A q-difference Baxter's operator, Bäcklund transformations and the Ablowitz-Ladik chain

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RAQIS14
IMB, *September 2, 2014*

Introduction and motivations

We consider the quantum version of the Ablowitz-Ladik model and the corresponding quantum version of its Bäcklund transformations. One of the main aims is to emphasize the significant relationships between the Baxter's operator and the Bäcklund transformations theory.

Outline

- Definition of the classical and quantum models.
- Bethe ansatz, quantum determinant and Baxter's equation.
- Classical and quantum Bäcklund transformations for the model.
- Baxter's operator and its q-integral representation.
- Discussion.

Classical equations of motion

$$\dot{q}_k = q_{k+1} + q_{k-1} - 2q_k - q_k r_k (q_{k+1} + q_{k-1}),$$
$$\dot{r}_k = -r_{k+1} - r_{k-1} + 2r_k + q_k r_k (r_{k+1} + r_{k-1}).$$

The model possesses a Lax matrix representation

$$L(\lambda) = \prod_{k=1}^{\overset{\curvearrowleft}{N}} L_k(\lambda), \qquad \text{with} \qquad L_k(\lambda) = \begin{pmatrix} \lambda & q_k \\ r_k & \lambda^{-1} \end{pmatrix}$$

with a Poisson algebra defined by a r-matrix structure
$\{L(\lambda) \otimes L(\nu)\} = [r(\lambda/\nu),\ L(\lambda) \otimes L(\nu)]$

$$r(\lambda/\nu) \doteq \begin{pmatrix} \frac{1}{2}\frac{\nu^2+\lambda^2}{\nu^2-\lambda^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\lambda\nu}{\nu^2-\lambda^2} & 0 \\ 0 & \frac{\lambda\nu}{\nu^2-\lambda^2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\frac{\nu^2+\lambda^2}{\nu^2-\lambda^2} \end{pmatrix}.$$

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The above relations are equivalent to the following Poisson brackets among the dynamical variables of the model

$$\{q_k, r_j\} = (1 - q_k r_k)\delta_{kj}, \qquad \{q_k, q_j\} = \{r_k, r_j\} = 0.$$

The quantum r matrix is given by the expression

$$R = (1 + \eta/2)\,\mathbb{1} - \eta r, \qquad \eta \doteq \mathrm{i}\hbar$$

defining the commutation relations

$$R(\lambda/\nu) \overset{1}{L}(\lambda) \overset{2}{L}(\nu) = \overset{2}{L}(\nu) \overset{1}{L}(\lambda) R(\lambda/\nu)$$

equivalent, at the level of the variables (q_k, r_k), to

$$[q_k, r_j] = \eta(1 - q_k r_k)\delta_{kj}.$$

The quantum r-matrix solves the Yang-Baxter equation

$$R_{ic,ja}(\lambda/\nu) R_{cm,kb}(\lambda) R_{an,br}(\nu) = R_{ja,kb}(\nu) R_{ic,br}(\lambda) R_{cm,an}(\lambda/\nu)$$

(convention on indices: if $T = A \otimes B$ then $T_{\alpha\beta,\gamma\delta} = A_{\alpha\beta} B_{\gamma\delta}$)

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Let us set

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

The quantum r-matrix defines the commutation relations among the elements of the monodromy matrix, e.g.

$$A(\nu)C(\lambda) = f(\nu,\lambda)C(\lambda)A(\nu) + g(\lambda,\nu)C(\nu)A(\lambda),$$
$$D(\nu)C(\lambda) = \tilde{f}(\nu,\lambda)C(\lambda)D(\nu) + \tilde{g}(\lambda,\nu)C(\nu)D(\lambda)$$

where $f(\nu,\lambda) = \frac{1}{1+\eta}\left(1 - \eta\frac{\nu^2}{\lambda^2-\nu^2}\right)$, $g(\lambda,\nu) = \frac{\eta}{1+\eta}\frac{\lambda\nu}{\lambda^2-\nu^2}$, $\tilde{f}(\nu,\lambda) = f(\lambda,\nu)$ and $\tilde{g}(\lambda,\nu) = -g(\lambda,\nu)$.

