eta) = -(1-P) = -2P^{(-)}. \tag{6}$$

where $\rho_1(u) = -\frac{\sinh(u+\eta)\sinh(u-\eta)}{\sinh\eta\sinh\eta}, \quad \rho_2(u) = -\frac{\sinh(u+2\eta)\sinh(u)}{\sinh\eta\sinh\eta}$

The QYBE leads to the following so-called RLL relation between the monodromy matrix

$$R_{12}(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R_{12}(u-v).$$

This leads to $[t(u), t(v)] = 0$. Moreover, (2) and (6) implies

$$T_1(\theta_j)T_2(\theta_j-\eta) = P_{12}^{(-)}T_1(\theta_j)T_2(\theta_j-\eta), \quad j = 1, \ldots, N. \tag{7}$$

This gives rise to the very operator identities

$$\begin{aligned} t(\theta_j)t(\theta_j-\eta) &= tr_{12}\left\{T_1(\theta_j)T_2(\theta_j-\eta)\right\} \\ &= tr_{12}\left\{P_{12}^{(-)}T_1(\theta_j)T_2(\theta_j-\eta)\right\} \\ &= \mathrm{Det}_q(T(\theta_j)), \quad j = 1, \ldots, N, \end{aligned} \tag{8}$$

where $\mathrm{Det}_q(T(u)) = tr\{P_{12}^{(-)}T_1(u)T_2(u-\eta)P_{12}^{(-)}\} = \Delta_q(u) \times \mathrm{id}$.

$\Rightarrow$ the T-Q ansatz $\Leftarrow$ Conventional BA.

Let a constant matrix g satisfies

$$R_{12}(u)\, g_1\, g_2 = g_2\, g_1\, R_{12}(u),$$

and introduce the twisted transfer matrix

$$t(u) = tr\{g\, T(u)\}.$$

It is easy to show that $[t(u), t(v)] = 0$. Moreover, it also satisfies the very operator identities

$$\begin{aligned} t(\theta_j)t(\theta_j - \eta) &= tr_{12}\left\{g_1 T_1(\theta_j) g_2 T_2(\theta_j - \eta)\right\} \\ &= tr_{12}\left\{g_1 g_2 P_{12}^{(-)} T_1(\theta_j) T_2(\theta_j - \eta)\right\} \\ &= \mathrm{Det}_q(T(\theta_j)) \times \det(g), \quad j = 1, \ldots, N. \end{aligned} \tag{9}$$

$\Rightarrow$ the inhomogeneous T-Q ansatz. [J. Cao et al, `arXiv:1305.7328`].

For open chains, one need to introduce two K-matrices $K^{\pm}(u)$. The K-matrix $K^-(u)$ satisfies the reflection equation (RE),

$$R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)K_2^-(u_2)$$
$$=K_2^-(u_2)R_{12}(u_1+u_2)K_1^-(u_1)R_{21}(u_1-u_2), \quad (10)$$

and $K^+(u)$ satisfies the dual RE

$$R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-2)K_2^+(u_2)$$
$$=K_2^+(u_2)R_{12}(-u_1-u_2-2)K_1^+(u_1)R_{21}(u_2-u_1). \quad (11)$$

Let us introduce double-row monodromy matrix

$$\mathcal{T}_1(u)=T_1(u)K^-(u)\widehat{T}_1(u),$$
$$\widehat{T}_0(u)=R_{10}(u+\theta_1)\ldots R_{N0}(u+\theta_N).$$

The properties (2), (6) and QYBE (1), RE (10) imply that

$$\mathcal{T}_{12}(\pm\theta_j) = \mathcal{T}_1(\pm\theta_j)R_{21}(\pm 2\theta_j-\eta)\mathcal{T}_2(\pm\theta_j-\eta) = P^{(-)}_{12}\mathcal{T}_{12}(\pm\theta_j), \quad (12)$$