Define a vacuum state $|0\rangle$ by $B(\lambda)|0\rangle = 0$. If it is unique, it is also an eigenvector of $A(\lambda)$ and $D(\lambda)$ with respective eigenvalues

$$a(\lambda) = \lambda^N, \quad d(\lambda) = \lambda^{-N}.$$

Looking for a set of eigenfunctions in the form (m and the λ_k's arbitrary)

$$\phi_m(\{\lambda\}) = \prod_{k=1}^{m} C(\lambda_k)|0\rangle,$$

and using the previous commutation relations, one gets

$$\mathrm{Tr}(L(\nu))\phi_m = \left(a(\nu)\prod_j f(\nu,\lambda_j) + d(\nu)\prod_j \tilde{f}(\nu,\lambda_j) \right)\phi_m + \\ + \sum_k \left(a(\lambda_k)\Lambda_k + d(\lambda_k)\tilde{\Lambda}_k \right) \prod_{j\neq k} C(\lambda_j)C(\nu)|0\rangle,$$

where $\Lambda_k = g(\lambda_k,\nu)\prod_{j\neq k} f(\lambda_k,\lambda_j)$. If the Bethe equations

$$a(\lambda_k)\Lambda_k + d(\lambda_k)\tilde{\Lambda}_k = 0 \Longrightarrow \prod_{j\neq k} \left(\frac{\lambda_j^2(1+\eta) - \lambda_k^2}{\lambda_j^2 - (1+\eta)\lambda_k^2} \right) = \lambda_k^{2N}, \quad k = 1 \ldots m,$$

are satisfied, then ϕ_m is an eigenvector of $\mathrm{Tr}(L(\nu))$.

The eigenvalue equation

$$\mathrm{Tr}(L(\nu))\phi_m = \frac{\nu^N}{(1+\eta)^m}\prod_{j=1}^{m}\left(1-\eta\frac{\nu^2}{\lambda_j^2-\nu^2}\right)\phi_m + \frac{\nu^{-N}}{(1+\eta)^m}\prod_{j=1}^{m}\left(1+\eta\frac{\lambda_j^2}{\lambda_j^2-\nu^2}\right)\phi_m$$

gives the Baxter's equation: indeed if we call $t(\nu)$ the eigenvalue of $\mathrm{Tr}L(\nu)$ and set

$$\psi(\nu,\{\lambda\}) \doteq \prod_{j=1}^{m}(\nu^2-\lambda_j^2)$$

then the eigenvalue equation is equivalent to

$$t(\nu)\psi(\nu,\{\lambda\}) = \frac{\nu^N}{(1+\eta)^m}\psi(\nu\sqrt{\eta+1},\{\lambda\}) + \frac{1}{\nu^N}\psi(\frac{\nu}{\sqrt{1+\eta}},\{\lambda\})$$

It is possible to understand deeper the structure of this equation.

We introduce the quantum determinant of the monodromy matrix. By using the commutation relations, it is possible to show that

$$\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \frac{D(\lambda\sqrt{\alpha})}{\sqrt{\alpha}} & -B(\lambda\sqrt{\alpha}) \\ -\frac{C(\lambda\sqrt{\alpha})}{\alpha} & \frac{A(\lambda\sqrt{\alpha})}{\sqrt{\alpha}} \end{pmatrix} = \left(\sqrt{\alpha}\right)^{N-1} \Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} \frac{D(\lambda\sqrt{\alpha})}{\sqrt{\alpha}} & -\frac{B(\lambda\sqrt{\alpha})}{\alpha} \\ -C(\lambda\sqrt{\alpha}) & \frac{A(\lambda\sqrt{\alpha})}{\sqrt{\alpha}} \end{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \left(\sqrt{\alpha}\right)^{N-1} \Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we pose

$$\alpha \doteq \frac{1}{1+\eta} = \frac{1}{1+i\hbar}$$

and we defined

$$\left(\sqrt{\alpha}\right)^{N-1} \Delta \doteq \frac{A(\lambda)D(\lambda\sqrt{\alpha})}{\sqrt{\alpha}} - \frac{B(\lambda)C(\lambda\sqrt{\alpha})}{\alpha} = \frac{D(\lambda)A(\lambda\sqrt{\alpha})}{\sqrt{\alpha}} - C(\lambda)B(\lambda\sqrt{\alpha}) =$$
$$= \frac{A(\lambda\sqrt{\alpha})D(\lambda)}{\sqrt{\alpha}} - C(\lambda\sqrt{\alpha})B(\lambda) = \frac{D(\lambda\sqrt{\alpha})A(\lambda)}{\sqrt{\alpha}} - \frac{B(\lambda\sqrt{\alpha})C(\lambda)}{\alpha}.$$

Remark 1

The operators $\Delta(\lambda)$ and $Tr(L(\mu))$ commute.

Remark 2

For our specific monodromy matrix one has $\Delta = \prod_{k=1}^{N}(1 - r_k q_k)$

Corollary 1

The eigenvalues of the quantum determinant Δ on the eigenfunctions ϕ_m are given by the relations

$$\Delta(\lambda)\prod_{j=1}^{m} C(\lambda_j)|0\rangle = \alpha^m \frac{d(\lambda)a(\lambda\sqrt{\alpha})}{(\sqrt{\alpha})^N}\prod_{j=1}^{m} C(\lambda_j)|0\rangle.$$

and, in our case

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The meaning of the Baxter's equation is now clearer: if we call δ_m the eigenvalue of the quantum determinant Δ, then we have

$$t(\nu)\psi_m(\nu,\{\lambda\}) = \delta_m \nu^N \psi_m(\frac{\nu}{\sqrt{\alpha}},\{\lambda\}) + \frac{1}{\nu^N}\psi_m(\nu\sqrt{\alpha},\{\lambda\}).$$

By choosing a different normalization for the function $\psi_m(\nu)$, that is $\psi_m(\nu) = \hat{\psi}_m(\nu)\nu^{2m}$, one gets

$$t(\nu)\hat{\psi}_m(\nu,\{\lambda\}) = \nu^N \hat{\psi}_m(\frac{\nu}{\sqrt{\alpha}},\{\lambda\}) + \frac{\delta_m}{\nu^N}\hat{\psi}_m(\nu\sqrt{\alpha},\{\lambda\})$$

that is the factor δ_m moved to one addend to the other.
In general the Baxter's equation will be (see also [Korff, '12])

$$t(\nu)\psi(\nu,\{\lambda\}) = \Lambda_+ \psi(\frac{\nu}{\sqrt{\alpha}},\{\lambda\}) + \Lambda_- \psi(\nu\sqrt{\alpha},\{\lambda\})$$

where Λ_+ and Λ_- are two scalar factors whose product gives δ_m, the eigenvalue of the corresponding quantum determinant.

Bäcklund transformations, Baxter's equation and Baxter's operator.

Classical Bäcklund transformations

A set of Bäcklund transformations for the classical model can be obtained through the dressing matrix technique.

$$\mathcal{D}_k(\lambda) = \begin{pmatrix} \lambda^2 + a_k & \lambda b_k \\ \lambda c_k & 1 \end{pmatrix}, \Longrightarrow \mathcal{D}_k(\lambda) = \begin{pmatrix} \lambda^2 - \mu^2(1 - b_k c_k) & \lambda b_k \\ \lambda c_k & 1 \end{pmatrix}$$

$$\tilde{L}_k \mathcal{D}_k - \mathcal{D}_{k+1} L_k = 0, \Longrightarrow b_k = q_k, \qquad c_k = \tilde{r}_{k-1}$$

$$1 - q_k r_k = \frac{(\tilde{r}_{k-1} - r_k)(\tilde{r}_k \mu^2 + r_k)}{\mu^2 \tilde{r}_k \tilde{r}_{k-1}}$$

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These equations define $\tilde{q}_k$ and q_k in terms of the variables $\tilde{r}_k$ and r_k.