$$K^+_{12}(u) = K^+_2(u-\eta)R_{12}(-2u-\eta)K^+_1(u). \quad (13)$$

Then

$$\begin{aligned}
t(\pm\theta_j)t(\pm\theta_j-\eta) &= \rho_2^{-1}(\pm 2\theta_j-\eta)\, tr_{12}\left\{K^+_{12}(\pm\theta_j)\,\mathcal{T}_{12}(\pm\theta_j)\right\} \\
&\overset{(12)}{=} \rho_2^{-1}(\pm 2\theta_j-\eta)\, tr_{12}\left\{K^+_{12}(\pm\theta_j)\,P^{(-)}_{12}\mathcal{T}_{12}(\pm\theta_j)\right\} \\
&\overset{(11)}{=} \rho_2^{-1}(\pm 2\theta_j-\eta)\, tr_{12}\left\{P^{(-)}_{12}K^+_{12}(\pm\theta_j)\,P^{(-)}_{12}\mathcal{T}_{12}(\pm\theta_j)\right\} \\
&= \rho_2^{-1}(\pm 2\theta_j-\eta)\, tr_{12}\left\{P^{(-)}_{12}K^+_{12}(\pm\theta_j)\,P^{(-)}_{12}\mathcal{T}_{12}(\pm\theta_j)P^{(-)}_{12}\right\} \\
&= \rho_2^{-1}(\pm 2\theta_j-\eta)\,\mathrm{Det}_q(K^+(\pm\theta_j))\,\mathrm{Det}_q(\mathcal{T}(\pm\theta_j)) \\
&= \rho_2^{-1}(\pm 2\theta_j-\eta)\,\Delta^{(o)}(\pm 2\theta_j)\times \mathrm{id}.
\end{aligned}$$

The most K-matrix $K^-(u)$ is given by [de Vega et al **1993** & Goshal et al **1994**]

$$K^-(u) = \begin{pmatrix} K^-_{11}(u) & K^-_{12}(u) \\ K^-_{21}(u) & K^-_{22}(u) \end{pmatrix},$$
$$K^-_{11}(u) = 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) + \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right),$$
$$K^-_{22}(u) = 2\left(\sinh(\alpha_-)\cosh(\beta_-)\cosh(u) - \cosh(\alpha_-)\sinh(\beta_-)\sinh(u)\right),$$
$$K^-_{12}(u) = e^{\theta_-}\sinh(2u), \quad K^-_{21}(u) = e^{-\theta_-}\sinh(2u).$$

The most general solution to the DRE is

$$K^+(u) = K^-(-u-\eta)\big|_{(\alpha_-,\beta_-,\theta_-)\to(-\alpha_+,-\beta_+,\theta_+)}.$$

In addition to reflection equations, the K-matric satisfies

$$K^-(0) = \frac{1}{2}tr(K^-(0)) \times \mathrm{id}, \quad K^-(\frac{i\pi}{2}) = \frac{1}{2}tr(K^-(\frac{i\pi}{2})) \times \sigma^z.$$

These properties and the quasi-periodic properties of R-matrix and K-matrices imply

$$
\begin{aligned}
t(-u-\eta) &= t(u), \quad t(u+i\pi) = t(u), \\
t(0) &= -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta \\
&\quad \times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\sinh(\eta+\theta_l)}{\sinh\eta \sinh\eta} \times \mathrm{id}, \\
t(\frac{i\pi}{2}) &= -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta \\
&\quad \times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta \sinh\eta} \times \mathrm{id}, \\
\lim_{u\to\pm\infty} t(u) &= -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} \times \mathrm{id} + \ldots,
\end{aligned}
$$

and the very operator identity

$$
t(\theta_j)t(\theta_j-\eta) = -\frac{\sinh^2\eta\,\Delta_q^{(o)}(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}, \quad \Delta_q^{(o)}(u) = \delta(u)\times \mathrm{id}.
$$

Let $|\Psi\rangle$ be a common eigenstate of the transfer matrix with an eigenvalue $\Lambda(u)$, then