Two of the main properties of the transformations:

1. the conserved quantities are invariant under the action of the maps.
2. the transformations are canonical.

The explicit form of the generating function of the canonical transformations is

$$F = \sum_k \int_{r_{k+1}+1}^{\tilde{r}_k} \frac{\ln(z - r_{k+1})}{z} dz + \int_{1/\mu^2}^{\tilde{r}_k} \frac{\ln(\mu^2 z + r_k)}{z} dz - \ln(\tilde{r}_k)\ln(\mu^2 \tilde{r}_{k-1}) - 2\ln(\mu)^2.$$

Remark 3

The derivative of F w.r.t. μ gives conserved quantities (spectrality property [Kuznetsov & Sklyanin, '99]).

$$\Phi \doteq \left.\frac{\partial F}{\partial \mu}\right|_{\tilde{r}=\tilde{r}(r,q)} = \frac{2}{\mu} \ln\left(\frac{\det(L(\mu))}{\mu^N \gamma}\right).$$

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By using the previous expression of Φ and the obvious relation

$$\mathrm{Tr}L(\mu) = \gamma + \frac{\det(L(\mu))}{\gamma}$$

we get

$$\mathrm{Tr}L(\mu) = \mu^N e^{\frac{\mu}{2}\Phi} + \frac{\det(L(\mu))}{\mu^N} e^{-\frac{\mu}{2}\Phi}.$$

This classical expression is exact.

From the quantum point of view, since Φ is c.c. to μ, we can substitute $\Phi \to \eta\frac{\partial}{\partial\mu}$, giving

$$\mathrm{Tr}L(\mu)\rho(\mu, r) = \mu^N e^{\frac{\mu}{2}\eta\frac{\partial}{\partial\mu}}\rho(\mu, r) + \frac{\Delta}{\mu^N} e^{-\frac{\mu}{2}\eta\frac{\partial}{\partial\mu}}\rho(\mu, r) =$$

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Quantization

Since $[q_k, r_j] = \eta(1 - q_k r_k)\delta_{kj}$, the action of q_k on a function $f(\{r_j\})$ is proportional to the Jackson derivative in the direction of q_k, that is $q_k f(\{r_j\}) = (1-\alpha) D_{\alpha,k} f(\{r_j\})$, where

$$D_{\alpha,k} f(\{r_j\}) \doteq \frac{f(r_1, \ldots, \alpha r_k, \ldots, r_N) - f(r_1, \ldots, r_k, \ldots, r_N)}{\alpha r_k - r_k}, \quad \alpha = \frac{1}{1+\eta}.$$

We are looking for an operator $\mathcal{Q}_\mu$ satisfying

$$\mathrm{Tr}(L(\mu))\mathcal{Q}_\mu = \mu^N \mathcal{Q}_{\frac{\mu}{\sqrt{\alpha}}} + \frac{\Delta}{\mu^N}\mathcal{Q}_{\mu\sqrt{\alpha}}, \tag{1}$$

Following [Pasquier & Gaudin, '92], [Sklyanin, '00], to evaluate the trace of the monodromy matrix we perform a "similarity" transformation

$$\hat{L}_k = M_{k+1}^{-1} L_k M_k$$

where

$$M_k = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \tilde{r}_{k-1} \end{pmatrix}$$

Remark 4

The column $|w_k\rangle = \begin{pmatrix} 1 \\ -\mu\tilde{r}_{k-1} \end{pmatrix}$ *is the kernel of the matrix* $\mathcal{D}_k(\mu) \Rightarrow$*, the matrix* $\hat{L}_k$ *is lower triangular.*