$$
\begin{aligned}
\Lambda(-u-\eta) &= \Lambda(u), \quad \Lambda(u+i\pi) = \Lambda(u), && (14)\\
\Lambda(0) &= -2^3 \sinh\alpha_- \cosh\beta_- \sinh\alpha_+ \cosh\beta_+ \cosh\eta && (15)\\
&\quad \times \prod_{l=1}^{N} \frac{\sinh(\eta-\theta_l)\sinh(\eta+\theta_l)}{\sinh\eta\,\sinh\eta}, && (16)\\
\Lambda(\frac{i\pi}{2}) &= -2^3 \cosh\alpha_- \sinh\beta_- \cosh\alpha_+ \sinh\beta_+ \cosh\eta && (17)\\
&\quad \times \prod_{l=1}^{N} \frac{\sinh(\frac{i\pi}{2}+\theta_l+\eta)\sinh(\frac{i\pi}{2}+\theta_l-\eta)}{\sinh\eta\,\sinh\eta}, && (18)\\
\lim_{u\to\pm\infty}\Lambda(u) &= -\frac{\cosh(\theta_- - \theta_+)e^{\pm[(2N+4)u+(N+2)\eta]}}{2^{2N+1}\sinh^{2N}\eta} + \ldots, && (19)\\
\Lambda(\theta_j)\Lambda(\theta_j-\eta) &= -\frac{\sinh^2\eta\,\delta(\theta_j)}{\sinh(2\theta_j+\eta)\sinh(2\theta_j-\eta)}. && (20)
\end{aligned}
$$

The function $\delta(u)$ is given by

$$\begin{aligned}\delta(u) &= -2^4\frac{\sinh(2u-2\eta)\sinh(2u+2\eta)}{\sinh\eta\,\sinh\eta}\sinh(u+\alpha_-)\sinh(u-\alpha_-)\cosh(u+\beta_-)\\ &\quad\times\cosh(u-\beta_-)\sinh(u+\alpha_+)\sinh(u-\alpha_+)\cosh(u+\beta_+)\cosh(u-\beta_+)\\ &\quad\times\prod_{l=1}^{N}\frac{\sinh(u+\theta_l+\eta)\sinh(u-\theta_l+\eta)\sinh(u+\theta_l-\eta)\sinh(u-\theta_l-\eta)}{\sinh(\eta)\sinh(\eta)\sinh(\eta)\sinh(\eta)}.\end{aligned}$$

Moreover, it follows that $\Lambda(u)$, as an entire function of u, is a trigonometric polynomial of degree $2N+4$. Hence (14)-(20) completely determine the function $\Lambda(u)$. For this purpose, let us introduce the following functions:

$$\begin{aligned}A(u) &= \prod_{l=1}^{N}\frac{\sinh(u-\theta_l+\eta)\,\sinh(u+\theta_l+\eta)}{\sinh\eta\,\sinh\eta},\\ a(u) &= -2^2\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh(u-\alpha_-)\cosh(u-\beta_-)\\ &\quad\times\sinh(u-\alpha_+)\cosh(u-\beta_+)A(u),\\ d(u) &= a(-u-\eta).\end{aligned}$$

The complete solutions to (14)-(20) can be parameterized by the most general inhomogeneous $T-Q$ ansatz [**J. Cao et al**, arXiv:1306.1742].

$$\begin{aligned}\Lambda(u) &= a(u)\frac{Q_1(u-\eta)Q(u-\eta)}{Q_2(u)Q(u)}+d(u)\frac{Q_2(u+\eta)Q(u+\eta)}{Q_1(u)Q(u)}\\ &\quad +\frac{c(u)\sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)Q(u)}A(u)A(-u-\eta), \qquad (21)\end{aligned}$$

where the functions $Q_1(u)$, $Q_2(u)$ and $Q(u)$are some trigonometric polynomials ,

$$\begin{aligned}Q_1(u) &= \prod_{j=1}^{2M}\frac{\sinh(u-\mu_j)}{\sinh(\eta)}, \quad Q_2(u)=Q_1(-u-\eta), \quad 2M+M_1=N+n,\\ Q(u) &= \prod_{j=1}^{M_1}\frac{\sinh(u-\lambda_j)\sinh(u+\lambda_j+\eta)}{\sinh\eta\,\sinh\eta}.\end{aligned}$$

The function $c(u)$ is a self-crossing trigonometric polynomials to make asymptotic behaviors satisfied. Regularity $\Rightarrow$ BAEs

Each choice gives rise to the complete set of eigenvalues. Here two typical choices.
I. $M = 0, M_1 = N$

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)} + \frac{c\sinh(2u)\sinh(2u+2\eta)}{Q(u)}A(u)A(-u-\eta), \quad (22)$$

where c is a constant and $Q(u)$ is

$$Q(u) = \prod_{j=1}^{N}\frac{\sinh(u-\lambda_j)\sinh(u+\lambda_j+\eta)}{\sinh\eta\,\sinh\eta}.$$

where the N parameters $\{\lambda_j\}$ satisfy the associated BAEs

$$\frac{a(\lambda_j)Q(\lambda_j-\eta)}{d(\lambda_j)Q(\lambda_j+\eta)} + 1 = -\frac{c\sinh 2\lambda_j\sinh(2\lambda_j+2\eta)A(\lambda_j)A(-\lambda_j-\eta)}{d(\lambda_j)Q(\lambda_j+\eta)}, \quad (23)$$

$$j = 1, \ldots, N.$$

II. $M_1 = 0,\ M = \frac{N+n}{2}$, $n = 0$ for an even N, $n = 1$ for an odd N

$$\Lambda(u) = a(u)\frac{Q_1(u-\eta)}{Q_2(u)} + d(u)\frac{Q_2(u+\eta)}{Q_1(u)} + \frac{c[\sinh(u)\sinh(u+\eta)]^n \sinh(2u)\sinh(2u+2\eta)}{Q_1(u)Q_2(u)}A(u)A(-u-\eta), \quad (24)$$

where the functions $Q_1(u)$, $Q_2(u)$ are

$$Q_1(u) = \prod_{j=1}^{2M}\frac{\sinh(u-\mu_j)}{\sinh(\eta)}, \quad Q_2(u) = Q_1(-u-\eta),$$

The $N+n$ parameters $\{\mu_j\}$ satisfy the associated BAEs

$$\frac{c\sinh(2\mu_j)\sinh(2\mu_j+2\eta)\,A(\mu_j)A(-\mu_j-\eta)}{d(\mu_j)Q_2(\mu_j)Q_2(\mu_j+\eta)} = -[\frac{\sinh\eta\sinh\eta}{\sinh\mu_j\sinh(\mu_j+\eta)}]^n,$$
$$j = 1,\ldots,N+n, \quad (25)$$

- QYBE& REs $\Rightarrow$ **Integrability** $\Leftrightarrow$ **Transfer matrix**

- *Intrinsic Prop.* of R(or K)-matrix $\Rightarrow$ **Operator Id.** $\Leftrightarrow$ **Solvability**

Moreover, almost all of standard R-matrices and K-matrices have such intrinsic properties.

The R-matrix reads

$$R_{12}(u)=\left(\begin{array}{ccc|ccc|ccc}
c(u) & & & & & & & & \\
 & b(u) & & e(u) & & & & & \\
 & & d(u) & & g(u) & & f(u) & & \\
\hline
 & \bar{e}(u) & & b(u) & & & & & \\
 & & \bar{g}(u) & & a(u) & & g(u) & & \\
 & & & & & b(u) & & e(u) & \\
\hline
 & & \bar{f}(u) & & \bar{g}(u) & & d(u) & & \\
 & & & & & \bar{e}(u) & & b(u) & \\
 & & & & & & & & c(u)
\end{array}\right).$$

It is the first simplest model beyond A-type.

The matrix elements are

$$a(u) = \sinh(u-3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh\eta, \quad b(u) = \sinh(u-3\eta) + \sin 3\eta,$$

$$c(u) = \sinh(u-5\eta) + \sinh\eta, \quad d(u) = \sinh(u-\eta) + \sinh\eta,$$

$$e(u) = -2e^{-\frac{u}{2}} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta), \quad \bar{e}(u) = -2e^{\frac{u}{2}} \sinh 2\eta \cosh(\frac{u}{2} - 3\eta),$$

$$f(u) = -2e^{-u+2\eta} \sinh\eta \sinh 2\eta - e^{-\eta} \sinh 4\eta,$$

$$\bar{f}(u) = 2e^{u-2\eta} \sinh\eta \sinh 2\eta - e^{\eta} \sinh 4\eta,$$

$$g(u) = 2e^{-\frac{u}{2}+2\eta} \sinh\frac{u}{2} \sinh 2\eta, \quad \bar{g}(u) = -2e^{\frac{u}{2}-2\eta} \sinh\frac{u}{2} \sinh 2\eta.$$

The associated non-diagonal K-matrices $K^-(u)$ is

$$K^-(u) = \begin{pmatrix} 1+2e^{-u-\epsilon}\sinh\eta & 0 & 2e^{-\epsilon+\sigma}\sinh u \\ 0 & 1-2e^{-\epsilon}\sinh(u-\eta) & 0 \\ 2e^{-\epsilon-\sigma}\sinh u & 0 & 1+2e^{u-\epsilon}\sinh\eta \end{pmatrix},$$

$$K^+(u) = \mathcal{M}K^-(-u+6\eta+i\pi)\big|_{(\epsilon,\sigma)\to(\epsilon',\sigma')},$$

$$\mathcal{M} = \mathrm{Diag}(e^{2\eta},1,e^{-2\eta}).$$

There four boundary parameters ϵ, σ, ϵ', σ'.