In the quantum case, from the element 21 of $\hat{L}_k\rho_k$ (ρ_k are the "column" of $\mathcal{Q}$) we get the q-difference equation

$$\mu^2\tilde{r}_k(1 - q_k\tilde{r}_{k-1})\rho_k = (\tilde{r}_{k-1} - r_k)\rho_k$$

More explicitly

$$\rho_k(\mu, r_k) = \frac{\mu^2\tilde{r}_k\tilde{r}_{k-1}}{(\mu^2\tilde{r}_k + r_k)(\tilde{r}_{k-1} - r_k)}\rho_k(\mu, \alpha r_k),$$

that is solved by

$$\rho_k(\mu, r_k) = G_k \prod_{p=0} \frac{1}{(1 + \alpha^p \frac{r_k}{\mu^2\tilde{r}_k})(1 - \alpha^p \frac{r_k}{\tilde{r}_{k-1}})} = \frac{G_k}{(\frac{r_k}{\tilde{r}_{k-1}}; \alpha)_\infty(-\frac{r_k}{\mu^2\tilde{r}_k}; \alpha)_\infty}$$

where G_k is independent of r_k.

By using the previous expression of the function ρ_k it is possible to show that the matrix $\hat{L}_k\rho_k$ can be written as

$$\hat{L}_k\rho_k = \begin{pmatrix} \frac{\mu^2\tilde{r}_k+r_k}{\mu\tilde{r}_{k-1}} & 0 \\ -q_k & \frac{\tilde{r}_{k-1}-r_k}{\mu\tilde{r}_k} \end{pmatrix}\rho_k$$

that is

$$\hat{L}_k\rho_k(\mu, r_k) = \begin{pmatrix} \frac{\mu\tilde{r}_k}{\tilde{r}_{k-1}}\rho_k(\frac{\mu}{\sqrt{\alpha}}, r_k) & 0 \\ -q_k\rho_k(\mu, r_k) & \frac{\tilde{r}_{k-1}}{\mu\tilde{r}_k}\rho_k(\mu\sqrt{\alpha}, \alpha\rho_k) \end{pmatrix}$$

from which it follows that the trace of $L(\mu)$ on $\rho = \prod_k \rho_k$ is given by

$$\mathrm{Tr}(L(\mu))\rho(\mu, r) = \mu^N\rho(\frac{\mu}{\sqrt{\alpha}}, r) + \frac{1}{\mu^N}\rho(\mu\sqrt{\alpha}, \alpha r).$$

or, from $\Delta\rho(r) = \prod_k(1 - r_kq_k)\rho(r) = \rho(\alpha r)$

$$\mathrm{Tr}(L(\mu))\rho(\mu, r) = \mu^N\rho(\frac{\mu}{\sqrt{\alpha}}, r) + \frac{\Delta}{\mu^N}\rho(\mu\sqrt{\alpha}, r).$$

Remark 5

The Baxter's equation involves three commuting operators (the determinant is not a c-number neither a Casimir). But a single equation is still enough to get the eigenvalues of both the trace and the determinant of the monodromy matrix.

Indeed considering the Baxter's eigenvalue equation

$$t(\mu)q(\mu) = \mu^N q\left(\frac{\mu}{\sqrt{\alpha}}\right) + \frac{\delta}{\mu^N} q\left(\mu\sqrt{\alpha}\right) \quad (2)$$

and expanding $q(\mu) = \prod_{j=1}^m (\frac{1}{\mu^2} - \frac{1}{\lambda_j^2})$, in the limit $\mu \to 0$ we get $\delta = \alpha^m$. The zeros of the right hand side of equation (2) gives the Bethe equations

$$\lambda_k^{2N} = \prod_{j \neq k} \left(\frac{\lambda_j^2(1+\eta) - \lambda_k^2}{\lambda_j^2 - (1+\eta)\lambda_k^2} \right)$$

Remark 6

The semi-classical limit of the quantum Bäcklund transformations is linked with the generating function of the corresponding classical transformations (e.g. [Pasquier & Gaudin, '92], [Sklyanin, '00]).