Intrinsic properties of the R-matrix

$$R_{12}(0) \propto \eta P_{12}, \quad R_{12}(u)\,R_{21}(-u) = \rho_1(u), \tag{26}$$

$$R^{t_1}_{12}(u)\,\mathcal{M}_1\,R^{t_1}_{21}(-u+12\eta)\,\mathcal{M}_1^{-1} = \rho_2(u)\ \times \mathrm{id}, \tag{27}$$

$$R_{12}(6\eta + i\pi) = P^{(1)}_{12} \times S^{(1)}_{12}, \quad R_{12}(4\eta) = P^{(3)}_{12} \times S^{(3)}_{12}. \tag{28}$$

and the corresponding properties of the K-matrices:

$$K^-(0) = (1 + 2e^{-\epsilon}\sinh\eta), \quad K^+(6\eta + i\pi) = (1 + 2e^{-\epsilon'}\sinh\eta)\mathcal{M}, \tag{29}$$

$$K^-(i\pi) = (1 - 2e^{-\epsilon}\sinh\eta), \quad K^+(6\eta) = (1 - 2e^{-\epsilon'}\sinh\eta)\mathcal{M}, \tag{30}$$

$$K^-(u)K^-(-u) \propto \mathrm{id}, \quad K^+(u)\,K^+(-u + 6\eta + i\pi) \propto \mathrm{id}. \tag{31}$$

$P^{(1)}_{12}$ and $P^{(3)}_{12}$ are projectors with rank 1 and 3 respectively.

These properties lead to the following operator identities which complete characterize the spectrum of the transfer matrix

$$t(\theta_j)t(\theta_j+6\eta+i\pi) = \frac{\delta_1(u)\ \times \mathrm{id}}{\rho_1(2u)}\Big|_{u=\theta_j}, \quad j=1,\ldots,N, \tag{32}$$

$$t(\theta_j)t(\theta_j+4\eta) = \frac{\delta_2(u)\ \times t(u+2\eta+i\pi)}{\rho_2(-2u+8\eta)}\Big|_{u=\theta_j}, \quad j=1,\ldots,N, \tag{33}$$

$$t(u) = t(-u+6\eta+i\pi), \quad t(u)=t(u+2i\pi). \tag{34}$$

where $\delta_i(u)$ are some functions.

and the values of the transfer matrix at 0, $i\pi$, ∞

$$t(0)=t(6\eta+i\pi)=(1+2e^{-\epsilon}\sinh\eta)tr\{K^{+}(0)\}\prod_{l=1}^{N}\rho_1(-\theta_l)\times\mathrm{id}, \quad (35)$$

$$t(i\pi)=t(6\eta)=(1-2e^{-\epsilon}\sinh\eta)tr\{K^{+}(i\pi)\}\prod_{l=1}^{N}\rho_1(i\pi-\theta_l)\times\mathrm{id}, \quad (36)$$

$$\lim_{u\to\pm\infty}t(u)=(\frac{1}{2})^{2N}e^{-\epsilon-\epsilon'}e^{\pm 2(N+1)(u-3\eta)}(1+2\cosh(\sigma'-\sigma+2\eta))\times\mathrm{id}+\ldots \quad (37)$$

$\Rightarrow$ T-Q relation

- *JHEP* 06 (2014), 128 [arXiv:1403.7915].

Conclusion and comments

So far, many typical $U(1)$-symmetry-broken models have been solved by the method:

- The spin torus.
- The XYZ closed spin chain.
- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields and its higher spin generalization.
- The open spin chains with general boundary condition associated with A-type algebras.
- The Hubbard model with unparallel boundary fields.
- The t-J model with unparallel boundary fields.
- The Izergin-Korepin model with non-diagonal boundary terms.

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Integrability $\Leftrightarrow$ Solvability

Thank for your attentions