Consider the eigenvalue equation

$$t(\mu)q(\mu) = \mu^N q\left(\frac{\mu}{\sqrt{\alpha}}\right) + \frac{\delta}{\mu^N} q\left(\mu\sqrt{\alpha}\right)$$

and look for a solution in the form

$$q(\mu) = e^{\frac{1}{\eta}\left(S_0 + \eta S_1 + \eta^2 S_2 + \dots\right)},$$

we get for S_0

$$t(\mu) = \mu^N e^{\frac{\mu}{2} S_0'} + \frac{\det(L(\mu))}{\mu^N} e^{-\frac{\mu}{2} S_0'}.$$

Confronting with the classical trace equation one identifies $S_0 = F$.

A q-integral formula for the Baxter's operator.

It is well known that the Baxter's operator can be expressed as the trace of a monodromy matrix (e.g. [Baxter, '89], [Bazhanov et al., '97], [Pronko, '00]). The monodromy matrix can be built as a product of elementary operators $\mathcal{R}^k_\mu$ from the spaces $\mathbb{C}[\tilde{r}_k, c_k]$ to $\mathbb{C}[r_k, c_{k+1}]$, where $\mathbb{C}[c_k]$ is the space of the so-called "auxiliary variables".
Since the Baxter's equation is a q-difference equation, we expect that the Baxter's operator can be represented by a q-integral formula. So we introduce the inverse of the operator q_k

$$(q_k)^{-1} f(\{r_j\}) \doteq \int d_\alpha r_k f(\{r_j\}) \doteq \sum_{n=0} \alpha^n r_k f(r_1, \ldots, \alpha^n r_k, \ldots, r_N).$$

Notice that $(1-\alpha)(q_k)^{-1}$ is the usual Jackson integral, the inverse of the Jackson derivative.

Also, we introduce the following q-integral representation for the operators $\mathcal{R}_\mu^k$

$$\mathcal{R}_\mu^k : \psi(c_k, \tilde{r}_k) \to \int d_\alpha c_k \int d_\alpha \tilde{r}_k P_k(\alpha c_k, \alpha \tilde{r}_k | c_{k+1}, r_k) \psi(c_k, \tilde{r}_k)$$

The $\mathcal{Q}$ operator is defined to be

$$\mathcal{Q}_\mu : \psi(\tilde{r}) \to \int d_\alpha \tilde{r}_N \dots \int d_\alpha \tilde{r}_1 \hat{Q}_\mu(\alpha \tilde{r} | r) \psi(\tilde{r})$$

where the kernel $\hat{Q}_\mu(\alpha \tilde{r} | r)$ is given by

$$\hat{Q}_\mu(\alpha \tilde{r} | r) = \int d_\alpha c_N \dots \int d_\alpha c_1 \prod_{k=1}^{N} P_k(\alpha c_k, \alpha \tilde{r}_k | c_{k+1}, r_k).$$

The commutativity between $\mathrm{Tr}(L(\lambda))$ and $\mathcal{Q}_\mu$ is ensured by the quantum analogue of the relation $\tilde{L}_k \mathcal{D}_k = \mathcal{D}_{k+1} L_k$, that is

$$\mathcal{R}_\mu^k \tilde{L}_k(\lambda) \mathcal{D}_k(\lambda, \mu) = \mathcal{D}_{k+1}(\lambda, \mu) L_k(\lambda) \mathcal{R}_\mu^k \qquad (3)$$

The YBE (3) gives the following set of equations for $\mathcal{R}_\mu^k$

$$\mathcal{R}_\mu^k \tilde{r}_k = c_{k+1}\mathcal{R}_\mu^k, \quad \mathcal{R}_\mu^k b_k = q_k \mathcal{R}_\mu^k,$$

$$\mathcal{R}_\mu^k \tilde{q}_k = \left(b_{k+1} - \mu^2(1 - b_{k+1}c_{k+1})q_k\right)\mathcal{R}_\mu^k, \quad \mathcal{R}_\mu^k\left(c_k - \mu^2 \tilde{r}_k(1 - b_k c_k)\right) = r_k \mathcal{R}_\mu^k,$$

$$\mathcal{R}_\mu^k\left(\tilde{q}_k c_k - \mu^2(1 - b_k c_k)\right) = \left(b_{k+1} r_k - \mu^2(1 - b_{k+1}c_{k+1})\right)\mathcal{R}_\mu^k.$$

The first equation gives a contribution proportional to a q-delta function, that is we can set

$$P_k(\alpha c_k, \alpha \tilde{r}_k | c_{k+1}, r_k) = \delta_\alpha(\tilde{r}_k - c_{k+1}) F_k(\alpha c_k, c_{k+1}, r_k).$$

Then, by using the the q-Leibniz rule and the q-integration by parts, the other equations gives

$$r_k F_k(\alpha c_k, c_{k+1}, r_k) + \mu^2 \alpha c_{k+1} F_k(\alpha c_k, \alpha c_{k+1}, \alpha r_k) = (r_k + \mu^2 \alpha c_{k+1}) F_k(\alpha c_k, \alpha c_{k+1}, r_k),$$

$$(r_k - c_k) F_k(\alpha c_k, c_{k+1}, r_k) = r_k F_k(c_k, c_{k+1}, r_k) - c_k F_k(\alpha c_k, c_{k+1}, \alpha r_k),$$

$$c_k F_k(\alpha c_k, c_{k+1}, r_k) = (r_k + \mu^2 \alpha c_{k+1}) F_k(\alpha c_k, \alpha c_{k+1}, r_k),$$

$$(c_k - r_k) F_k(\alpha c_k, c_{k+1}, r_k) = \mu^2 c_{k+1} F_k(c_k, c_{k+1}, r_k)$$

The equations are compatible and explicitly solvable for $F_k(\alpha c_k, c_{k+1}, r_k)$. The final result, putting all together, is the following q-integral equation for $\mathcal{Q}_\mu$

$$\mathcal{Q}_\mu : \psi(\tilde{r}) \to \int d_\alpha \tilde{r}_N \ldots \int d_\alpha \tilde{r}_1 \hat{Q}_\mu(\alpha\tilde{r}|r)\psi(\tilde{r})$$

where

$$\hat{Q}_\mu(\alpha\tilde{r}|r) = A \prod_{k=1}^{N} \frac{1}{\tilde{r}_k(\frac{r_k}{\tilde{r}_{k-1}};\alpha)_\infty(-\frac{r_k}{\mu^2\tilde{r}_k};\alpha)_\infty}$$

and A is a constant of normalization that may depend on α and μ.

Conclusions and discussion

- We built the Baxter's equation in two different ways: by using the algebraic Bethe ansatz technique and with the help of the quantum analogue of the classical Bäcklund transformations.
- This confirms the deep relationship between the Bäcklund transformations and the Baxter's operator.
- The Baxter's equation is a q-difference equation whose semi-classical limit is linked with the generating function of the classical Bäcklund transformations.
- The quantum determinant of the monodromy matrix is a conserved quantity but not a Casimir of the Poisson algebra defined by the commutation relations and it plays an explicit role in the Baxter's equation.

- The construction leading to the formulae for the classical Bäcklund transformations has a well defined and precise quantum counterpart.
- We gave a q-integral formula for the Baxter's operator and proved the commutativity properties of $\mathcal{Q}_\mu$ with the other conserved quantities of the model encoded into the trace and quantum determinant of the monodromy matrix.
- We need of a detailed study of the analytic properties of the q-integral representation of the $\mathcal{Q}$ operator
- It would be interesting to investigate on the relationships between the same q-integral representation and the Green's function of the Schrödinger equation corresponding to the interpolating flow of the Bäcklund transformations [Ragnisco & Zullo, '12].

Thanks!